TRIGONOMETRY

Trigonometry is a branch of mathematics that studies relationships involving lengths and angles of a triangle. It comes from two Greek words – *trigonom* (triangle) and *metron* (measure).

There is an enormous number of the uses of trigonometry and trigonometric functions. For instance, the technique of triangulation is used in astronomy to measure the distance between land marks. Although it was first applied in spheres, it had a greater application to planes. Surveyors have used trigonometry for many centuries.

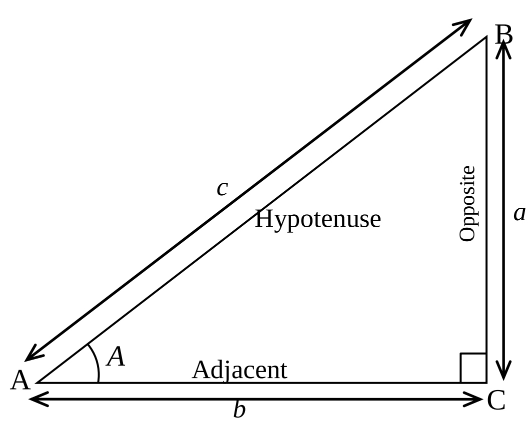
Within mathematics, it is used in calculus (perhaps its greatest application), linear algebra, and statistics.

Trigonometric tables were created over 2000 years ago for computation in astronomy.

A student is expected to be familiar with the definitions of trigonometric ratios for acute angles.

If one angle is 90° and one of the other angles is known, the third can be determined because the three angles of any triangle add up to 180°. The two acute angles therefore add up to 90° (complimentary angles).

Once the angles are known, the ratios of the sides are determined regardless of the overall size of the triangle. If the length of one side is known, the other two are determined. These ratios are given by the following trigonometric functions of known angle, *A*; where *a*, *b*, and *c* refer to the lengths of the sides accompanying the figure.



Sine function (sin)
This is the ratio of the opposite side of the triangle to its hypotenuse.

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}$$

Cosine function (cos)
This is the ratio of the adjacent side to the hypotenuse

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}$$

Tangent function (tan)

This is the ratio of the opposite to the adjacent side.

$$\begin{aligned} \tan A &= \frac{a}{b} = \frac{a}{c} \times \frac{c}{b} \\ &= \left(\frac{a}{c}\right) \div \left(\frac{b}{c}\right) \\ &= \frac{\sin A}{\cos A} \end{aligned}$$

$$\tan A = \frac{\sin A}{\cos A}$$

The hypotenuse is the side opposite to the 90° angle. It is the longest side of a triangle and one of the sides adjacent to A.

The term perpendicular and base are sometimes used for opposite and adjacent sides respectively.

Many people find it easy to remember what sides of the right angle are equal to sine, cosine, or tangent by memorising the mnemonic **SOH-CAH-TOA**.

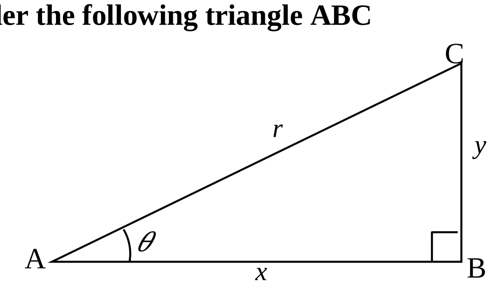
The reciprocals of the functions are named cosecant (cosec), secant (sec) and cotangent (cot)

$$\operatorname{cosec} A = \frac{1}{\sin A} = \frac{\text{Hypotenuse}}{\text{Opposite}} = \frac{c}{a}$$

$$\sec A = \frac{1}{\cos A} = \frac{\text{Hypotenuse}}{\text{Adjacent}} = \frac{c}{b}$$

$$\cot A = \frac{1}{\tan} = \frac{\text{Adjacent}}{\text{Opposite}} = \frac{\cos A}{\sin A} = \frac{b}{a}$$

Consider the following triangle ABC



$$\sin\theta = \frac{y}{r}\ ,\ \cos\theta = \frac{x}{r};\ \text{and}\ \tan\theta = \frac{y}{r}$$

$$y = r\sin\theta; \qquad x = r\cos\theta$$

Applying the Pythagoras' theorem to triangle ABC;

$$\Rightarrow (r\cos\theta)^2 + (r\sin\theta)^2 = r^2$$
$$r^2\cos^2\theta + r^2\sin^2\theta = r^2$$
$$\cos^2\theta + \sin^2\theta = 1$$
$$\mathbf{\cos^2\theta + \sin^2\theta = 1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$\boxed{\mathbf{\cos^2\theta + \sin^2\theta = 1}}$$

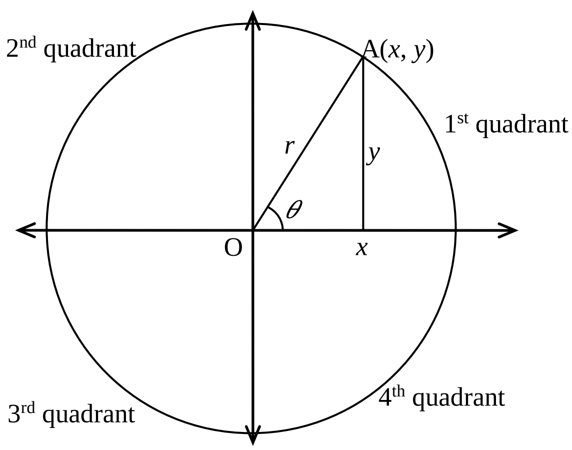
Dividing equation (i) by $\cos^2\theta$

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$\mathbf{1 + \tan^2\theta = \sec^2\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$
$$\boxed{\mathbf{1 + \tan^2\theta = \sec^2\theta}}$$

Dividing Eqn (i) by $\sin^2\theta$

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$
$$\cot^2\theta + 1 = \text{cosec}^2\theta$$
$$\mathbf{1 + \cot^2\theta = cosec^2\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$
$$\boxed{\mathbf{1 + \cot^2\theta = cosec^2\theta}}$$

Trigonometric Ratios for general angle



Angles measured from the x-axis in the anti-clockwise sense are termed as positive angles while those measured in the clockwise sense are negative angles.

When A is in the 1st quadrant, x and y are positive. When A is in the 2nd quadrant, x is negative and y is positive. When A is in the third quadrant, x and y are all negative. When A is in the 4th quadrant, x is positive and y is negative. r is taken to be positive for all positions of the line OA.

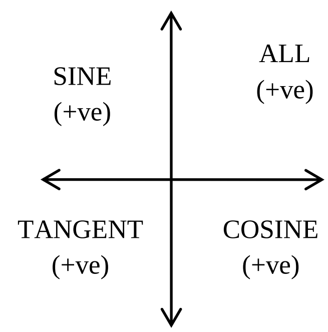
The trigonometrical ratios for angles xOA of any magnitude are defined precisely in the same way as for acute angles.

Thus $\sin\theta = \dfrac{y}{r}$, $\quad \cos\theta = \dfrac{x}{r}$ and $\tan\theta = \dfrac{y}{x}$

The appropriate signs are attached to x and y according to the position of point A. hence for angles in which OA lies in the 1st quadrant; since x and y and r are positive, the sine, cosine, and tangent will all be positive.

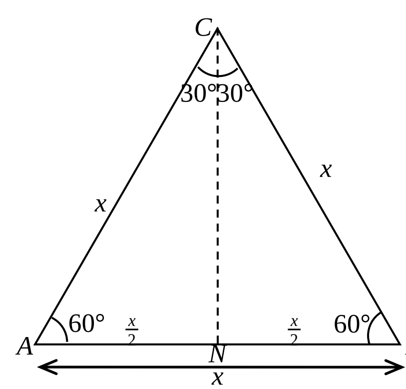
For angles in which OA lies in the 2nd quadrant, since y and r are positive and x negative, the sine is positive. Cosine and tangent are negative.

For angles in which OA is in the 3rd quadrant, sine and cosine are both negative but tangent is positive. In the 4th quadrant, sine and tangent are negative while cosine is positive. This is illustrated below.

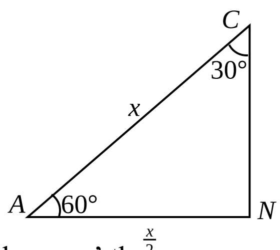


Trigonometric ratios of 30°, 45°, and 60°

Consider the equilateral triangle ABC of side x



Considering triangle CAN:



Applying the Pythagoras' theorem:

$$\left(\tfrac{x}{2}\right)^2 + (\overline{CN})^2 = x^2$$

$$\frac{x^2}{4} + (\overline{CN})^2 = x^2$$

$$(\overline{CN})^2 = x^2 - \frac{x^2}{4}$$

$$\overline{CN}^2 = \frac{3x^2}{4}$$

$$\overline{CN} = \frac{\sqrt{3}x}{2}$$

Using ***SOH-CAH-TOA***

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\sin\theta = \frac{\text{CN}}{\text{AC}}$$

$$\sin 60° = \frac{\frac{\sqrt{3}}{2}x}{x}$$

$$= \frac{\sqrt{3}}{2}$$

$$\sin 30° = \frac{x/2}{x} = \frac{1}{2}$$

$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\sin 60° = \frac{\frac{x\sqrt{3}}{2}}{x} = \frac{\sqrt{3}}{2}$$

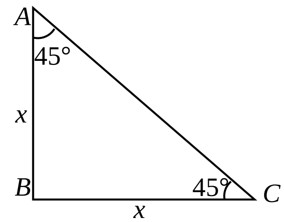
$$\cos 30° = \frac{\frac{\sqrt{3}}{2}x}{x} = \frac{\sqrt{3}}{2}$$

$$\tan 60° = \frac{\frac{x\sqrt{3}}{2}}{x/2} = \sqrt{3}$$

$$\tan 30 = \frac{x/2}{x\sqrt{3}/2}$$

$$\tan 30 = \frac{1}{\sqrt{3}}$$

Consider a right isosceles triangle with two sides of lengths x units.



Applying the Pythagoras' theorem on ABC:

$$x^2 + x^2 = AC^2$$

$$2x^2 = AC^2$$

$$AC = x\sqrt{2}$$

Applying *SOH-CAH-TOA*

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\sin 45° = \frac{x}{AC}$$

$$\sin 45° = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\sin 45° = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$$

$$\sin 45° = \frac{x}{x} = 1$$

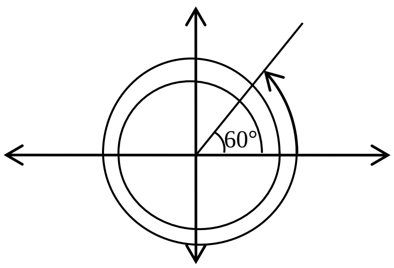
Example I

Write down the values of the following, leaving surds in your answers (*the calculator should not be used*).

- **(a)** cos 780°
- **(b)** sin 780°
- **(c)** tan 780°
- **(d)** sin 540°
- **(e)** cos 540°
- **(f)** cos 210°
- **(g)** sin 150°
- **(h)** sin(-270°)
- **(i)** sin 225°
- **(j)** sin 405°
- **(k)** tan(-60°)

Solution

(a) cos 780.



$\cos 780° = \cos 60°$

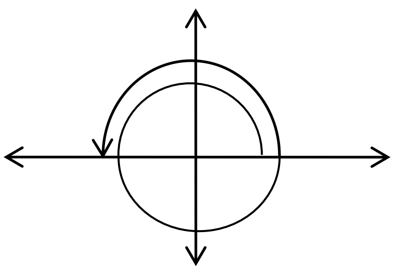
$= \frac{1}{2}$

$\sin 780° = \sin 60°$

$= \frac{\sqrt{3}}{2}$

$\tan 780° = \tan 60° = \sqrt{3}$

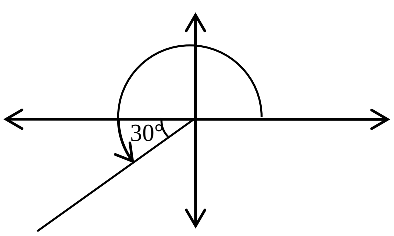
sin 540°



$\sin 540° = \sin 180° = 0°$

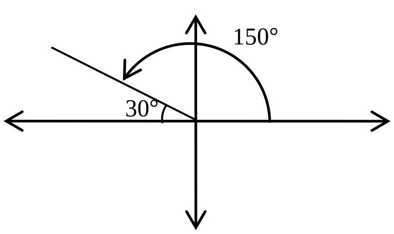
$\cos 540° = \cos 180° = 0°$

cos 210°



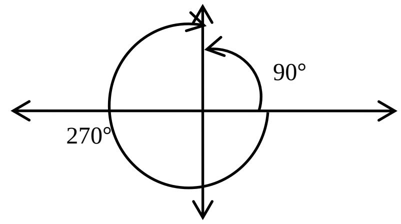
$\cos 210° = -\cos 30° = \frac{-\sqrt{3}}{2}$

sin 150°



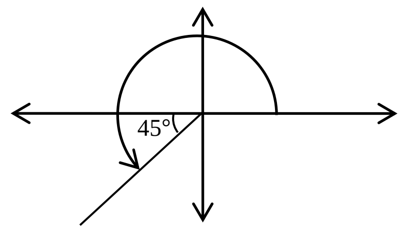
$\sin 150 = +\sin 30 = \frac{1}{2}$

sin -270°



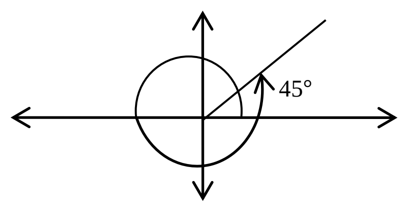
$\sin -270 = +\sin 90° = 1$

sin 225°



$\sin 225° = -\sin 45° = \frac{-1}{\sqrt{2}}$

sin 405°



$\sin 405° = \sin 45° = \frac{1}{\sqrt{2}}$

Trigonometric Curves

For any angle θ, a single value of sin θ or cos θ can be found. The same applies to tan θ unless when $\theta = \pm 90°$ and $\pm 270°$ for which the values of tan θ are not defined. Thus sin θ and cos θ are functions which are defined for all negative values of θ.

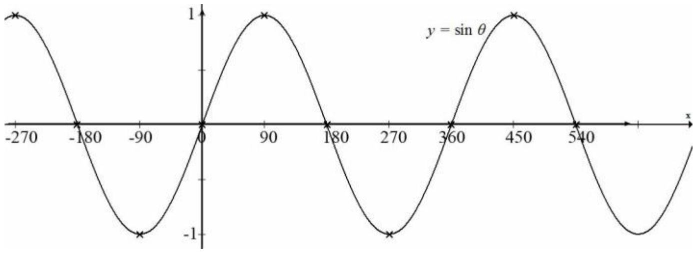
Tan θ is a function which is defined for all positive and negative values of θ except $\pm 90°$ and $\pm 270°$.

To draw the graphs of $\sin\theta$, $\cos\theta$ and $\tan\theta$, we construct a table of values, giving ordered pairs of these functions and hence plot the graph.

Example

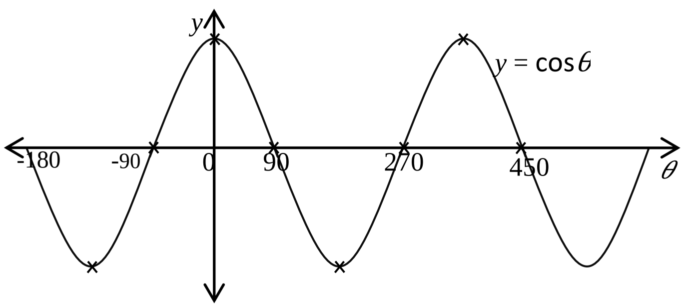
$y = \sin\theta$

θ	-270	-180	-90	0	90	180	270	360	450	540
$y = \sin\theta$	1	0	-1	0	1	0	-1	0	1	0



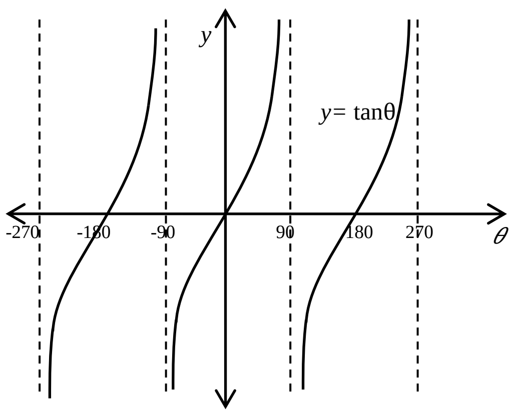
$y = \cos\theta$

θ	-180	-90	0	90	180	270	360	450
$y = \cos\theta$	1	0	1	0	-1	0	1	0



$y = \tan\theta$

θ	-270	-180	-90	0	90	180	270	360	450
$y = \tan\theta$	∞	0	∞	0	∞	0	∞	0	∞



From the graph of sin θ and cos θ, the maximum values of cos θ and sin θ are 1 and 1 respectively. The minimum value of cos θ and sin θ are -1 and -1 respectively.

The graphs for sinθ and cosθ repeat themselves at regular intervals of 360° while that of tanθ repeat itself at regular interval of 180°. These intervals are called periods. These trigonometric functions are examples of periodic functions.

Trigonometric Equations

Trigonometric equations differ from algebraic equations in that they often have unlimited number of solutions.

Example I

Solve the following equations for $0 \leq \theta \leq 360°$

(a) $\sin\theta = \frac{-1}{2}$

(b) $\sec\theta = 2$

(c) $\tan\theta = -\sqrt{3}$

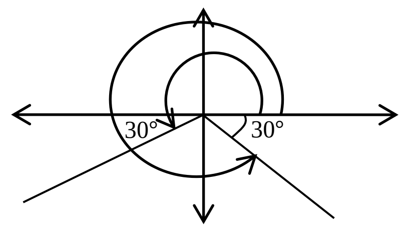
(d) $\sin^2\theta = \frac{1}{2}$

Solutions

$\sin\theta = \frac{-1}{2}$

The acute angle whose sine is $\frac{1}{2}$ is 30°. But sinθ is negative in the 3rd and 4th quadrants.

(a)



$\Rightarrow$ For $\sin\theta = \frac{-1}{2}$

$\theta = 210°$

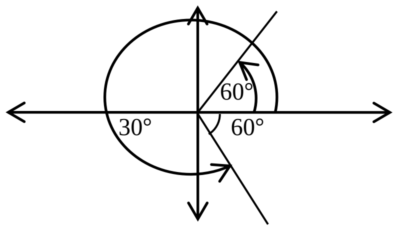
$\theta = 330°$

(b) $\sec\theta = 2$

$$\frac{1}{\cos\theta} = 2$$

$$\Rightarrow \frac{1}{2} = \cos\theta \quad \Rightarrow \cos\theta = \frac{1}{2}$$

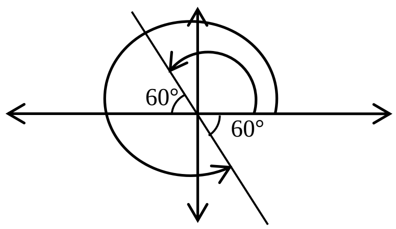
The acute angle whose cosine is $\frac{1}{2}$ is 60° but cos θ° is positive in the 1st and 4th quadrants.



For $\cos\theta = \frac{1}{2}$, $\theta = 60°, 300°$

(c) $\tan\theta = -\sqrt{3}$

The acute angle whose tangent is $\sqrt{3}$ is 60° but $\tan\theta$ is negative in the 2^{nd} and 4^{th} quadrants.



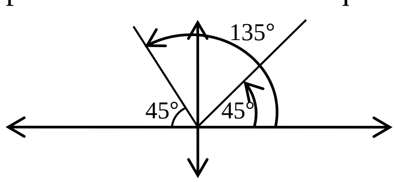
$\Rightarrow$ For $\tan\theta = -\sqrt{3}$, θ = 120°, 300°

(d) $\sin^2\theta = \frac{1}{2}$

$\sin\theta = \pm\frac{1}{\sqrt{2}}$

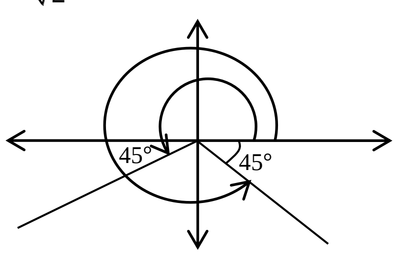
$\Rightarrow \sin\theta = \frac{1}{\sqrt{2}}$ and $\sin\theta \ \frac{-1}{\sqrt{2}}$

The acute angle whose sine is $\frac{1}{\sqrt{2}}$ is 45° but $\sin\theta$ is positive in the 1^{st} and 2^{nd} quadrants.



$\Rightarrow$ For $\sin\theta = \frac{1}{\sqrt{2}}$, θ = 45, 135

For $\sin\theta = \frac{-1}{\sqrt{2}}$



For sin θ = $\frac{-1}{\sqrt{2}}$

$\theta°$ = 225°, 315°

For $\sin^2\theta = \frac{1}{2}$, θ = 45, 135°, 225°, 315°

Example II

Solve the following equations for -180° ≤ θ ≤ 180°.

(a) Sin(2θ + 30) = 0.8
(b) $\tan^2\theta + \tan\theta = 0$
(c) $\sin^2\theta + \sin\theta = 0$
(d) $2\sin^2\theta - \sin\theta - 1 = 0$

Solution

(a) $\sin(2\theta + 30°) = 0.8$

$2\theta + 30° = \sin^{-1}(0.8)$

$2\theta + 30° = 53.1°, 126.9°$

$\Rightarrow$ 2θ = 23.1, 96.9

θ = 11.55, 48.45

For $\sin(2\theta + 30°) = 0.8$, θ = 11.55, 48.45.

(b) $\tan^2\theta + \tan\theta = 0$

tan θ(tan θ + 1) = 0

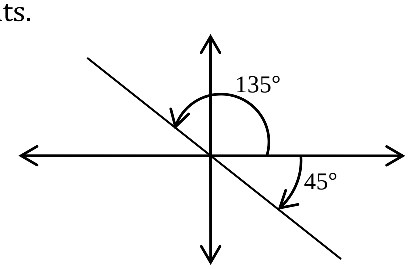
tan θ = 0 **OR** tan θ = -1

For tan θ = 0,

θ = $\tan^{-1}0$

θ = 0, -180, 180

For tan θ = -1, the acute angle whose tangent is 1 is 45°. But tan θ is negative in the 2^{nd} and 4^{th} quadrants.



For tan θ = -1, θ = 135°, -45°

$\Rightarrow$ $\tan^2\theta + \tan\theta = 0$

θ = -180°, -45°, 0, 135°, 180°

(c) $\sin^2\theta + \sin\theta = 0$

sin θ(sin θ +1) = 0

sin θ = 0, sin θ = -1

For sin θ = 0°, θ = 0, 180°, -180°

For sin θ = -1,

The acute angle whose sine is 1 is 90°. Sine is negative in the 3^{rd} and 4^{th} quadrants.

For sin θ = -1, θ = -90

For $\sin^2\theta + \sin\theta = 0°$, θ = -180°, -90°, 0°, 180°

(d) $2\sin^2\theta - \sin\theta - 1 = 0$

$$\sin\theta = \frac{1 \pm \sqrt{(-1)^2 - 4(2)(-1)}}{2 \times 2}$$

$$\sin\theta = \frac{1 \pm 3}{4}$$

$\Rightarrow$ $\sin\theta = 1$, $\sin\theta = \frac{-1}{2}$

For $\sin\theta = 1$,
$\theta = \sin^{-1}(1)$
$\theta = 90°$

For $\sin\theta = \frac{-1}{2}$,

θ = -30°, -150°
$\Rightarrow$ θ = -30°, -150°, 90°

Example III
Solve the following equations from 0° to 360° inclusive.

(a) $\cos 3\theta = \frac{\sqrt{3}}{2}$

(b) $\tan(3\theta - 45°) = \frac{1}{2}$

(c) $\sec 2\theta = 3$
(d) $4\cos 2\theta = 1$

(e) $\tan^2\theta = \frac{1}{3}$

(f) $\sin^2 2\theta = 1$

Solutions

(a) **cos 3θ** $= \frac{\sqrt{3}}{2}$

$$3\theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

3θ = 30°, 330°, 390°, 690°, 750°, 1050°
$\Rightarrow$ θ = 10°, 110°, 130°, 230°, 250°, 350°

(b) $\tan(3\theta - 45°) = \frac{1}{2}$

$$3\theta - 45 = \tan^{-1}\left(\frac{1}{2}\right)$$

$3\theta - 45$ = 26.6, 206.6, 386.6, 566.6, 746.6, 926.6
$\Rightarrow$ θ = 23.9°, 83.9°, 143.9°, 203.9°, 263.9°, 323.9°

(c) **sec 2θ = 3**

$$\frac{1}{\cos 2\theta} = 3$$

$$\frac{1}{3} = \cos 2\theta$$

$$2\theta = \cos^{-1}(\tfrac{1}{3})$$

2θ = 70.5°, 289.5°, 430.5°, 649.5°
θ = 35.25°, 144.75°, 215.25°, 324.75°

(d) **tan^{2}θ** $= \frac{1}{3}$

$$\tan\theta = \pm\frac{1}{\sqrt{3}}$$

$\tan\theta = \frac{1}{\sqrt{3}}$ or $\tan\theta = -\frac{1}{\sqrt{3}}$

For $\tan\theta = \frac{1}{\sqrt{3}}$, θ = 30°, 210°

For tan θ $-\frac{1}{\sqrt{3}}$, θ = 150°, 330°

$\Rightarrow$ When $\tan^2\theta = \frac{1}{3}$, θ = 30°, 150°, 210°, 230°

(e) **sin^{2}2θ = 1**
$\sin 2\theta = \pm 1$
For $\sin 2\theta = 1$,
2θ = 90°, 450° $\Rightarrow$ θ = 45°, 225°
$\sin 2\theta$ = -1,
2θ = 270, 630 $\Rightarrow$ θ = 135°, 315°
$\Rightarrow$ When $\sin^2 2\theta = 1$,
θ = 45°, 135°, 225°, 315°

Example IV
Solve the following equations for values of θ from -180° to 180°
(a) $\tan\theta = \cot\theta + 3$
(b) $\sec\theta = 2\cos\theta$
(c) $5\sin\theta + 6\operatorname{cosec}\theta = 17$
(d) $3\cos\theta + 2\sec\theta + 7 = 0$

Solution
(a) $\tan\theta = 4\cot\theta + 3$

$$\tan\theta = \frac{4}{\tan\theta} + 3$$

$\tan^2\theta = 4 + 3\tan\theta$.
$\tan^2\theta - 3\tan\theta - 4 = 0$

$$\tan\theta = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(-4)}}{2(1)}$$

$$\tan\theta = \frac{3 \pm 5}{2}$$

$\tan\theta = 4$, $\tan\theta = -1$

When $\tan\theta = 4$,
$\theta = \tan^{-1}(4)$
θ = 76°, -104° (for -180° ≤ θ ≤ 180°)
When $\tan\theta = -1$,
$\theta = \tan^{-1}(-1)$ = -45°, 135° (for -180° ≤ θ ≤ 180°)
$\Rightarrow$ For $\tan\theta = 4\cot\theta + 3°$,
θ = -104°, -145°, 76°, 135°

(b) **secθ = 2cosθ**

$\frac{1}{\cos\theta} = 2\cos\theta$

$1 = 2\cos^2\theta$

$\cos^2\theta = \frac{1}{2}$

$\cos\theta = \pm\frac{1}{\sqrt{2}}$

For $\cos\theta = \frac{1}{\sqrt{2}}$, θ = 45°, -45°.

For $\cos\theta = \frac{-1}{\sqrt{2}}$, θ = 135°, -135°

∴ For sec θ = 2cos θ, θ = -135°, -45°, 45°, 135°.

(c) 5sin θ + 6cosec θ = 17

Solution

5sinθ + 6cosecθ = 17

$5\sin\theta + \frac{6}{\sin\theta} = 17$

$5\sin^2\theta + 6 = 17\sin\theta$

$5\sin^2\theta - 17\sin\theta + 6$

$\sin\theta = \frac{17 \pm \sqrt{(-17)^2 - 4(5)\times 6}}{2\times 5}$

$\sin\theta = \frac{17 \pm \sqrt{289 - 120}}{10}$

$\sin\theta = \frac{17 \pm 13}{10}$

sinθ = 3

sinθ = 0.4

$\theta = \sin^{-1}(0.4) \Rightarrow \theta$ = 23.6, 156.4

$\theta = \sin^{-1}(3) \Rightarrow \theta$ has no value since sinθ is maximum when it is 1

(d) 3cos θ +2sec θ +7 = 0

$3\cos\theta + \frac{2}{\cos\theta} + 7 = 0$

$3\cos^2\theta + 2 + 7\cos\theta = 0$

$3\cos^2\theta + 7\cos\theta + 2 = 0$

$\cos\theta = \frac{-7 \pm \sqrt{(7)^2 - 4\times 3\times 2}}{2\times 3}$

$\cos\theta = \frac{-7 \pm 5}{6}$

$\cos\theta = \frac{-1}{3}$

cosθ = -2

For cosθ = -2, θ has no values because the minimum of cosθ is -1

For cos $\theta = \frac{-1}{3}$

θ = 109.5°, -109.5°.

Example IV

Solve the following equations from 0° to 360°

(a) $3 - \cos\theta = 2\sin^2\theta$

(b) $\cos^2\theta + \sin\theta + 1 = 0$

(c) $\sec^2\theta = 3\tan\theta - 1$

(d) $\mathrm{cosec}^2\theta = 3 + \cot\theta$

(e) $3\tan^2\theta + 5 = 7\sec\theta$

Solutions

(a) $3 - \cos\theta = 2\sin^2\theta$

$3 - 3\cos\theta = 2(1 - \cos^2\theta)$

$3 - 3\cos\theta = 2 - 2\cos^2\theta$

$2\cos^2\theta - 3\cos\theta + 1 = 0$

$\cos\theta = \frac{3 \pm \sqrt{(-3)^2 - 4\times 2\times 1}}{2\times 2}$

$\cos\theta = \frac{3 \pm \sqrt{9-8}}{4}$

cos θ = 1, **OR** cos $\theta = \frac{1}{2}$

For cos θ = 1,

$\theta = \cos^{-1}(1)$

θ = 0°, 360°

For cos $\theta = \frac{1}{2}$,

$\theta = \cos^{-1}(1/2)$

θ = 60°, 300°

⇒ For $3 - 3\cos\theta = 2\sin^2\theta$, θ = 0°, 60°, 300°, 360°

(b) $\cos^2\theta + \sin\theta + 1 = 0$

$1 - \sin^2\theta + \sin\theta + 1 = 0$

$\sin^2\theta - \sin\theta - 2 = 0$

$\sin\theta = \frac{1 \pm \sqrt{(-1)^2 - 4\times 1\times -2}}{2}$

$\sin\theta = \frac{1 \pm 3}{2}$

sin θ = 2 **OR** sin θ = -1

For sin θ = 2, the value of θ is not defined because sin θ is maximum at 1

For sin θ= -1, θ = 270°

(c) **sec²**θ = 3tan θ – 1

$\sec^2\theta = 1 + \tan^2\theta$

$\Rightarrow 1 + \tan^2\theta = 3\tan\theta - 1$

$\tan^2\theta - 3\tan\theta + 2 = 0$

$$\tan\theta = \frac{3 \pm \sqrt{(-3)^2 - 4\times1\times2}}{2\times1}$$

$$\tan\theta = \frac{3 \pm 1}{2}$$

$\tan\theta = 2$ **OR** $\tan\theta$ **= 1**

For $\tan\theta = 2$,

$\theta = \tan^{-1}(2) = 63.4°, 243.4°$

For $\tan\theta = 1$,

$\theta = \tan^{-1}(1) = 45°, 225°$

$\therefore$ For $\sec^2\theta = 3\tan\theta - 1$, $\theta = 45°, 63.4°, 243.4°, 225°$.

(d) **cosec**$^2\theta = 3 + \cot\theta$

But $\text{cosec}^2\theta = 1 + \cot^2\theta$

$\Rightarrow 1 + \cot^2\theta = 3 + \cot\theta$

$\cot^2\theta - \mathbf{cot}\theta - 2 = 0$

$$\cot\theta = \frac{1 \pm \sqrt{(-1)^2 - 4\times1\times(-2)}}{2\times1}$$

$$\cot\theta = \frac{1 \pm 3}{2}$$

$\cot\theta = 2$ **OR** $\cot\theta = -1$

$\Rightarrow \tan\theta = \frac{1}{2}$ **OR** $\tan\theta = -1$

For $\tan\theta = \frac{1}{2}$, $\theta = \tan^{-1}(\frac{1}{2})$

$\theta = 26.6°, 206.6°$

For $\tan\theta = -1$, $\theta = 135°, 315°$

$\Rightarrow$ For $\text{cosec}^2\theta = 3 + \cot\theta$,

$\theta = 26.6°, 135°, 206.6°, 315°$

(e) 3tan$^2\theta$ + 5 = 7secθ

$3(\sec^2\theta - 1) + 5 = 7\sec\theta$

$3\sec^2\theta - 3 + 5 = 7\sec\theta$

$3\sec^2\theta - 7\sec\theta + 2 = 0$

$$\sec\theta = \frac{7 \pm \sqrt{(-7)^2 - 4\times3\times2}}{3\times2}$$

$$\sec\theta = \frac{7 \pm 5}{6}$$

$\sec\theta = 2$ **OR** $\sec\theta = \frac{1}{3}$

$\Rightarrow \cos\theta = \frac{1}{2}$ **OR** $\cos\theta = 3$

For $\cos\theta = \frac{1}{2}$, $\theta = 60°, 300°$

For $\cos\theta = 3$, θ is not defined because $\cos\theta$ is maximum at 1.

(f) 2cot$^2\theta$ + 8 = 7cosecθ

$1 + \cot^2\theta = \text{cosec}^2\theta$

$\cot^2\theta = \text{cosec}^2\theta - 1$

$2(\text{cosec}^2\theta - 1) + 8 = 7\text{cosec}\theta$

$2\text{cosec}^2\theta - 2 + 8 = 7\text{cosec}\theta$

$2\text{cosec}^2\theta - 7\text{cosec}\theta + 6 = 0$

$$\text{cosec}\theta = \frac{7 \pm \sqrt{(-2)^2 - 4\times2\times2}}{2\times2}$$

$$\text{cosec}\theta = \frac{7 \pm 5}{4}$$

$\text{cosec}\theta = 3$, **OR** $\text{cosec}\theta = \frac{1}{2}$

$\Rightarrow \sin\theta = \frac{1}{3}$, **OR** $\sin\theta = 2$

For $\sin\theta = \frac{1}{3}$, $\theta = 19.5, 160.5$

For $\sin\theta = 2$, $\theta = \sin^{-1}(2)$

The values of θ are not defined.

Example I (UNEB Questions)

Find all the values of θ, $0^0 \le \theta \le 360^0$, which satisfy the equation

$$\sin^2\theta - \sin 2\theta - 3\cos^2\theta = 0.$$

Solution

a) $\sin^2\theta - 2\sin\theta\cos\theta - 3\cos^2\theta = 0$

Dividing through by $\cos^2\theta$,

$\tan^2\theta - 2\tan\theta - 3 = 0$

$\tan^2\theta - 3\tan\theta + \tan\theta - 3 = 0$

$\tan\theta(\tan\theta - 3) + 1(\tan\theta - 3) = 0$

$(\tan\theta - 3)(\tan\theta + 1) = 0$

Either $\tan\theta - 3 = 0$

$\tan\theta = 3$

$\theta = \tan^{-1}(3)$

$\theta = 71.6^0, 251.6^0$

Or $\tan\theta + 1 = 0$

$\tan\theta = -1$

$\theta = \tan^{-1}(-1)$

$\theta = 135^0, 315^0$

Example II (UNEB Question)

Solve $\cos\theta + \sin 2\theta = 0$ for $0^0 \le \theta \le 360^0$.

$\cos\theta + \sin 2\theta = 0$

$\cos\theta + 2\sin\theta\cos\theta = 0$

$\cos\theta(1 + 2\sin\theta) = 0$

Either $\cos\theta = 0$

$\theta = \cos^{-1}(0)$

$\theta = 90^0, 270^0$

Or $1 + 2\sin\theta = 0$

$2\sin\theta = -1$

$\sin\theta = -\frac{1}{2}$

$\theta = \sin^{-1}(-\frac{1}{2})$

$\theta = 210^0, 330^0$

For $0^0 \leq \theta \leq 360^0$, $\theta = 90^0, 210^0, 270^0, 330^0$

Example III (UNEB Question)

Solve $\cot^2\theta = 5(\text{cosec}\,\theta + 1)$ for $0° \leq \theta \leq 360°$

Solution

(a) $\cot^2\theta = 5(\text{cosec}\,\theta + 1)$

But $\cot^2\theta = \text{cosec}^2\theta - 1$

$\text{cosec}^2\theta - 1 = 5(\text{cosec}\,\theta + 1)$

$\text{cosec}^2\theta - 1 = 5\,\text{cosec}\,\theta + 5$

$\text{cosec}^2\theta - 5\,\text{cosec}\,\theta - 6 = 0$

$\text{cosec}^2\theta - 6\,\text{cosec}\,\theta + \text{cosec}\,\theta - 6 = 0$

$\text{cosec}\,\theta(\text{cosec}\,\theta - 6) + 1(\text{cosec}\,\theta - 6) = 0$

$\text{cosec}\,\theta - 6)(\text{cosec}\,\theta + 1) = 0$

Either $\text{cosec}\,\theta = 6$

$\frac{1}{\sin\theta} = 6$

$\sin\theta = \frac{1}{6}$

$\theta = 9.6°, 170.4°$

Or $\text{cosec}\,\theta + 1 = 0$

$\frac{1}{\sin\theta} = -1$

$\theta = 270°$

Hence $\theta = 9.6°, 170.4°$ and $270°$

Example IV (UNEB Question)

Solve $2\sin 2x = 3\cos x$, for $-180^0 \leq x \leq 180°$.

Solution

$2\sin 2x = 3\cos x$

$2\sin 2x - 3\cos x = 0$

But $\sin 2x = 2\sin x\cos x$

$4\sin x\cos x - 3\cos x = 0$

$\cos x(4\sin x - 3) = 0$

$\cos x = 0$

$x = \cos^{-1}(0)$

$x = 90^0, -90$

$4\sin x - 3 = 0$

$\sin x = \frac{3}{4}$

$x = \sin^{-1}\left(\frac{3}{4}\right)$

$x = 48.6°, 131.4°$

$\Rightarrow x = (^-90°, 48.6°, 90°, 131.4°)$ are the solutions to the equation $2\sin 2x = 3\cos x$

Example V (UNEB Question)

Solve the equation $\cos x + \cos 2x = 1$ for values of x from 0^0 to 360^0 inclusive

Solution

$\cos x + \cos 2x = 1$

But $\cos 2x = 2\cos^2 x - 1$

By substitution, we have

$\cos x + 2\cos^2 x - 1 = 1$

$2\cos^2 x + \cos x - 2 = 0$

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4\times 2\times(-2)}}{2\times 2}$$

$$\cos x = \frac{-1 \pm \sqrt{1^2 + 16}}{4}$$

$$= \frac{-1 \pm \sqrt{17}}{4}$$

Taking $\cos x = \frac{-1 + \sqrt{17}}{4}$

$x = 38.7°, 321.3°$

Taking $\cos x = \frac{-1 - \sqrt{17}}{4}$

$\cos x = -1.280776406$

(The values of x are not defined because x is maximum at 1)

Hence $x = 38.7^0, 321.3^0$

Example VI (UNEB Question)

Solve $7\tan\theta + \cot\theta = 5\sec\theta$ for $0° \leq \theta \leq 180°$.

Solution

(a) $7\tan\theta + \cot\theta = 5\sec\theta$

$$7\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{5}{\cos\theta}$$

Multiplying through by $\cos\theta\sin\theta$

$7\sin^2\theta + \cos^2\theta = 5\sin\theta$

$7\sin^2\theta + 1 - \sin^2\theta = 5\sin\theta$

$6\sin^2\theta - 5\sin\theta + 1 = 0$

$6\sin^2\theta - 3\sin\theta - 2\sin\theta + 1 = 0$

$3\sin\theta(2\sin\theta - 1) - 1(2\sin\theta - 1) = 0$

$(2\sin\theta - 1)(3\sin\theta - 1) = 0$

Either $2\sin\theta = 1$

$\sin\theta = \frac{1}{2}$

$\theta = \sin^{-1}\left(\frac{1}{2}\right)$

$\theta = 30^0, 150^0$

Or $3\sin\theta - 1 = 0$

$\sin\theta = \frac{1}{3}$

$\theta = \sin^{-1}\left(\frac{1}{3}\right)$

$\theta = 19.5^0, 160.5^0$

$\Rightarrow 19.5^0, 30^0, 150^0, 160.5^0$ are the solutions to the equation

Example VII (UNEB Question)

Solve the equation $4\cos x - 2\cos 2x = 3$ for $0^0 \le x \le \pi$.

Solution

$4\cos x - 2(2\cos^2 x - 1) = 3$

$4\cos x - 4\cos^2 x + 2 = 3$

$4\cos x - 4\cos^2 x - 1 = 0$

$4\cos^2 x - 4\cos x + 1 = 0$

$4\cos^2 x - 2\cos x - 2\cos x + 1 = 0$

$2\cos x(2\cos x - 1) - 1(2\cos x - 1) = 0$

$(2\cos x - 1)(2\cos x - 1) = 0$

$\Longrightarrow 2\cos x - 1 = 0$

$2\cos x = 1$

$\cos x = \frac{1}{2}$

$x = 60°, 300°$

$x = \frac{\pi}{3}, \frac{3\pi}{3}.$

Elimination of θ from a set of equations

Example

Eliminate θ from the following equations:

(i) $x = a\cos\theta, y = b\sin\theta$

(ii) $x = a\cot\theta, y = b\sec\theta$

(iii) $x = a\tan\theta, y = b\tan\theta$

(iv) $x = 1 - \sin\theta, y = 1 + \cos\theta$

(v) $x = \sin\theta + \tan\theta, y = \tan\theta - \sin\theta$

(vi) $x\cos\theta + y\sin\theta = a, x\sin\theta - y\cos\theta = b$

Solution

(i) $x = a\cos\theta, y = b\sin\theta$

$\frac{x}{a} = \cos\theta, \ \frac{y}{b} = \sin\theta$

$\sin^2\theta + \cos^2\theta = 1$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(ii) $x = a\cot\theta, y = b\,\mathrm{cosec}\theta$

$\frac{x}{a} = \cot\theta, = \frac{y}{b} = \mathrm{cosec}\theta$

$1 + \cot^2\theta = \mathrm{cosec}^2\theta$

$1 + \left(\frac{x}{a}\right)^2 = \left(\frac{y}{b}\right)^2$

$1 + \frac{x^2}{a^2} = \frac{y^2}{b^2}$

(iii) $x = a\tan\theta, y = b\cos\theta$

$\frac{x}{a} = \tan\theta, \ \frac{y}{b} = \cos\theta \ \Rightarrow \frac{b}{y} = \sec\theta$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \left(\frac{x}{a}\right)^2 = \frac{b^2}{y^2}$

$1 + \frac{x^2}{a^2} = \frac{b^2}{y^2}$

(iv) $x = 1 - \sin\theta, \quad y = 1 + \cos\theta$

$\sin\theta = 1 - x, \quad y - 1 = \cos\theta$

$\sin^2\theta + \cos^2\theta = 1$

$(1 - x)^2 + (y - 1)^2 = 1$

$\Rightarrow (x - 1)^2 + (y - 1)^2 = 1$

(v) $\boldsymbol{x = \sin\theta + \tan\theta}$ …………………… (i)

$y = \tan\theta - \sin\theta$ ………………..…… (ii)

Eqn (i) + Eqn (ii);

$\Rightarrow x + y = 2\tan\theta$

$\tan\theta = \frac{x+y}{2}$

Eqn (i) − Eqn (ii);

$x - y = 2\sin\theta$

$\frac{x-y}{2} = \sin\theta$

From $\tan\theta = \frac{x+y}{2}$

$\Rightarrow \cot\theta = \frac{2}{x+y}$

From $\sin\theta = \frac{x-y}{2}$

$\Rightarrow \mathrm{cosec}\theta = \frac{2}{x-y}$

$1 + \cot^2\theta = \mathrm{cosec}^2\theta$

$1 + \left(\frac{2}{x+y}\right)^2 = \left(\frac{2}{x-y}\right)^2$

$1 + \frac{4}{(x+y)^2} = \frac{4}{(x-y)^2}$

$\Rightarrow (x^2 - y^2)^2 = 16xy$

(vi) $x\cos\theta + y\sin\theta = a$ ..………………..(i)

$x\sin\theta - y\cos\theta = b$ ………………….. (ii)

From Eqn (i);

$\cos\theta = \dfrac{a - y\sin\theta}{x}$ …………….. (iii)

Substituting Eqn (iii) in Eqn (ii);

$$x\sin\theta - y\left(\frac{a - y\sin\theta}{x}\right) = b$$

$\Rightarrow x^2\sin\theta - ay + y^2\sin\theta = xb$

$(x^2 + y^2)\sin\theta = xb + ay$

$\sin\theta = \dfrac{bx + ay}{x^2 + y^2}$ …………………… (iv)

Substitute Eqn (iv) in Eqn (iii)

$$\Rightarrow \cos\theta = \frac{a - y(\frac{bx+ay}{x^2+y^2})}{x}$$

$$\cos\theta = \frac{ax^2 + ay^2 - bxy - ay^2}{x(x^2 + y^2)}$$

$$\cos\theta = \frac{ax^2 - bxy}{x(x^2 + y^2)}$$

$$\cos\theta = \frac{ax - by}{x^2 + y^2}$$

$\sin^2\theta + \cos^2\theta = 1$

$$\frac{(bx + ay)^2}{(x^2 + y^2)^2} + \frac{(ax - by)^2}{(x^2 + y^2)^2} = 1$$

$(bx + ay)^2 + (ax - by)^2 = (x^2 + y^2)^2$

$b^2x^2 + 2abxy + a^2y^2 + a^2x^2 - 2abxy + b^2y^2$
$= (x^2 + y^2)^2$

$(a^2 + b^2)x^2 + (a^2 + b^2)y^2 = (x^2 + y^2)^2$

$(x^2 + y^2)(a^2 + b^2) = (x^2 + y^2)^2$

$a^2 + b^2 = x^2 + y^2$

Proving Trigonometric Identities

(i) $\sec\theta + \text{cosec}\theta\cot\theta = \sec\theta\,\text{cosec}^2\theta$

(ii) $\sin^2\theta(1 + \sec^2\theta) = \sec^2\theta - \cos^2\theta$

(iii) $\dfrac{1 - \cos\theta}{\sin\theta} = \dfrac{1}{\cos ec\theta + \cot\theta}$

(iv) $\dfrac{\tan\theta + \cot\theta}{\sec\theta + \text{cosec}\theta} = \dfrac{1}{\sin\theta + \cos\theta}$

(v) $\sec^2\theta = \dfrac{\text{cosec}\theta}{\text{cosec}\theta - \sin\theta}$

(vi) $\dfrac{1 + \sin\theta}{\cos\theta} = \sec\theta + \tan\theta$

(vii) $\dfrac{1 + \sin\theta}{1 - \sin\theta} = (\sec\theta + \tan\theta)^2$

(viii) $\dfrac{\cot\alpha + \tan\beta}{\cot\beta + \tan\alpha} = \cot\alpha\tan\beta$

Solution

(a) $\sec\theta + \text{cosec}\theta\cot\theta$

$$= \frac{1}{\cos\theta} + \frac{1}{\sin\theta}\left(\frac{\cos\theta}{\sin\theta}\right)$$

$$= \frac{1}{\cos\theta} + \frac{\cos\theta}{\sin^2\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta\cos^2\theta}$$

$$= \frac{1}{\sin^2\theta\cos^2\theta}$$

$$= \frac{1}{\cos\theta} \times \frac{1}{\sin^2\theta}$$

$$= \sec\theta\,\text{cosec}\theta$$

(b) $\sin^2\theta(1 + \sec^2\theta)$

$$= \sin^2\theta + \sin^2\theta\sec^2\theta$$

$$= \sin^2\theta + \frac{\sin^2\theta}{\cos^2\theta}$$

$= \sin^2\theta + \tan^2\theta$

$= \sin^2\theta + \sec^2\theta - 1$

$= 1 - \cos^2\theta + \sec^2\theta - 1$

$= \sec^2\theta - \cos^2\theta$

(c) $\dfrac{1 - \cos\theta}{\sin\theta}$

$$= \frac{1 - \cos\theta}{\sin\theta} \cdot \frac{1 + \cos\theta}{1 + \cos\theta}$$

$$= \frac{1 - \cos^2\theta}{\sin\theta + \sin\theta\cos\theta}$$

$$= \frac{\sin^2\theta}{\sin\theta + \sin\theta\cos\theta}$$

$$= \frac{\frac{\sin^2\theta}{\sin^2\theta}}{\frac{\sin\theta}{\sin^2\theta} + \frac{\sin\theta\cos\theta}{\sin^2\theta}}$$

$$= \frac{1}{\text{cosec}\theta + \cot\theta}$$

(d) $\dfrac{\tan\theta + \cot\theta}{\sec\theta + \text{cosec}\theta}$

$$= \frac{\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}}{\frac{1}{\cos\theta} + \frac{1}{\sin\theta}}$$

$$= \frac{\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}}{\frac{\sin\theta + \cos\theta}{\sin\theta\cos\theta}} = \frac{1}{\sin\theta + \cos\theta}$$

(e) $\sec^2\theta = \dfrac{1}{\cos^2\theta}$

$$\sec^2\theta = \frac{1}{1-\sin^2\theta}$$

$$= \frac{\frac{1}{\sin\theta}}{\frac{1}{\sin\theta} - \frac{\sin^2\theta}{\sin\theta}}$$

$$= \frac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta - \sin\theta}$$

(f) $\dfrac{1+\sin\theta}{\cos\theta} = \sec\theta + \tan\theta$

$$\frac{1+\sin\theta}{\cos\theta} = \frac{(1+\sin\theta)\cos\theta}{\cos\theta\times\cos\theta}$$

$$= \frac{\cos\theta + \sin\theta\cos\theta}{\cos^2\theta}$$

$$= \frac{\cos\theta}{\cos^2\theta} + \frac{\sin\theta\cos\theta}{\cos^2\theta}$$

$$= \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$$

$$= \sec\theta + \tan\theta$$

(g) $\dfrac{1+\sin\theta(1+\sin\theta)}{1-\sin\theta(1+\sin\theta)}$

$$= \frac{(1+\sin\theta)^2}{1-\sin^2\theta}$$

$$= \frac{(1+\sin\theta)^2}{\cos^2\theta}$$

$$= \frac{1+2\sin\theta+\sin^2\theta}{\cos^2\theta}$$

$$= \frac{1}{\cos^2\theta} + \frac{2\sin\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta}$$

$$= \sec^2\theta + 2\tan\theta\sec\theta + \tan^2\theta$$

$$= (\sec\theta + \tan\theta)^2$$

(h) $\dfrac{\cot\alpha + \tan\beta}{\cot\beta + \tan\alpha}$

$$= \frac{\frac{\cos\alpha}{\sin\alpha} + \frac{\sin\beta}{\cos\beta}}{\frac{\cos\beta}{\sin\beta} + \frac{\sin\alpha}{\cos\alpha}}$$

$$= \frac{\frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\sin\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\cos\alpha\sin\beta}}$$

$$= \frac{\cos\alpha\sin\beta}{\cos\beta\sin\alpha}$$

$$= \frac{\cos\alpha}{\sin\alpha}\cdot\frac{\sin\beta}{\cos\beta}$$

$$= \cot\alpha\tan\beta$$

Formulae for sin(A ± B), cos(A ± B), and tan(A ± B)

sin(*A* + *B*) = sin*A*cos*B* + cos*A*sin*B*
sin(*A* – *B*) = sin*A*cos*B* – cos*A*sin*B*
cos(*A* + *B*) = cos*A*cos*B* – sin*A*sin*B*
cos(*A* – *B*) = cos*A*cos*B* + sin*A*sin*B*
tan(*A* + *B*) = $\dfrac{\tan A + \tan B}{1 - \tan A + \tan B}$
tan(*A* – *B*) = $\dfrac{\tan A - \tan B}{1 + \tan A + \tan B}$

Examples

Find the values of the following:

(a) cos(45° – 30°)
(b) cos 105°
(c) cos 75°
(d) sin(60° + 45°)
(e) sin 15°

Solution

(a) cos(45° – 35°)

= cos45° cos30° + sin45°sin30°

$$= \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\times\frac{1}{2}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$$

$$= \frac{1+\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{(1+\sqrt{3})2\sqrt{2}}{(2\sqrt{2})(2\sqrt{2})}$$

$$= \frac{2\sqrt{2}+2\sqrt{6}}{8} = \frac{\sqrt{2}+\sqrt{6}}{4}$$

(b) sin(30° +45°)

= sin30 cos45 + cos30 sin 45

$$= \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right) + \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}$$
$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$
$$= \frac{2\sqrt{2}(\sqrt{3}+1)}{4} = \frac{\sqrt{6}+\sqrt{2}}{2}$$

(c) cos 105°

= cos(60° + 45°)
= cos60 cos45 – sin60 sin45

$$= \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$$
$$= \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}}$$
$$= \frac{\sqrt{3}-1}{2\sqrt{2}} \quad = \frac{(1-\sqrt{3})\sqrt{2}}{2\sqrt{2}\cdot\sqrt{2}}$$
$$= \frac{\sqrt{2}-\sqrt{6}}{4}$$

(d) cos 75°

= cos(30° + 45°)
= cos30° cos45° – sin30° sin45°

$$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{2} \times \frac{1}{\sqrt{2}}$$
$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$$
$$= \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{(\sqrt{3}-1)2\sqrt{2}}{2\sqrt{2}\times 2\sqrt{2}}$$
$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

(f) sin(60° + 45°)

= sin60 cos45 + cos60 sin45

$$= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} + \frac{1}{2} \cdot \frac{1}{\sqrt{2}}$$
$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$$
$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$
$$= \frac{(\sqrt{3}+1)2\sqrt{2}}{2\sqrt{2}\times 2\sqrt{2}}$$
$$= \frac{2\sqrt{2}(\sqrt{3}+1)}{8} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

(f) sin 15°

= sin(45 – 30)
= sin45 cos30 – cos45 sin30

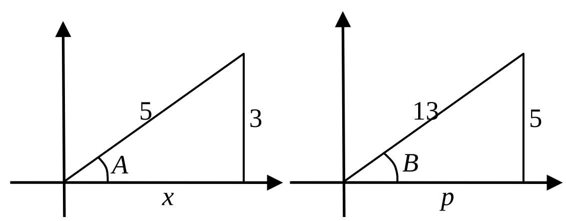
$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$$
$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$$
$$= \frac{2\sqrt{3}-1}{2\sqrt{2}}$$
$$= \frac{(\sqrt{3}-1)2\sqrt{2}}{8} \quad = \frac{\sqrt{6}-\sqrt{2}}{4}$$

Example II

If sin $A = \frac{3}{5}$ and sin$B = \frac{5}{13}$, where A and B are acute angles, find the values of the following:

(a) **sin(A + B)**
(b) **cos(A + B)**
(c) **cot(A + B)**

Solution



$x^2 + 3^2 = 5^2$	$p^2 + 5^2 = 13^2$
$x^2 + 9 = 25$	$p^2 + 25 = 169$
$x^2 = 16$	$p^2 = 144$
$x = 4$	$p = 12$

$\Rightarrow$ sin $A = \frac{3}{5}$; cos $A = \frac{4}{5}$; tan $A = \frac{3}{4}$

sin $B = \frac{5}{13}$; cos $B = \frac{12}{13}$; tan $B = \frac{5}{12}$

sin(A + B) = sin$A$ cos$B$ + cos$A$ sin$B$

$$= \frac{3}{5} \times \frac{12}{13} + \frac{4}{5} \times \frac{5}{13}$$
$$= \frac{36}{65} + \frac{20}{65}$$
$$= \frac{56}{65}$$

(b) cos(A + B) = cos$A$ cos$B$ – sin$A$ sin$B$

$$= \frac{4}{5}\times\frac{12}{13}-\frac{3}{5}\times\frac{5}{13}$$
$$= \frac{48}{65}-\frac{15}{65}$$
$$= \frac{33}{65}$$

(c) cot(A + B) $= \dfrac{1}{\tan(A+B)}$

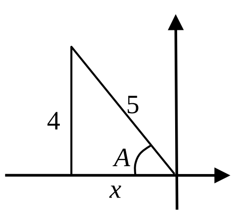
$$\tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B}$$
$$\Rightarrow \cot(A+B) = \frac{1-\tan A\tan B}{\tan A+\tan B}$$
$$= \frac{1-\frac{3}{4}\times\frac{5}{12}}{\frac{3}{4}+\frac{5}{12}}$$
$$= \frac{1-\frac{15}{48}}{7/6} = \frac{33/16}{7/6}$$
$$= \frac{33}{16}\times\frac{6}{7}$$
$$= \frac{99}{56}$$

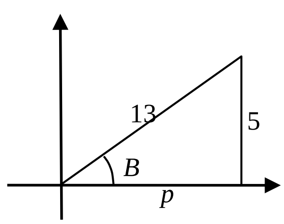
Example III

If $\sin A = \dfrac{4}{5}$, $\cos B = \dfrac{12}{13}$, where A is obtuse and B is acute, find the values of:

(a) sin(A – B)
(b) tan(A – B)
(c) tan(A + B)

Solutions





$x^2 + 4^2 = 5^2$
$x^2 + 16 = 25$
$x^2 = 9$
$x = 3$

$p^2 + 12^2 = 13^2$
$p^2 + 144 = 169$
$p^2 = 25$
$p = 5$

A is obtuse

$\Rightarrow \sin A = \dfrac{4}{5}$; $\cos A = \dfrac{-3}{5}$; $\tan A = \dfrac{-4}{3}$

B is acute

$\Rightarrow \sin B = \dfrac{5}{13}$; $\cos B = \dfrac{12}{13}$; $\tan B = \dfrac{5}{12}$

$\text{Sin}(A - B) = \sin A \cos B - \cos A \sin B$

$$= \frac{4}{5}\times\frac{12}{13}-\frac{-3}{5}\times\frac{5}{13}$$
$$= \frac{48}{65}+\frac{15}{65}$$
$$= \frac{63}{65}$$

(b) tan(*A* – *B*) $= \dfrac{\tan A-\tan B}{1+\tan A\tan B}$

$$= \frac{\frac{-4}{3}-\frac{5}{12}}{1+\frac{-4}{3}\times\frac{5}{12}}$$
$$= \frac{-7/4}{4/9} = \frac{-63}{16}$$

(c) tan(*A* + *B*) $= \dfrac{\tan A+\tan B}{1-\tan A\tan B}$

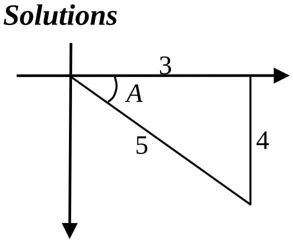
$$= \frac{\frac{-4}{3}+\frac{5}{12}}{1-\frac{-4}{3}\times\frac{5}{12}}$$
$$= \frac{\frac{-11}{12}}{1+\frac{20}{36}}$$
$$= \frac{\frac{-11}{12}}{\frac{56}{36}} = \frac{-33}{56}$$

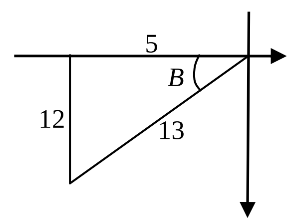
Example III

If $\cos A = \dfrac{3}{5}$ and $\tan B = \dfrac{12}{5}$; where A and B are reflex angles. Find the values of:

(a) sin(*A* – *B*)
(b) tan(*A* – *B*)
(c) cos(*A* + *B*)

Solutions





A and B are reflex

$\Rightarrow \cos A = \dfrac{3}{5}$; $\sin A = \dfrac{-4}{5}$; $\tan A = \dfrac{-4}{3}$

$\cos B = \dfrac{-5}{13}$, $\sin B = \dfrac{-12}{13}$; $\tan B = \dfrac{12}{5}$

(a) sin(*A* – *B*) $= \sin A \cos B - \cos A \sin B$

$$= \frac{-4}{5}\times\frac{-5}{13}-\frac{3}{5}\times\frac{-12}{13}$$

$= \frac{20}{65} + \frac{36}{65}$

$= \frac{56}{65}$

(b) tan(*A* – *B*) $= \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$= \frac{\frac{-4}{3} - \frac{12}{5}}{1 + \frac{-4}{3} \times \frac{12}{5}}$

$= \frac{-56/15}{-11/5} = \frac{56}{33}$

(c) cos(A + B) = cos *A* cos *B* – sin *A* sin*B*

$= \frac{3}{5} \times \frac{-5}{13} - \frac{-4}{5} \times \frac{-12}{13}$

$= \frac{-15}{65} - \frac{48}{65}$

$= \frac{-63}{65}$

Example IV

From the following, find the values of tan *x*

(a) sin(*x* + 45°) = 2cos(*x* + 45°)

(b) 2sin(*x* – 45°) = cos(*x* + 45°)

(c) tan(*x* – *A*) = $\frac{3}{2}$**,** where tan *A* = 2

(d) sin(*x* + 30°) = cos(*x* + 30°)

Solution

(a) sin(*x* + 45°) = 2cos(*x* + 45°)

sin*x* cos45°+cos*x* sin45° = 2(cos*x* cos45°–sin*x* sin45°)

$\sin x \cdot \frac{1}{\sqrt{2}} + \cos x \cdot \frac{1}{\sqrt{2}} = 2(\cos x \cdot \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \sin x)$

$\frac{\sqrt{2}}{2} \sin x + \frac{\sqrt{2}}{2} \cos x = 2(\frac{\sqrt{2}}{2} \cos x - \frac{\sqrt{2}}{2} \sin x)$

$\frac{\sqrt{2}}{2} \sin x + \frac{\sqrt{2}}{2} \cos x = \sqrt{2} \cos x - \sqrt{2} \sin x$

$\left(\frac{\sqrt{2}}{2} + \sqrt{2}\right) \sin x = \sqrt{2} \cos x - \frac{\sqrt{2}}{2} \cos x$

$\frac{3\sqrt{2}}{2} \sin x = \frac{\sqrt{2}}{2} \cos x$

3sin *x* = cos *x*

$\frac{3 \sin x}{\cos x} = \frac{\cos x}{\cos x}$

3 tan *x* = 1

tan *x* = $\frac{1}{3}$

(b) 2sin(*x* – 45°) = cos(*x* + 45°)

2(sin*x* cos45 – cos*x*sin45) = cos*x* cos45 – sin*x*sin45

$2\left(\sin x(\frac{\sqrt{2}}{2}) - \cos x(\frac{\sqrt{2}}{2}) = \cos x(\frac{\sqrt{2}}{2}) - \sin x(\frac{\sqrt{2}}{2})\right)$

$\sqrt{2} \sin x - \sqrt{2} \cos x = \frac{\sqrt{2}}{2} \cos x - \frac{\sqrt{2}}{2} \sin x$

$\sqrt{2} \sin x + \frac{\sqrt{2}}{2} \sin x = \sqrt{2} \cos x + \frac{\sqrt{2}}{2} \cos x$

$\frac{3\sqrt{2}}{2} \sin x = \frac{3\sqrt{2}}{2} \cos x$

tan *x* = 1

(c) tan(*x* – *A*) = $\frac{3}{2}$**, tan *A* = 2**

$\frac{\tan x - \tan A}{1 + \tan x \tan A} = \frac{3}{2}$

$\frac{\tan x - 2}{1 + 2 \tan x} = \frac{3}{2}$

2(tan *x* – 2) = 3(1 + 2tan *x*)

2tan *x* – 4 = 3 + 6tan *x*

4tan *x* = -7

tan *x* = $\frac{-7}{4}$

(d) sin(*x* + 30) = cos(*x* + 30)

sin *x* cos 30 + cos *x* sin 30 = cos *x* cos 30 – sin *x* sin 30

$\frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \cos x = \frac{\sqrt{3}}{2} \cos x - \frac{1}{2} \sin x$

$\frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \sin x = \frac{\sqrt{3}}{2} \cos x - \frac{1}{2} \cos x$

$\sin x\,(\frac{\sqrt{3}+1}{2}) = \cos x(\frac{\sqrt{3}-1}{2})$

$\frac{\sin x}{\cos x} = \frac{\frac{\sqrt{3}-1}{2}}{\frac{\sqrt{3}+1}{2}}$

tan *x* = $\frac{\sqrt{3}-1}{1+\sqrt{3}}$

tan *x* = $\frac{(\sqrt{3}-1)(1-\sqrt{3})}{(\sqrt{3}+1)(1-\sqrt{3})}$

tan *x* = $\frac{\sqrt{3}-3-1+\sqrt{3}}{-2}$

tan *x* = 2 – $\sqrt{3}$

Example V

Solve the following equations for 0° ≤ θ ≤ 360°

(a) 2sin *x* = cos(*x* + 60°)

(b) cos(*x* + 45°) = cos *x*

(c) sin(*x* – 30°) = $\frac{1}{2}$ **cos *x***

(d) 3sin(*x* + 10°) = 4cos(*x* – 10°)

Solutions

(a) 2sin *x* = cos(*x* + 60°)

2sin *x* = cos *x* cos 60° – sin *x* sin 60°

$2\sin x = \frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$

$2\sin x + \frac{\sqrt{3}}{2}\sin x = \frac{1}{2}\cos x$

$(4 + \sqrt{3})\sin x = \cos x$

$\tan x = \frac{1}{4+\sqrt{3}}$

$x = 9.9°, 189.9°$

(b) cos(*x* + 45°) = cos *x*

$\cos x \cos 45° - \sin x \sin 45° = \cos x$

$\frac{\sqrt{2}}{2}\cos x - \frac{\sqrt{2}}{2}\sin x = \cos x$

$\frac{\sqrt{2}}{2}\cos x + \cos x + \cos x = \frac{\sqrt{2}}{2}\sin x$

$(\frac{\sqrt{2}}{2} + 1)\cos x = \frac{\sqrt{2}}{2}\sin x$

$(\frac{\sqrt{2}+2}{2})\cos x = \frac{\sqrt{2}}{2}\sin x$

$\frac{\sqrt{2}+2}{\sqrt{2}} = \frac{\sin x}{\cos x}$

$\frac{\sqrt{2}+2}{\sqrt{2}} = \tan x$

$x = 67.5°, 247.5°$

(c) sin(*x* + 30) = $\frac{1}{2}$ cos *x*

$\sin x \cos 30 - \cos x \sin 30 = \frac{1}{2}\cos x$

$\frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x = \frac{1}{2}\cos x$

$\frac{\sqrt{3}}{2}\sin x = \cos x$

$\frac{\sin x}{\cos x} = \frac{2}{\sqrt{3}}$

$\tan x = \frac{2}{\sqrt{3}}$

$x = 49.1°, 229.1°$

(d) 2sin(*x* + 10°) = 4cos(*x* – 10°)

$2(\sin x \cos 10 - \cos x \sin 10)$
$= 4(\cos x \cos 10° + \sin x \sin 10°)$

$2\sin x\cos 10 - 2\cos x \sin 10 = 4\cos x \cos 10 + 4\sin x \sin 10$

$2\sin x \cos 10 - 4\sin x \sin 10$
$= 4\cos x \cos 10 + 2\cos x \sin 10$

$\sin x(2\cos 10 - 4\sin 10) = \cos x(4\cos 10 + 2\sin 10)$

$\frac{\sin x}{\cos x} = \frac{4\cos 10 + 2\sin 10}{2\cos 10 - 4\sin 10}$

$\tan x = \frac{4\cos 10 + 2\sin 10}{2\cos 10 - 4\sin 10}$

$x = 73.4°, x = 253.4°$

Example VI

If tan(*x* + 45°) = 2, find the value of tan *x*

Solution

$\tan(x + 45°) = 2.$

$\frac{\tan x + \tan 45°}{1 - \tan x \tan 45°} = 2$

$\frac{\tan x + 1}{1 - \tan x} = 2$

$\tan x + 1 = 2(1 - \tan x)$

$\tan x + 1 = 2 - 2\tan x$

$3\tan x = 1$

$\tan x = \frac{1}{3}$

Example VII

If tan(*A* + *B*) = $\frac{1}{7}$ and tan *A* = 3, find the value of tan *B*.

$\tan(A + B) = \frac{1}{7}$

$\frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{1}{7}$

$\tan A = 3$

$\frac{3 + \tan B}{1 - 3\tan B} = \frac{1}{7}$

$7(3 + \tan B) = 1 - 3\tan B$

$21 + 7\tan B = 1 - 3\tan B$

$10\tan B = -20$

$\tan B = -2$

Example VIII

Express the following as single trigonometric ratios.

(a) $\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$

(b) $\frac{\sqrt{3} + \tan x}{1 - \sqrt{3}\tan x}$

(c) $\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x$

(d) $\frac{1}{\cos 24 \cos 15 - \sin 24 \sin 15}$

(e) $\frac{1}{2}\cos 75 + \frac{\sqrt{3}}{2}\sin 75$

(f) $\frac{1 - \tan 15}{1 + \tan 15}$

Solutions

(a) $\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$

$= \cos 60 \cos x - \sin 60 \sin x$

$\cos(60 + x)$

$$\Rightarrow \frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x = \cos(60 + x)$$

(b) $\dfrac{\sqrt{3}+\tan x}{1-\sqrt{3}\tan x}$

$= \dfrac{\tan 60 + \tan x}{1 - \tan 60 \tan x}$

$= \tan(60 + x)$

(c) $\dfrac{1}{\sqrt{2}}\sin x + \dfrac{1}{\sqrt{2}}\cos x$

$= \cos 45 \sin x + \sin 45 \cos x$

$= \cos(45 - x)$

(d) $\dfrac{1}{\cos 24 \cos 15 - \sin 24 \sin 15}$

$= \dfrac{1}{\cos(24+15)}$

$= \dfrac{1}{\cos 39}$

$= \sec 39°$

(e) $\dfrac{1}{2}\cos 75 + \dfrac{\sqrt{3}}{2}\sin 75$

$\cos 60° \cos 75° + \sin 60° \sin 75°$

$\cos 75° \cos 60° + \sin 75° \sin 60°$

$\cos(75° - 60°)$

$\cos 15°$

(f) $\dfrac{1-\tan 15}{1+\tan 15} = \dfrac{\tan 45 - \tan 15}{1 + \tan 45 \tan 15}$

$= \tan(45 - 15)$

$= \tan(30)$

Example IX

Prove the following identities:

(i) $\sin(A + B) + \sin(A - B) = 2\sin A \cos B$

(ii) $\cos(A + B) - \cos(A - B) = -2\sin A \sin B$

(iii) $\tan A + \tan B = \dfrac{\sin(A+B)}{\cos A \cos B}$

(iv) $\tan(A + B + C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan A \tan C - \tan A \tan B}$

Hence prove that if A, B, and C are angles of a triangle, then $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

Solution

$\sin(A + B) + \sin(A - B)$

$\sin A \cos B + \cos A \sin B + \sin A \cos B - \cos A \sin B$

$= 2\sin A \cos B$

$\Rightarrow \sin(A + B) + \sin(A - B) = 2\sin A \cos B$

(ii) $\cos(A + B) - \cos(A - B)$

$\cos A \cos B - \sin A \sin B - (\cos A \cos B + \sin A \sin B)$

$= -2\sin A \sin B$

$\Rightarrow \cos(A + B) - \cos(A - B) = -2\sin A \sin B$

(iii) $\tan A + \tan B$

$= \dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}$

$= \dfrac{\sin A \cos B + \cos A \sin B}{\cos A \cos B}$

$= \dfrac{\sin(A+B)}{\cos A \cos B}$

$\Rightarrow \tan A + \tan B = \dfrac{\sin(A+B)}{\cos A \cos B}$

(iv) $\tan(A + B + C)$

Let $B + C = D$

$\tan(A + D) = \dfrac{\tan A + \tan D}{1 - \tan A \tan D}$

$= \dfrac{\tan A + \tan(B+C)}{1 - \tan A \tan(B+C)}$

$= \dfrac{\tan A + \frac{\tan B + \tan C}{1 - \tan B \tan C}}{1 - \tan A\left(\frac{\tan B + \tan C}{1 - \tan B \tan C}\right)}$

$= \dfrac{\frac{\tan A - \tan A \tan B \tan C + \tan B + \tan C}{1 - \tan B \tan C}}{\frac{1 - \tan B \tan C - \tan A \tan B - \tan A + \tan C}{1 - \tan B \tan C}}$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan A \tan C - \tan A \tan B}$$

Since A, B, and C are angles of a triangle, then

$A + B + C = 180°$

$\tan(A + B + C) = \tan 180°$

$\tan(A + B + C) = 0$

$\Rightarrow \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan A \tan C - \tan A \tan B} = 0$

$\tan A + \tan B + \tan C - \tan A \tan B \tan C = 0$

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

Example (UNEB Question)

Without using tables or calculator, evaluate tan 15°

Solution

$\tan 15° = \tan(45° - 30°)$

$= \dfrac{\tan 45 - \tan 30}{1 + \tan 45 \tan 30}$

$= \dfrac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}}$

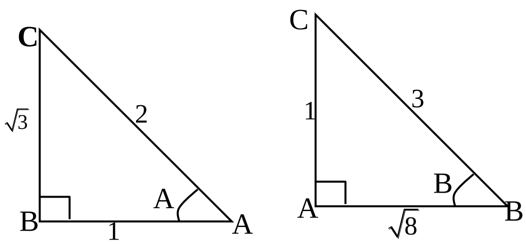
$= \dfrac{\sqrt{3} - 1}{\sqrt{3} + 1}$

$$= \frac{(\sqrt{3}-1)(1-\sqrt{3})}{(\sqrt{3}+1)(1-\sqrt{3})}$$
$$= \frac{\sqrt{3}-3-1+\sqrt{3}}{1-3}$$
$$= \frac{2\sqrt{3}-4}{-2} = 2-\sqrt{3}$$

Example **(UNEB Question)**
The acute angles A and B are such that cos***A*** = ½, sin ***B*** = 1/3. Show without the use of tables or calculator, show that

$$\tan(A+B) = \frac{9\sqrt{3}+8\sqrt{2}}{5}$$

Solution



$$\tan B = \frac{1}{\sqrt{8}} = \frac{\sqrt{8}}{8}$$
$$= \frac{2\sqrt{2}}{8} = \frac{\sqrt{2}}{4}$$
$$\tan B = \frac{\sqrt{3}}{2}$$

From compound angle formula,

$$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A \tan B}$$
$$= \frac{\sqrt{3}+\frac{\sqrt{2}}{4}}{1-(\sqrt{3}\times\frac{\sqrt{2}}{4})}$$
$$= \frac{\frac{4\sqrt{3}+\sqrt{2}}{4}}{\frac{4-\sqrt{3}\times\sqrt{2}}{4}}$$
$$= \frac{4\sqrt{3}+\sqrt{2}}{4} \times \frac{4}{4-\sqrt{6}}$$
$$= \frac{\left(4\sqrt{3}+\sqrt{2}\right)}{\left(4-\sqrt{6}\right)}$$
$$= \frac{\left(4\sqrt{3}+\sqrt{2}\right)\left(4+\sqrt{6}\right)}{\left(4-\sqrt{6}\right)\left(4+\sqrt{6}\right)}$$
$$= \frac{16\sqrt{3}+4\sqrt{18}+4\sqrt{2}+\sqrt{12}}{16-6}$$
$$= \frac{16\sqrt{3}+4\times3\sqrt{2}+4\sqrt{2}+2\sqrt{3}}{10}$$
$$= \frac{18\sqrt{3}+16\sqrt{2}}{10}$$
$$= \frac{9\sqrt{3}+8\sqrt{2}}{5}$$

Double angle & Triple angle formulae

By writing $A = B$ in the additional formulae for sine, cosine, and tangent, we obtain the double angle formula for each of them.

$$\sin(A + B) = \sin A \cos B + \cos A \cos B$$

$\Rightarrow \sin 2A = \sin(A + A)$
$= \sin A \cos A + \cos A \sin A$
$= 2\sin A \cos A$

$$\boxed{\mathbf{\sin 2A = 2\sin A \cos A}}$$

$\cos(A + B) = \cos A \cos B - \sin A \sin B$
$\cos(A + A) = \cos A \cos A - \sin A \sin A.$
$= \cos^2 A - \sin^2 A$
But $\cos^2 A = 1 - \sin^2 A$
$\Rightarrow \cos 2A = 1 - \sin^2 A - \sin^2 A$
$= 1 - 2\sin^2 A$
But when $\sin^2 A = 1 - \cos^2 A$
$\cos^2 A = \cos^2 A - \sin^2 A$
$= \cos^2 A - (1 - \cos^2 A)$
$= 2\cos^2 A - 1$

$$\boxed{\begin{array}{l}\mathbf{\cos {}^2A = 2\cos^2 A - 1 \quad OR} \\ \mathbf{\cos {}^2A = 1 - 2\sin^2 A}\end{array}}$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1-\tan A \tan B}\text{; where } A = B$$
$$\tan(A + A) = \frac{\tan A + \tan A}{1-\tan A \tan A}$$
$$= \frac{2\tan A}{1-\tan^2 A}$$
$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

sin 3*A* = sin(*A* + 2*A*)
$= \sin A \cos 2A + \cos A \sin 2A$
$= \sin A(1 - 2\sin^2 A) + \cos A(2\sin A \cos A)$
$= \sin A - 2\sin^3 A + 2\cos^2 A \sin A$
$= \sin A - 2\sin^3 A + 2(1 - \sin^2 A)\sin A$
$= \sin A - 2\sin^3 A + 2\sin A - 2\sin^3 A$
$= 3\sin A - 4\sin^3 A$

$\boxed{\sin 3A = 3\sin A - 4\sin^3 A}$

cos 3*A* = cos(2*A* + *A*)

$= \cos 2A \cos A - \sin 2A \sin A$
$= (2\cos^2 A - 1)\cos A - (2\sin A \cos A)\sin A$
$= 2\cos^3 A - \cos A - 2\sin^2 A \cos A$
$= 2\cos^3 A - \cos A - 2(1 - \cos^2 A)\cos A$
$= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A$
$= 4\cos^3 A - 3\cos A$
$\Rightarrow$ **$\cos 3A = 4\cos^3 A - 3\cos A$**

tan 3*A* = tan(*A* + 2*A*)

$$= \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A}$$

$$= \frac{\tan A + \frac{2\tan A}{1-\tan^2 A}}{1 - \tan A\left(\frac{2\tan A}{1-\tan^2 A}\right)}$$

$$= \frac{\tan A + \frac{2\tan A}{1-\tan^2 A}}{1 - \tan A\left(\frac{2\tan A}{1-\tan^2 A}\right)}$$

$$= \frac{\frac{\tan A - \tan^3 A + 2\tan A}{1-\tan^2 A}}{\frac{1-\tan^2 A - 2\tan^2 A}{1-\tan^2 A}}$$

$$= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

$$\boxed{\Rightarrow \tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}}$$

Example I

Simplify the following expressions

(i) **2sin 17 cos 17**

(ii) $\frac{2\tan 30°}{1 - \tan^2 30°}$

(iii) **$2\cos^2 42 - 1$**

(iv) $2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$

(v) $1 - 2\sin^2 22\frac{1}{2}°$

(vi) $\frac{2\tan\frac{1}{2}\theta}{1 - \tan^2\frac{\theta}{2}}$

(vii) **$1 - 2\sin^2 3\theta$**

(viii) $\frac{1 - \tan^2 20}{\tan 20}$

(ix) **sec θ cosec θ**

(x) **2sin 2*A* cos 2*A***

Solutions

(i) **sin 2(17°)** = 2sin 17° cos 17°

sin 34° = 2sin 17° cos 17°
$\Rightarrow$ 2sin 17 cos 17 = sin 34

(ii) **tan(30° + 30°) =** $\frac{\tan 30° + \tan 30°}{1 - \tan 30° \tan 30°}$

$$\tan 60° = \frac{2\tan 30°}{1 - \tan^2 30°}$$

$$\frac{2\tan 30°}{1 - \tan^2 30°} = \tan 60°$$

(iii) **$2\cos^2 42° - 1$**

$\cos 2\theta = 2\cos^2\theta - 1$
$\cos 2(42°) = 2\cos^2 42° - 1$
$\cos 84° = 2\cos^2 42° - 1$
$2\cos^2 42° - 1 = \cos 84°$

(iv) **$2\sin\frac{1}{2}\boldsymbol{\theta}\cos\frac{1}{2}\boldsymbol{\theta}$**

$\sin 2\theta = 2\sin\theta\cos\theta$
$\sin 2\left(\frac{1}{2}\theta\right) = 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$
$\sin\theta = 2\sin\frac{1}{2}\theta\ \cos\frac{1}{2}\theta$
$\Rightarrow \sin\theta = 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$
$2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta = \sin\theta$

(v) **$1 - 2\sin^2 22½°$**

$\cos 2A = 1 - 2\sin^2 A$
$\cos 2(22½) = 1 - 2\sin^2 22½$
$\cos 45 = 1 - 2\sin^2 22½$
$1 - 2\sin^2 22½ = \cos 45$

(vi) $\frac{2\tan\frac{1}{2}\theta}{1 - 2\tan^2\frac{1}{2}\theta} = \frac{\tan\frac{1}{2}\theta + \tan\frac{1}{2}\theta}{1 - \tan\frac{1}{2}\theta\tan\frac{1}{2}\theta}$

$= \tan\left(\frac{1}{2}\theta + \frac{1}{2}\theta +\right)$
$= \tan\theta$

(vii) $1 - 2\sin^2\theta$

$\cos 2(3\theta) = 1 - 2\sin^2 3\theta$
$\cos 6\theta = 1 - 2\sin^2 3\theta$
$1 - 2\sin^2 3\theta = \cos 6\theta$

(viii) $\frac{1 - \tan^2 20}{\tan 20}$

$\tan 40 = \tan(20 + 20)$

$$= \frac{2\tan 20}{1 - \tan^2 20}$$

$$\frac{1}{\tan 40} = \frac{1 - \tan^2 20}{2\tan 20}$$

$$\frac{2}{\tan 40} = \frac{1 - \tan^2 20}{\tan 20}$$

$$2\cot 40 = \frac{1-\tan^2 20}{\tan 20}$$
$$\frac{1-\tan^2 20}{\tan 20} = 2\cot 40$$

(ix) $\sec\theta\operatorname{cosec}\theta = \frac{1}{\cos\theta}\times\frac{1}{\sin\theta}$

$$= \frac{1}{\cos\theta\sin\theta}$$

But sin 2θ = $2\sin\theta\cos\theta$

$\frac{1}{2}$ sin 2θ = sin θ cos θ

$$\sec\theta\operatorname{cosec}\theta = \frac{1}{\frac{1}{2}\sin 2\theta}$$
$$\sec\theta\operatorname{cosec}\theta = \frac{2}{\sin 2\theta}$$
$$\sec\theta\operatorname{cosec}\theta = 2\operatorname{cosec} 2\theta$$

(x) 2sin 2*A* cos 2*A*

sin 4*A* = sin2(2*A*)

= 2sin 2*A* cos2*A*

$\Rightarrow$ 2sin 2*A* cos 2*A* = sin 4*A*

Example II

Evaluate the following without using tables or calculator:

(a) 2sin 15° cos 15°

(b) $2\cos^2 75° - 1$

(c) $\cos^2 22½° - \sin^2 22 ½°$

(d) $\frac{1-2\cos^2 25}{1-2\sin^2 65}$

(e) $\frac{2\tan 22\frac{1}{2}}{1-\tan^2 22\frac{1}{2}}$

(f) $1-2\sin^2 67\frac{1}{2}$

Solution

(a) 2sin 15° cos 15° = sin 2(15°)

= sin 30°

$= \frac{1}{2}$

(b) $2\cos^2 75° - 1$ = cos 150°

= -cos 30°

$= \frac{-\sqrt{3}}{2}$

(c) $\cos^2 22½° - \sin^2 22 ½°$

= cos($22\frac{1}{2}$° + $22\frac{1}{2}$°)

= cos 45

$= \frac{\sqrt{2}}{2}$

(d) $\frac{1-2\cos^2 25}{1-2\sin^2 65} = \frac{-1(2\cos^2 25-1)}{1-2\sin^2 65}$

$$= \frac{-1(\cos 50°)}{\cos 130°}$$
$$= \frac{-1(\cos 50°)}{-\cos 50°} = 1$$

(e) $\frac{2\tan 22\frac{1}{2}}{1-\tan^2 22\frac{1}{2}} = \tan(22\tfrac{1}{2}° + 22\tfrac{1}{2}°)$

$$\tan 45° = 1$$
$$\frac{2\tan 22\frac{1}{2}}{1-\tan^2 22\frac{1}{2}} = \tan 45° = 1$$

(f) $1-2\sin^2 67\frac{1}{2}$ = cos 135°

= -cos 45°

$= \frac{-\sqrt{2}}{2}$

Example III

Solve the following equations from $0 \le \theta \le 360°$

(a) cos 2θ + cos *θ* + 1 = 0

(b) sin 2*θ* cos *θ* + sin²*θ* = 1

(c) 2sin θ(5cos 2*θ* + 1) = 3 sin 2*θ*

(d) 3cot 2θ + cot θ = 1

(e) 4tan *θ* tan 2*θ* = 1

Solution

(a) cos 2θ + cos *θ* + 1 = 0

$$2\cos^2\theta - 1 + \cos\theta + 1 = 0$$
$$2\cos^2\theta + \cos\theta = 0$$
$$\cos\theta(2\cos\theta + 1) = 0$$
$$\cos\theta = 0, \quad \cos\theta = \frac{-1}{2}$$

For $\cos\theta = 0$, $\theta = 90°, 270°$

For $\cos\theta = \frac{-1}{2}$, θ = 120°, 240°

$\Rightarrow$ The solutions to the equation cos 2θ + cos θ + 1 = 0 are 90°, 120°, 240° and 270°.

(b) sin 2*θ* cos *θ* + sin²*θ* = 1

$$(2\sin\theta\cos\theta)\cos\theta + \sin^2\theta = 1$$
$$2\cos^2\theta\sin\theta + \sin^2\theta = 1$$
$$2(1-\sin^2\theta)\sin\theta + \sin^2\theta = 1$$
$$2\sin\theta - 2\sin^3\theta + \sin^2\theta = 1$$

$$2\sin^3\theta - \sin^2\theta - 2\sin\theta + 1 = 0$$
$$\sin\theta = 1, \quad \sin\theta = -1$$
$$\sin\theta = \frac{1}{2}$$

For $\sin\theta = 1$, $\theta = 90°$
For $\sin\theta = -1$, $\theta = 270°$
For $\sin\theta = \frac{1}{2}$, θ = 30°, 150°

$\Rightarrow$ 30°, 90°, 150°, 270° are the solutions to the equation sin 2θ cos $\boldsymbol{\theta}$ + sin²$\boldsymbol{\theta}$ = 1

(c) 2sin θ(5cos 2θ + 1) = 3 sin 2θ

$$2\sin\theta[5(2\cos^2\theta - 1) + 1] = 3\cdot 2\sin\theta\cos\theta$$
$$2\sin\theta(10\cos^2\theta - 5 + 1) = 6\sin\theta\cos\theta$$
$$20\cos^2\theta\sin\theta - 8\sin\theta = 6\sin\theta\cos\theta$$
$$20\cos^2\theta\sin\theta - 8\sin\theta - 6\sin\theta\cos\theta = 0$$
$$2\sin\theta[10\cos^2\theta - 3\cos\theta - 4] = 0$$
$$\sin\theta = 0, \ \cos\theta = 0.8, \ \cos\theta = \frac{-1}{2}$$

For sin θ = 0, θ = 0°, 180°, 360°
For cos $\theta = \frac{-1}{2}$, θ = 120°, 240°
For cos θ = 0.8, θ = 36.9°, 323.1°

$\Rightarrow$ 0, 36.9, 120, 180, 240, 323.1, 360 are the solutions to the equation
2sin θ(5cos 2θ + 1) = 3 sin 2θ

(d) 3cot 2$\boldsymbol{\theta}$ + cot $\boldsymbol{\theta}$ = 1

$$\frac{3}{\tan 2\theta} + \frac{1}{\tan\theta} = 1$$
$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$
$$\Rightarrow 3\left(\frac{1 - \tan^2\theta}{2\tan\theta}\right) + \frac{1}{\tan\theta} = 1$$
$$3 - 3\tan^2\theta + 2 = 2\tan\theta$$
$$3\tan^2\theta + 2\tan\theta - 5 = 0$$
$$\tan\theta = 1, \quad \tan\theta = \frac{-5}{3}$$

For tanθ = 1, θ = 45°, 225°
For tan$\theta = \frac{-5}{3}$, θ = 121°, 301°.

(e) 4tan θ tan 2$\boldsymbol{\theta}$ = 1

$$4\tan\theta\left(\frac{2\tan\theta}{1 - \tan^2\theta}\right) = 1$$
$$\frac{8\tan^2\theta}{1 - \tan^2\theta} = 1$$
$$8\tan^2\theta = 1 - \tan^2\theta$$
$$9\tan^2\theta = 1$$
$$\tan\theta = \pm\frac{1}{3}$$

When $\tan\theta = \frac{1}{3}$, θ = 18.4°, 198.4°
When $\tan\theta = \frac{-1}{3}$, θ = 161.6°, 341.6°

t-formula

If *t* = $\tan\frac{x}{2}$**,**

$$\sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2}$$

And if *t* = tan *x*

$$\sin 2x = \frac{2t}{1+t^2}, \qquad \cos 2x = \frac{1-t^2}{1+t^2}$$

Proof

If $t = \tan\frac{x}{2}$,

$$\sin x = \sin\left(\tfrac{x}{2} + \tfrac{x}{2}\right)$$
$$= \sin\tfrac{x}{2}\cos\tfrac{x}{2} + \cos\tfrac{x}{2}\sin\tfrac{x}{2}$$
$$= 2\sin\tfrac{x}{2}\cos\tfrac{x}{2}$$
$$\sin x = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1}$$
$$\sin x = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2} + \cos^2\frac{x}{2}}$$

Dividing through by $\cos^2\frac{x}{2}$

$$\sin x = \frac{\dfrac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}}}{\dfrac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} + \dfrac{\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}}}$$
$$\sin x = \frac{2\tan\frac{x}{2}}{\tan^2\frac{x}{2} + 1}$$
$$= \frac{2t}{1+t^2}$$
$$\boxed{\sin x = \frac{2t}{1+t^2}}$$

$$\cos x = \cos(\tfrac{x}{2} + \tfrac{x}{2})$$
$$= \cos^2 \tfrac{x}{2} - \sin^2 \tfrac{x}{2}$$
$$= \cos x = \frac{\cos^2 \tfrac{x}{2} - \sin^2 \tfrac{x}{2}}{1}$$
$$\cos x = \frac{\cos^2 \tfrac{x}{2} - \sin^2 \tfrac{x}{2}}{\sin^2 \tfrac{x}{2} + \cos^2 \tfrac{x}{2}}$$

Dividing through by $\cos^2 \tfrac{x}{2}$

$$\cos x = \frac{\dfrac{\cos^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} - \dfrac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}}}{\dfrac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} + \dfrac{\cos^2 \frac{x}{2}}{\cos^2 \frac{x}{2}}}$$
$$\cos x = \frac{1 - \tan^2 \tfrac{x}{2}}{\tan^2 \tfrac{x}{2} + 1}$$
$$\cos x = \frac{1 - t^2}{1 + t^2}$$

For $t = \tan x$

$$\sin 2x = \frac{2 \sin x \cos x}{1}$$
$$= \frac{2 \sin x \cos x}{\sin^2 x + \cos^2 x}$$

Dividing through by $\cos^2 x$

$$\sin 2x = \frac{\frac{2 \sin x \cos x}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x}}$$
$$\sin 2x = \frac{2 \tan x}{\tan^2 x + 1}$$
$$\sin 2x = \frac{2t}{1 + t^2}$$

$$\cos 2x = 2\cos^2 x - 1$$
$$= \frac{2}{\sec^2 x} - 1$$
$$= \frac{2 - \sec^2 x}{\sec^2 x}$$
$$= \frac{2 - (1 + \tan^2 x)}{\sec^2 x}$$
$$\cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$$
$$\cos 2x = \frac{1 - t^2}{1 + t^2}$$

Note: The *t*-formula is used to solve equations of the form
$a \cos\theta + b \sin\theta = c$

Example I

Solve the following equations for $0 \le \theta \le 360°$

(a) $2\cos\theta + 3\sin\theta - 2 = 0$
(b) $3\cos\theta - 4\sin\theta + 1 = 0$
(c) $3\cos\theta + 4\sin\theta = 2$
(d) $4\cos\theta \sin\theta + 15\cos 2\theta = 10$

Solution

(a) $2\cos\theta + 3\sin\theta = 2$

$$\cos\theta = \frac{1 - t^2}{1 + t^2}, \quad \sin\theta = \frac{2t}{1 + t^2}, \text{ for } t = \tan\tfrac{\theta}{2}$$
$$2\left(\frac{1 - t^2}{1 + t^2}\right) + 3\left(\frac{2t}{1 + t^2}\right) = 2$$
$$2(1 - t^2) + 3(2t) = 2(1 + t^2)$$
$$2 - 2t^2 + 6t = 2 + 2t^2$$
$$4t^2 - 6t = 0$$
$$2t(2t - 3) = 0$$
$$t = 0$$
$$\Rightarrow \tan\tfrac{\theta}{2} = 0 \text{ and } \tan\tfrac{\theta}{2} = \frac{3}{2}$$

For $\tan\tfrac{\theta}{2} = 0$, $\dfrac{\theta}{2} = \tan^{-1}(0)$

$$\frac{\theta}{2} = 0°, 180°, \ldots$$
$$\theta = 0, 360.$$

For $\tan\tfrac{\theta}{2} = \dfrac{3}{2}$, $\dfrac{\theta}{2} = 56.3°$

$$\theta = 112.6°$$

$\Rightarrow$ 0°, 112.6°, and 360° are solutions to the equation
$2\cos\theta + 3\sin\theta - 2 = 0$

(b) $3\cos\theta - 4\sin\theta + 1 = 0$

$$3\left(\frac{1 - t^2}{1 + t^2}\right) - 4\left(\frac{2t}{1 + t^2}\right) + 1 = 0$$
$$3 - 3t^2 - 8t + 1 + t^2 = 0$$
$$-2t^2 - 8t + 4 = 0$$
$$t^2 + 4t - 2 = 0$$
$$t = \frac{-4 \pm \sqrt{(4)^2 - 4 \times 1 \times (-2)}}{2 \times 1}$$
$$t = \frac{-4 \pm \sqrt{16 + 8}}{2}$$
$$t = -2 \pm \sqrt{6}$$
$$\tan\tfrac{\theta}{2} = -2 - \sqrt{6}$$
$$\tan\tfrac{\theta}{2} = -2 + \sqrt{6}$$

For $t = -2 - \sqrt{6}$, $\tan\tfrac{\theta}{2} = -2 - \sqrt{6}$

$$\frac{\theta}{2} = 102.7, 282.7$$
$$\theta = 205.4°$$

When $t = \tan\frac{\theta}{2} = -2+\sqrt{6}$

$$\frac{\theta}{2} = \tan^{-1}(-2 + \sqrt{6})$$

$$\frac{\theta}{2} = 24.2°$$

$\theta = 48.4°$

$\Rightarrow \theta = 48.4°$ and 205.4° are the solutions to the equation

(c) $3\cos\theta + 4\sin\theta = 2$

$$3\left(\frac{1-t^2}{1+t^2}\right) + 4\left(\frac{2t}{1+t^2}\right) = 2$$

$3 - 3t^2 + 8t = 2(1 + t^2)$

$3 - 3t^2 + 8t = 2 + 2t^2$

$5t^2 - 8t - 1 = 0$

$$t = \frac{8 \pm \sqrt{(-8)^2 - 4\times5\times(-1)}}{2\times5}$$

$$t = \frac{8 \pm \sqrt{64+20}}{10}$$

$$t = \frac{8 \pm \sqrt{84}}{10}$$

$t = -0.11652$

$t = 1.71652$

For $t = -0.11652$, $\tan\frac{\theta}{2} = -0.11652$

$$\frac{\theta}{2} = 173.4 \quad \Rightarrow \theta = 346.7°$$

$$\tan\frac{\theta}{2} = 1.71652$$

$$\frac{\theta}{2} = 59.8° \quad \Rightarrow \theta = 119.6°$$

$\Rightarrow$ 119.6° and 346.7° are solutions to the above equation.

(d) $4\cos\theta\sin\theta + 15\cos2\theta = 10$

$2 \times 2\sin\theta\cos\theta + 15\cos2\theta = 10$

$2\sin2\theta + 15\cos2\theta = 10$

$2\sin2\theta + 15\cos2\theta = 0$

Let $t = \tan\theta$

$$\sin\theta = \frac{2t}{1+t^2} \text{ and } \cos\theta = \frac{1-t^2}{1+t^2}$$

$$2\left(\frac{2t}{1+t^2}\right) + 15\left(\frac{1-t^2}{1+t^2}\right) = 10$$

$4t + 15 - 15t^2 = 10 + 10t^2$

$25t^2 - 4t - 5 = 0$

$$t = \frac{4 \pm \sqrt{(-4)^2 - 4\times25\times(-5)}}{2\times25}$$

$t = 0.5343$

$t = -0.3743$

For $t = 0.5343$

$\tan\theta = 0.5343$

$\theta = 28.1°$

$\theta = 208.1°$

For $t = -0.3743$, $\tan\theta = -0.3743$

$\theta = \tan^{-1}(0.3743)$

$\theta = 159.5°$, 200.5°

$\Rightarrow$ 28.1°, 208.1°, 159.5° and 200.5° are the solutions to the above equation

The *R*- Formula

The *R*-formula is used to solve equations of the form $a\cos\theta + b\sin\theta = c$.

$$\boxed{\begin{array}{c}\mathbf{R\cos(\theta \pm \alpha) = c} \\ \mathbf{R\sin(\theta \pm \alpha) = c}\end{array}}$$

Where $R = \sqrt{a^2+b^2}$ and $\alpha = \tan^{-1}\left(\frac{a}{b}\right)$

Example I

Solve the equation $3\cos\theta + 4\sin\theta = 2$ for $0 \le \theta \le 360°$

Solution

$R\cos(\theta - \alpha) = 2$

$R(\cos\theta\cos\alpha + \sin\theta\sin\alpha) = 2$

$R\cos\theta\cos\alpha + R\sin\theta\sin\alpha = 2$

By comparison

$R\cos\theta\cos\alpha = 3\cos\theta$

$R\sin\theta\sin\alpha = 4\sin\theta$

$\Rightarrow R\cos\alpha = 3$ ……………………….. (i)

$R\sin\alpha = 4$ ……………………….. (ii)

Eqn (ii) ÷ Eqn (1);

$$\Rightarrow \tan\alpha = \frac{4}{3}$$

$$\alpha = \tan^{-1}\left(\frac{4}{3}\right)$$

$\alpha = 53.1°$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 4^2$

$R^2(\cos^2\alpha + \sin^2\alpha) = 25$

$R = 5$

$R\cos(\theta - \alpha) = 2$

$5\cos(\theta - 53.1) = 2$

$$\theta - 53.1 = \cos^{-1}\left(\frac{2}{5}\right)$$

$\theta - 53.1° = 66.4°, 293.6°$

$\theta - 119.5°, 346.7°$

Alternatively

$3\cos\theta + 4\sin\theta = 2$

$R\cos(\theta - \alpha) = 2$

$R = \sqrt{a^2 + b^2}$

$= \sqrt{(3)^2 + 4^2}$

$= 5$

$\alpha = \tan^{-1}\left(\frac{b}{a}\right)$

$\alpha = \tan^{-1}\left(\frac{4}{3}\right)$

$\alpha = 53.1$

$5\cos(\theta - 53.1) = 2$

$\cos(\theta - 53.1°) = \frac{2}{5}$

$\theta - 53.1° = 66.4°, 293.6°$

$\theta = 119.5°, 346.7°$

Example II

$\sin\theta + \sqrt{3}\cos\theta = 1$ for $0 \leq \theta \leq 360$

Solution

$R\sin(\theta + \alpha) = 1$

$R = \sqrt{(1)^2 + (\sqrt{3})^2} = 2$

$\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = 60°$

$R\sin(\theta + \alpha) = 1$

$2\sin(\theta + 60°) = 1$

$\sin(\theta + 60°) = \frac{1}{2}$

$\theta + 60° = \sin(½)$

$\theta + 60° = 30, 150°$

$\theta = -30, 90°$

$\Rightarrow \theta = 90°$, and $330°$.

Example III

$\cos\theta - 7\sin\theta = 2$ for $0° \leq \theta \leq 360°$

Solution

$\cos\theta - 7\sin\theta = 2$

$R\cos(\theta + \alpha) = 2$

$R = \sqrt{1^2 + (-7)^2} = \sqrt{50}$

$\alpha = \tan^{-1}\left(\frac{7}{1}\right) \qquad \Rightarrow \alpha = 81.9°$

$\sqrt{50}\cos(\theta + 81.9°) = 2$

$\cos(\theta + 81.9°) = \frac{2}{\sqrt{50}}$

$\theta + 81.9° = 73.6°, 286.4°$

$\theta = -8.3°, 204.5°$

$\Rightarrow \theta = 204.5°, 351.7°$

Example IV

Solve: $5\sin\theta - 12\cos\theta = 6$

Solution

$R\sin(\theta - \alpha) = 6$

$R = \sqrt{5^2 + 12^2} = 13$

$\alpha = \tan^{-1}\left(\frac{12}{5}\right)$

$\alpha = 67.4°$

$13\sin(\theta - 67.4) = 6$

$\sin(\theta - 67.4) = \frac{6}{13}$

$\theta - 67.4° = 27.5°, 152.5°$

$\theta = 94.9°, 219.9°$

Example V

Solve $\cos\theta + \sin\theta = \sec\theta$ for $0 \leq \theta \leq 360°$

Solution

$\cos\theta + \sin\theta = \frac{1}{\cos\theta}$

$\cos^2\theta + \sin\theta\cos\theta = 1$.............................(i)

But $\cos 2\theta = 2\cos^2\theta - 1$

$\Rightarrow \cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

Subsituting for $\cos^2\theta$ and $\sin\theta\cos\theta$ in Eqn (i);

$\sin 2\theta = 2\sin\theta\cos\theta$

$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$

$\frac{1}{2}(1 + \cos 2\theta) + \frac{1}{2}\sin 2\theta = 1$

$\frac{1}{2}\cos 2\theta + \frac{1}{2}\sin 2\theta = \frac{1}{2}$

$\cos 2\theta + \sin 2\theta = 1$

$R\cos(2\theta - \alpha) = 1$

$R = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\alpha = \tan^{-1}\left(\frac{1}{1}\right) = 45°$

$\sqrt{2}\cos(2\theta - 45°) = 1$

$\cos(2\theta - 45°) = \frac{1}{\sqrt{2}}$

$2\theta - 45° = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$2\theta - 45° = 45°, 315°, 405°$

$\theta = 45°, 180°, 225°$

Example VI

Solve the equation $4\cos\theta\sin\theta + 15\cos 2\theta = 10$

Solution

$4\cos\theta\sin\theta + 15\cos 2\theta = 10$

$2(2\sin\theta\cos\theta) + 15\cos 2\theta = 10$
$2\sin 2\theta + 15\cos 2\theta = 10$
$R\sin(2\theta + \alpha) = 10$
$R = \sqrt{2^2 + 15^2}$
$= \sqrt{229}$
$\sqrt{229}\sin(2\theta + \alpha) = 10$
$\alpha = \tan^{-1}\left(\frac{15}{2}\right) = 82.4°$
$\sqrt{229}\sin(2\theta + 82.4°) = 10$
$\sin(2\theta + 82.4°) = \frac{10}{\sqrt{229}}$
$2\theta + 82.4° = \sin^{-1}\left(\frac{10}{\sqrt{229}}\right)$
$2\theta + 82.4° = 41.4°, 138.6°, 401.4°, 498.4°$
$\theta = 339.5°, 28.1°, 159.5°, 208°$

Example VII
Show that $3\cos\theta + 2\sin\theta$ can be written as $\sqrt{13}\cos(\theta - \alpha)$. Hence find the minimum and maximum values of the function, giving the corresponding values of θ from -180° to 180°

Solution
$3\cos\theta + 2\sin\theta$
$R\cos(\theta - \alpha)$
$R = \sqrt{a^2 + b^2}$
$R = \sqrt{3^2 + 2^2} = \sqrt{13}$
$\alpha = \tan^{-1}\left(\frac{2}{3}\right) = 33.7°$
$\Rightarrow 3\cos\theta + 2\sin\theta = R\cos(\theta - \alpha)$
$= \sqrt{13}\cos(\theta - 33.7)$
Let $y = \sqrt{13}\cos(\theta - 33.7)$
For the maximum value of y, $\cos(\theta - 33.7) = 1$
$\Rightarrow y_{max} = \sqrt{13}$
And for minimum value of y, $\cos(\theta - 33.7) = -1$
$\Rightarrow y_{min} = -\sqrt{13}$
For y_{max} $\cos(\theta - 33.7°) = 1$,
$\Rightarrow \theta - 33.7° = \cos^{-1}(1)$
$\theta - 33.7° = 0, 360°.$
$\theta = 33.7°$
For y_{min} $\cos(\theta - 33.7°) = -1$,
$\theta - 33.7° = 180°.$
$\theta = 213.7°$

Example VII
Find the maximum and minimum values of the following expressions, stating the value of θ for which they occur (from 0° to 360°)

(a) $8\cos\theta - 15\sin\theta$
(b) $4\sin\theta - 3\cos\theta$
(c) $\sin\theta - 6\cos\theta$
(d) $\cos(\theta + 60) - \cos\theta$

Solution
(a) $8\cos\theta - 15\sin\theta$
$\boldsymbol{R}\cos(\theta - \alpha)$
$R = \sqrt{8^2 + 15^2} = 17$
$\alpha = \tan^{-1}\left(\frac{15}{8}\right) = 61.9°$
$17\cos(\theta - 61.9°)$
Let $y = 17\cos(\theta - 61.9°)$
For y_{max}, $\cos(\theta - 61.9°) = 1$
$\Rightarrow y_{max} = 17$
$\theta - 61.9° = \cos^{-1}(1)$
$\theta - 61.9° = 0, 360°$
$\theta = 61.9°$

For y_{min}, $\cos(\theta - 61.9) = -1$
$\Rightarrow y_{min} = -17$
$\theta - 61.9° = \cos^{-1}(-1)$
$\theta - 61.9° = 180°$
$\theta = 241.9°$

(b) $4\sin\theta - 3\cos\theta$
$R = \sqrt{4^2 + 3^2} = 5$
$R\sin(\theta - \alpha)$
$5\sin(\theta - \alpha)$
$\alpha = \tan^{-1}\left(\frac{3}{4}\right) = 36.9°$
$5\sin(\theta - 36.9°)$
Let $y = 5\sin(\theta - 36.9°)$
$y_{min} = -5$
$y_{max} = 5$
For y_{min}, $\sin(\theta - 36.9°) = -1$
$\theta - 36.9° = 270°$
$\theta = 306.9°$
For y_{max}, $\sin(\theta - 36.9°) = 1$
$\theta - 36.9° = 90°$
$\theta = 126.9°$

(c) $\boldsymbol{\sin\theta - 6\cos\theta}$
$R = \sqrt{1^2 + (-6)^2} = \sqrt{37}$
$\sqrt{37}\sin(\theta - \alpha)$
$\alpha = \tan^{-1}\left(\frac{6}{1}\right) = 80.5°$
$y = \sqrt{37}\sin(\theta - 80.1)$
$y_{max} = \sqrt{37}$ and it occurs when $\sin(\theta - 80.1) = 1$
$\theta - 80.1° = 90°$

$\theta = 170.5°$

$y_{min} = -\sqrt{37}$ and it occurs when

$\sin(\theta - 80.1) = -1$

$\theta - 80.1° = 270°$

$\theta = 350.5°$

(d) $\cos(\theta + 60) - \cos\theta$

$= \cos\theta\cos 60 - \sin\theta\sin 60 - \cos\theta$

$= \frac{1}{2}\cos\theta - \sin\theta\frac{\sqrt{3}}{2} - \cos\theta$

$= \frac{-1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta$

$y = -\left[\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right]$

$y = -\left[R\cos(\theta - \alpha)\right]$

$R = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2} = 1$

$y = -\left[\cos(\theta - \alpha)\right]$

$\alpha = \tan^{-1}\left(\sqrt{3}\right) = 60°$

$y = -[\cos(\theta - 60)]$

y_{min} occurs when $\cos(\theta - 60) = 1$

$\theta - 60° = 0, 360$

$\theta = 60°$

$y_{max} = 1$ and occurs when $\cos(\theta - 60°) = -1$

$\theta - 60° = \cos^{-1}(-1)$

$\theta = 240°$

Example VIII (UNEB Question)

Solve $\cos\theta + \sqrt{3}\sin\theta = 2$ for $0 \leq \theta \leq \pi$

Solution

$\cos\theta + \sqrt{3}\sin\theta = 2$

$R\cos(\theta - \alpha) = 2$

$R = \sqrt{1^2 + (\sqrt{3})^2} = 2$

$2\cos(\theta - \alpha) = 2$

$\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = 60°$

$2\cos(\theta - 60°) = 2$

$\cos(\theta - 60°) = 1$

$\theta - 60° = \cos^{-1}(1)$

$\theta - 60° = 0$

$\theta = 60°$

$\theta = \frac{\pi}{3}$

Since 180 = π radians, $\Rightarrow \theta = \frac{60\pi}{180} = \frac{\pi}{3}$

Example IX (UNEB Question)

(a) Express $4\cos\theta - 5\sin\theta$ in the form $R\cos(\theta + \beta)$, where R is a constant and β an acute angle.
Determine the maximum value of the expression and the value of θ for which it occurs

(b) Solve the equation $4\cos\theta - 5\sin\theta = 2.2$, for $0^0 < \theta < 360^0$.

Solution

$4\cos\theta - 5\sin\theta$

$R\cos(\theta + \beta)$

$\beta = \tan^{-1}\left(\frac{5}{4}\right) = 51.3°$

$R = \sqrt{4^2 + 5^2} = \sqrt{41}$

$\sqrt{41}\cos(\theta + 51.3°)$

Let $y = \sqrt{41}\cos(\theta + 51.3°)$

$y_{max} = \sqrt{41}$ and it occurs when $\cos(\theta + 51.3°) = 1$

$\theta + 51.3° = 0$

$\theta = -51.3°$

$\Rightarrow \theta = 308.7°$ $(0^0 < \theta < 360^0)$

$4\cos\theta - 5\sin\theta = 2.2$

$\Rightarrow \sqrt{41}\cos(\theta + 51.3°) = 2.2$

$\cos(\theta + 51.3°) = \frac{2.2}{\sqrt{41}}$

$\theta + 51.3° = 69.9°, 290.1°$

$\theta = 18.6°, 238.8°$

Example XI (UNEB Question)

Express $y = 8\cos x + 6\sin x$ in the form $R\cos(x - \alpha)$ where R is positive and α is acute . Hence find the maximum and minimum values of $\frac{1}{8\cos x + 6\sin x + 15}$

Solution

$8\cos x + 6\sin x = R\cos(x - \alpha)$

$8\cos x + 6\sin x = R\cos x\cos\alpha + R\sin x\sin\alpha$

By comparison

$R\cos\alpha = 8$ …..........………………… (i)

$R\sin\alpha = 6$……………..........……….(ii)

Eqn (i)^2 + Eqn (ii)^2;

$R^2 = 8^2 + 6^2 = 100$

$R = 10$

Eqn (ii) $\div$ Eqn (i)

$\tan\alpha = \frac{6}{8}$

$\alpha = 36.87^0$

Hence $8\cos x + 6\sin x = 10\cos(x - 36.87^0)$

$Now \frac{1}{8\cos x + 6\sin x + 5} = \frac{1}{10\cos(x - 36.87) + 15}$

Note: For y to be maximum, the denominator must be minimum and for y to be minimum, the denominator must be maximum.

Let $m = \frac{1}{10\cos(x-36.87)+15}$

$$M_{\max} = \frac{1}{10\times(-1)+15}$$

$$= \frac{1}{-10+15} = \frac{1}{5} = 0.2$$

$$M_{\min} = \frac{1}{10\times 1+15}$$

$$= \frac{1}{25} = 0.04$$

The maximum and minimum values of $\frac{1}{8\cos x + (\sin x + 15)}$ are 0.2 and 0.04 respectively.

Factor Formula

1. **sin P + sin Q =** $2\sin(\frac{P+Q}{2})\cos(\frac{P-Q}{2})$
2. **sin P − sin Q =** $2\cos(\frac{P+Q}{2})\sin(\frac{P-Q}{2})$
3. **cos *P* + cos Q =** $2\cos(\frac{P+Q}{2})\cos(\frac{P-Q}{2})$
4. **cos *P* – cos Q = -** $2\sin(\frac{P+Q}{2})\sin(\frac{P-Q}{2})$

Application of the factor formula

Example 1

Express the following in factors:

(a) $\sin 7\theta + \sin 5\theta$
(b) $\sin 4x - \sin 2x$
(c) $\cos 7x + \cos 5x$
(d) $\cos 3A - \cos 5A$
(e) $\sin(x + 30) + \sin(x - 30)$
(f) $\cos(x + 30) - \cos(x - 30)$
(g) $\cos \frac{3}{2}x - \cos \frac{x}{2}$
(h) $\frac{1}{2} + \cos 2\theta$
(i) $1 + \sin 2x$
(j) $\text{Sin } 2(x + 40) + \sin 2(x - 40)$

Solution

(a) $\sin 7\theta + \sin 5\theta$

From $\sin P + \sin Q = 2\sin(\frac{P+Q}{2})\cos(\frac{P-Q}{2})$

$\sin 7\theta + \sin 5\theta = 2\sin(\frac{7\theta+5\theta}{2})\cos\frac{7\theta-5\theta}{2}$

$= 2\sin 6\theta \cos\theta$

(b) $\sin 4x - \sin 2x$

From $\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$

$\sin 4x - \sin 2x = 2\cos\frac{4x+2x}{2}\sin\frac{4x-2x}{2}$

$\sin 4x - \sin 2x = 2\cos 3x \sin x$

(c) $\cos 7x + \cos 5x$

From $\cos P + \cos Q = 2\cos(\frac{P+Q}{2})\cos(\frac{P-Q}{2})$

$\Rightarrow \text{Cos } 7x + \cos 5x = 2\cos(\frac{7x+5x}{2})\cos(\frac{7x-5x}{2})$

$= 2\cos 6x \cos x$

(d) $\cos 3A - \cos 5A$

From $\cos P - \cos Q = -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}$

$\cos 3A - \cos 5A = -2\sin(\frac{3A+5A}{2})\sin(\frac{3A-5A}{2})$

$= -2\sin 4A \sin(-A)$

$= 2\sin 4A \sin A$

(e) $\sin(x + 30) + \sin(x - 30)$

$= -2\sin(\frac{(x+30)+(x-30)}{2})\cos(\frac{(x+30)-(x-30)}{2})$

$= -2\sin x \cos 30$

(f) $\cos(x + 30) - \cos(x - 30)$

$= 2\sin(\frac{(x+30)+(x-30)}{2})\sin(\frac{(x+30)-(x-30)}{2})$

$= 2\sin x \sin 30$

(g) $\cos(\frac{3x}{2}) - \cos\frac{x}{2} = -2\sin\left(\frac{\frac{3x}{2}+\frac{x}{2}}{2}\right)\sin\left(\frac{\frac{3x}{2}-\frac{x}{2}}{2}\right)$

$= 2\sin x \sin\frac{x}{2}$

(h) $\frac{1}{2} + \cos 2\theta$

$\cos 60 + \cos 2\theta$

$= 2\cos\frac{60+2\theta}{2}\cos\frac{60-2\theta}{2}$

$= 2\cos(30 + \theta)\cos(30 - \theta)$

(i) $1 + \sin 2x$

$\sin 90 + \sin 2x$

$2\sin\frac{90+2x}{2}\cos\frac{90-2x}{2}$

$= 2\sin(45 + x)\cos(45 - x)$

(j) $\sin 2(x + 40) + \sin 2(x - 40)$

$= 2\sin\frac{2(x+40)+2(x-40)}{2}\cos\frac{2(x+40)-2(x-40)}{2}$

$= 2\sin 2x \cos 80$

Example II

Solve the following equations from $x = 0°$ to $360°$ inclusive.

(a) $\cos x + \cos 5x = 0$
(b) $\sin 3x - \sin x = 0$
(c) $\sin(x + 10) + \sin x = 0$
(d) $\cos(2x + 10) + \cos(2x - 10) = 0$
(e) $\cos(x + 20) - \cos(x - 70) = 0$

Solution

(a) $\cos x + \cos 5x = 0$

$2\cos\frac{x+5x}{2}\cos\frac{x-5x}{2} = 0$

$2\cos 3x \cos -2x = 0$

$2\cos 3x \cos 2x = 0$

$\cos 3x \cos 2x = 0$
$\Rightarrow$ $\cos 2x = 0$ **OR**
$\cos 3x = 0$
For $\cos 2x = 0$;
$2x = \cos^{-1}(0)$
$2x = 90°, 270°, 450°, 630°, 810°$
$\Rightarrow$ $x = 45°, 135°, 225°, 315°$.
For $\cos 3x = 0$;
$3x = \cos^{-1}(0)$
$3x = 90°, 270°, 450°, 630°, 810°, 990°, 1170°$
$x = 30°, 90°, 150°, 210°, 270°, 330°$.
$\therefore$ The solutions to the equation $\cos x + \cos 5x = 0$ are 30°, 45°, 90°, 135°, 150°, 210°, 225°, 270°, 315°, 330°.

(b) $\sin 3x - \sin x = 0$
$2\cos\frac{3x+x}{2}\sin\frac{3x-x}{2} = 0$
$2\cos 2x \sin x = 0$
$\cos 2x \sin x = 0$
$\Rightarrow 2x = \cos^{-1}(0)$
$2x = 90°, 270°, 450°, 630°, 810°, 990°$
$\Rightarrow x = 45°, 135°, 225°, 315°$
And for $\sin x = 0$;
$x = \sin^{-1}(0)$
$x = 0, 180°, 360°$
$\Rightarrow$ The solutions to the equation sin 3x – sin x = 0 are 0, 45, 135, 180, 225, 315, 360.

(c) $\sin(x + 10) + \sin x = 0$
$2\sin\left(\frac{x+10+x}{2}\right)\cos\left(\frac{x+10-x}{2}\right) = 0$
$2\sin(x + 5)(\cos 5) = 0$
$\sin(x + 5) = 0$
$x + 5 = \sin^{-1}0)$
$x + 5 = 0, 180°, 360°$
$x = 355°, 175°$.
$\Rightarrow x = 175°, 335°$ are solutions to the equation $\sin(x + 10) + \sin x = 0$

(d) $\cos(2x + 10) + \cos(2x - 10) = 0$
$2\cos\left(\frac{(2x+10)+(2x-10)}{2}\right)\cos\left(\frac{(2x+10)-(2x-10)}{2}\right)$
$2\cos 2x \cos 10 = 0$
$\cos 2x = 0$
$2x = \cos^{-1}(0)$
$2x = 90°, 270°, 450°, 630°$.
$x = 45°, 135°, 225°, 315°$
$\Rightarrow$ The solutions to the equation $\cos(2x + 20) + \cos(2x - 10) = 0$ are $x = 45°, 135°, 225°$ and $315°$

(f) $\cos(x + 20) - \cos(x - 70) = 0$
$-2\sin\frac{(x+20)+(x-70)}{2}\sin\frac{(x+20)-(x-70)}{2} = 0$
$-2\sin(x - 25)\sin 45 = 0$
$\sin(x - 25) = 0$
$x - 25 = \sin^{-1}(0)$
$x - 25 = 0, 180°, 360°$
$x = 25, 205°, 385°$

Example II

Prove the following identities:

(a) $\dfrac{\cos B + \cos C}{\sin B - \sin C} = \cot\frac{B-C}{2}$

(b) $\dfrac{\cos B - \cos C}{\sin B + \sin C} = -\tan\frac{B-C}{2}$

(c) $\dfrac{\sin B - \sin C}{\sin B + \sin C} = \cot\frac{B+C}{2}\tan\frac{B-C}{2}$

(d) $\dfrac{\sin B + \sin C}{\cos B + \cos C} = \tan\frac{B+C}{2}$

Solution

(a) $\dfrac{\cos B + \cos C}{\sin B - \sin C}$

$$= \frac{2\cos(\frac{B+C}{2})\cos(\frac{B-C)}{2})}{2\cos(\frac{B+C}{2})\sin(\frac{B-C)}{2})}$$

$$= \frac{\cos\frac{B-C}{2}}{\sin\frac{B-C}{2}}$$

$$= \cot(\tfrac{B-C}{2})$$

(b) $\dfrac{\cos B - \cos C}{\sin B + \sin C}$

$$= \frac{-2\sin\frac{B+C}{2}\sin\frac{B-C}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}$$

$$= \frac{-\sin\frac{B-C}{2}}{\cos\frac{B-C}{2}}$$

$$= -\tan(\tfrac{B-C}{2})$$

$$\Rightarrow \frac{\cos B - \cos C}{\sin B + \sin C} = -\tan(\tfrac{B-C}{2})$$

(c) $\dfrac{\sin B - \sin C}{\sin B + \sin C}$

$$\frac{\sin B - \sin C}{\sin B + \sin C} = \frac{2\cos(\frac{B+C}{2})\sin(\frac{B-C}{2})}{2\sin(\frac{B+C}{2})\cos(\frac{B-C}{2})}$$

$$= \frac{\cos(\frac{B+C}{2})}{\sin(\frac{B+C}{2})} \times \frac{\sin(\frac{B-C}{2})}{\cos(\frac{B-C}{2})}$$

$$= \cot(\tfrac{B+C}{2})\tan(\tfrac{B-C}{2})$$

(d) $\dfrac{\sin B + \sin C}{\cos B + \cos C}$

$$= \frac{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{2\cos\frac{B+C}{2}\cos\frac{B-C}{2}}$$

$$= \tan\frac{B-C}{2}$$

$$\Rightarrow \frac{\sin B + \sin C}{\cos B + \cos C} == \tan\frac{B-C}{2}$$

Example IV

Prove the following

(a) $\sin x + \sin 2x + \sin 3x = \sin 2x(2\cos x + 1)$

(b) $\cos x + \sin 2x - \cos 3x = \sin 2x(2\sin x + 1)$

(c) $\cos\theta - 2\cos 3\theta + \cos 5\theta = 2\sin\theta\,(\sin 2\theta - \sin 4\theta)$

(d) $\sin x - \sin(x + 60) + \sin(x + 120) = 0$

(e) $1 + 2\cos 2\theta + \cos 4\theta = 4\cos^2\theta \cos 2\theta$

Solutions

(a) $\sin x + \sin 2x + \sin 3x$

$= \sin x + \sin 3x + \sin 2x$

$= 2\sin \frac{x+3x}{2} \cos \frac{x-3x}{2} + \sin 2x$

$= 2\sin 2x \cos(-x) + \sin 2x$

$= 2\sin 2x \cos x + \sin 2x$

$= \sin 2x(2\cos x + 1)$

$\Rightarrow \sin x + \sin 2x + \sin 3x = \sin 2x(2\cos x + 1)$

(b) $\cos x + \sin 2x - \cos 3x$

$= \cos x - \cos 3x + \sin 2x$

$= -2\sin \frac{x+3x}{2} \sin \frac{x-3x}{2} + \sin 2x$

$= -2\sin 2x \sin(-x) + \sin 2x$

$= 2\sin 2x \sin x + \sin 2x$

$= \sin 2x[2\sin x + 1]$

$\Rightarrow \cos x + \sin 2x - \cos 3x = \sin 2x[2\sin x + 1]$

(c) $\cos\theta - 2\cos 3\theta + \cos 5\theta$

$= \cos \theta - \cos 3\theta + \cos 5\theta - \cos 3\theta$

$= -2\sin 2\theta \sin(-\theta) + {}^{-}2\sin 4\theta \sin \theta$

$= 2\sin 2\theta \sin \theta - 2\sin 4\theta \sin\theta$

$= 2\sin \theta\,(\sin 2\theta - \sin 4\theta)$

$\Rightarrow \cos\theta - 2\cos 3\theta + \cos 5\theta = 2\sin \theta\,(\sin 2\theta - \sin 4\theta)$

(d) $\sin x - \sin(x + 60) + \sin(x + 120)$

$= \sin x + \sin(x + 120) - \sin(x + 60)$

$= 2\sin(x + 60)\cos -60 - \sin(x + 60)$

$= \sin(x + 60) - \sin(x + 60)$

$= 0$

$\Rightarrow \sin x - \sin(x + 60) + \sin(x + 120) = 0$

(e) $1 + 2\cos 2\theta + \cos 4\theta$

Since $\cos 4\theta = \cos^2 2\theta - 1$,

$\Rightarrow 1 + 2\cos 2\theta + 2\cos^2 2\theta - 1$

$= 2\cos 2\theta + 2\cos^2 2\theta$

$= 2\cos 2\theta\,[1 + \cos 2\theta]$

$= 2\cos 2\theta\,[1 + 2\cos^2\theta - 1]$

$= 4\cos^2\theta \cos 2\theta$

$\Rightarrow 1 + 2\cos 2\theta + \cos 4\theta = 4\cos^2\theta \cos 2\theta$

Example V

Solve the following equations for values of θ from 0° to 180° inclusive

(a) $\cos \theta + \cos 3\theta + \cos 5\theta = 0$

(b) $\sin \theta - 2\sin 2\theta + \sin 3\theta = 0$

(c) $\sin \theta + \cos 2\theta - \sin 3\theta = 0$

(d) $\sin 2\theta + \sin 4\theta + \sin 6\theta = 0$

(e) $\cos \frac{1}{2}\theta + 2\cos \frac{3}{2}\theta + \cos \frac{5}{2}\theta = 0$

Solution

(a) $\cos \theta + \cos 3\theta + \cos 5\theta = 0$

$\cos \theta + \cos 5\theta + \cos 3\theta = 0$

$2\cos 3\theta \cos -2\theta + \cos 3\theta = 0$

$\cos 3\theta(2\cos 2\theta + 1] = 0$

Either $\cos 3\theta = 0$ **OR**

$\cos 2\theta = -\dfrac{1}{2}$

For $\cos 3\theta = 0$;

$3\theta = \cos^{-1}(0)$

$3\theta = 90°, 270°, 450°, 630°, 810°, 990°$

$\theta = 30°, 90°, 150°, 210°, 270, 330°$

$\Rightarrow \theta = 30°, 90°, 150°$ (for $0° \le \theta \le 180°$)

For $\cos 2\theta = -\dfrac{1}{2}$;

$2\theta = \cos^{-1}(\frac{-1}{2})$

$2\theta = 120°, 240°.$

$\theta = 60°, 120°$

$\Rightarrow$ 30°, 60°, 90°, 120°, 150° are the solutions to the equation $\cos 3\theta + \cos 3\theta + \cos 5\theta = 0$

(b) $\sin \theta - 2\sin 2\theta + \sin 3\theta = 0$

$\sin \theta + \sin 3\theta - 2\sin 2\theta = 0$

$2\sin 2\theta \cos(-\theta) - 2\sin 2\theta = 0$

$2\sin 2\theta \cos \theta - 2\sin 2\theta = 0$

$2\sin 2\theta\,(\cos \theta - 1) = 0$

Either $\sin 2\theta = 0$ **OR** $\cos \theta = 1$

For $\sin 2\theta = 0$;

$2\theta = \sin^{-1}0$

$2\theta = 0°, 180°, 360°$

$\Rightarrow \theta = 0°, 90°, 180°$

(c) $\sin \theta + \cos 2\theta - \sin 3\theta = 0$

$\sin \theta - \sin 3\theta + \cos 2\theta = 0$

$2\cos 2\theta \sin -\theta + \cos 2\theta = 0$

$\cos 2\theta(-2\sin \theta + 1) = 0$

$\cos 2\theta = 0$ **OR** $\sin\theta = \dfrac{1}{2}$

For $\cos 2\theta = 0$

$2\theta = \cos^{-1}0$

$2\theta = 90°, 270°, 450°$

$= 45°, 135°$

For $\sin \theta = \dfrac{1}{2}$;

$\theta = \sin^{-1}(\frac{1}{2})$

$\theta = 30°, 150°$

$\Rightarrow$ 30°, 45°, 135°, 150° are the solutions to the equation $\sin\theta + \cos 2\theta - \sin 3\theta = 0$

(d) $\sin 2\theta + \sin 4\theta + \sin 6\theta = 0$

$(\sin 2\theta + \sin 6\theta) + \sin 4\theta = 0$

$2\sin 4\theta \cos -2\theta + \sin 4\theta = 0$

$2\sin 4\theta \cos 2\theta + \sin 4\theta = 0$

$\sin 4\theta (2\cos 2\theta + 1) = 0$

For $\sin 4\theta = 0$;

$4\theta = \sin^{-1}0$

$4\theta = 0, 180, 360, 540, 720$

$= 0, 45, 90, 135, 180$

For $2\cos 2\theta + 1 = 0$

$\cos 2\theta = -\frac{1}{2}$

$2\theta = 120°, 240°$

$\theta = 60°, 120°$

$\Rightarrow$ 0°, 45°, 60°, 90°, 120°, 135°, 180° are the solutions to the equation $\sin 2\theta + \sin 4\theta + \sin 6\theta = 0$

(e) $\cos\frac{1}{2}\theta + 2\cos\frac{3}{2}\theta + \cos\frac{5}{2}\theta = 0$

$\cos\frac{1}{2}\theta + \cos\frac{5}{2}\theta + 2\cos\frac{3}{2}\theta = 0$

$2\cos\frac{6\theta}{2}\cos(-\theta) + 2\cos\frac{3\theta}{2} = 0$

$2\cos\frac{3\theta}{2}(\cos\theta + 1) = 0$

$\cos\frac{3\theta}{2} = 0$

$\frac{3\theta}{2} = \cos^{-1}(0)$

$\frac{3\theta}{2} = 90, 270, 450$

$\theta = 60, 180$

For $(\cos\theta + 1) = 0$;

$\cos\theta = {}^{-}1$

$\theta = \cos^{-1}({}^{-}1)$

$\theta = 180$

$\Rightarrow$ 60, 180 are the solutions to the equation $\cos\frac{1}{2}\theta + 2\cos\frac{3}{2}\theta + \cos\frac{5}{2}\theta = 0$

Example V

Prove the following identities if A, B and C are taken to be angles of a triangle.

(a) $\sin A + \sin(B - C) = 2\sin B \cos C$

(b) $\cos A - \cos(B - C) = -2\cos B \cos C$

(c) $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(d) $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$

(e) $\cos A + \cos B + \cos C - 1$

$= 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Solutions

$\sin A + \sin(B - C)$

$= 2\sin\frac{A+B-C}{2}\cos\frac{A-(B-C)}{2}$

$= 2\sin\frac{A+B-C}{2}\cos\frac{A+C-B}{2}$

But $A + B + C = 180$

$A + B + C - 2C = 180 - 2C$

$A + B - C = 180 - 2C$

$\frac{A + B - C}{2} = 90 - C$

$\Rightarrow \sin\frac{A+B-C}{2} = \sin(90 - C)$

$= \sin 90\cos C - \cos 90\sin C$

$= \cos C$

$A + B + C = 180$

$A + C + B - 2B = 180 - 2B$

$A + C - B = 180 - 2B$

$\cos\frac{A+C-B}{2} = \cos\frac{180-2B}{2}$

$\cos\frac{A+C-B}{2} = \cos(90 - B)$

$= \cos 90\cos B + \sin 90\sin B$

$= \sin B$

$\Rightarrow \sin A + \sin(B - C) = 2\sin B\cos C$

(c) $\sin A + \sin B + \sin C$

$= \sin\frac{A+B}{2}\cos\frac{A-B}{2} + \sin C$

$= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} + 2\sin\frac{C}{2}\cos\frac{C}{2}$

But $A + B + C = 180$

$C = 180 - (A + B)$

$\frac{C}{2} = \sin(90 - \frac{A+B}{2})$

$\sin\frac{C}{2} = \sin 90\cos\frac{A+B}{2} - \cos 90\sin\frac{A+B}{2}$

$= \cos\frac{A+B}{2}$

$\cos\frac{C}{2} = \cos(90 - \frac{A+B}{2})$

$\cos\frac{C}{2} = \cos 90\cos\frac{A+B}{2} + \sin 90\sin\frac{A+B}{2}$

$= \sin\frac{A+B}{2}$

$\Rightarrow 2\cos\frac{C}{2}\cos\frac{A-B}{2} + 2\cos\frac{A+B}{2}\cos\frac{C}{2}$

$= 2\cos\frac{C}{2}\cos\frac{A-B}{2} + 2\cos\frac{A+B}{2}\cos\frac{C}{2}$

$= 2\cos\frac{C}{2}\left[\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\right]$

$= 2\cos\frac{C}{2}[2\cos\frac{A}{2}\cos\frac{B}{2}]$

$= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

(d) $\sin 2A + \sin 2B + \sin 2C$.

$= 2\sin(A + B)\cos(A - B) + 2\sin C\cos C$

But $A + B + C = 180$

$C = 180 - (A + B)$

$\Rightarrow \sin C = \sin[180 - (A + B)]$

$\sin C = \sin 180 \cos (A + B) - \cos 180 \sin(A + B)$

$\sin C = \sin(A + B)$

$\cos C = \cos(180 - (A+B))$

$\cos C = \cos 180 \cos(A + B) + \sin 180 \sin(A + B)$

$= -\cos(A + B)$

$\Rightarrow 2\sin(A + B)\cos(A - B) + 2\sin C \cos C$

$= 2\sin C \cos (A - B) + 2\sin C(-\cos(A+B))$

$= 2\sin C[\cos(A - B) - \cos(A + B)]$

$= 2\sin C[-2\sin A \sin -B]$

$= 4\sin A \sin B \sin C$

$\Rightarrow \sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$

(e) $\cos A + \cos B + \cos C - 1$

$\cos C = 2\cos^2 \frac{C}{2} - 1$

$\cos C = 1 - 2\sin^2 \frac{C}{2}$

$\Rightarrow 2\sin^2 \frac{C}{2} = 1 - \cos C$

$\cos A + \cos B + \cos C - 1 = \cos A + \cos B - 2\sin^2 \frac{C}{2}$

$= 2\cos \frac{A+B}{2} \cos \frac{A-B}{2} - 2\sin^2 \frac{C}{2}$

$A + B + C = 180$

$\frac{C}{2} = 90 - \frac{A+B}{2}$

$\sin \frac{C}{2} = \sin (90 - \frac{A+B}{2})$

$\sin \frac{C}{2} = \sin 90 \cos \frac{A+B}{2} - \cos 90 \sin \frac{A+B}{2}$

$\Rightarrow \sin \frac{C}{2} = \cos \frac{A+B}{2}$

$\Rightarrow 2\cos \frac{A+B}{2} \cos \frac{A-B}{2} - 2\sin \frac{C}{2} \sin \frac{C}{2}$

$= 2\cos \frac{A+B}{2} \cos \frac{A-B}{2} - 2\sin \frac{C}{2} \cos \frac{A+B}{2}$

$= 2\sin \frac{C}{2} \cos \frac{A-B}{2} - 2 \sin \frac{C}{2} \cos \frac{A+B}{2}$

$= 2\sin \frac{C}{2} [\cos \frac{A-B}{2} - \cos \frac{A+B}{2}]$

$= 2\sin \frac{C}{2} [-2\sin \frac{A}{2} \sin \frac{-B}{2}]$

$= 2\sin \frac{C}{2} [2\sin \frac{A}{2} \sin \frac{B}{2}]$

$= 4\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

$\Rightarrow \cos A + \cos B + \cos C - 1 = 4\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

Example VI (UNEB 2007)

Show that $\frac{\sin\theta - 2\sin 2\theta + \sin 3\theta}{\sin\theta + 2\sin 2\theta + \sin 3\theta} = -\tan^2 \frac{\theta}{2}$

Solution

$$\frac{\sin\theta - 2\sin 2\theta + \sin 3\theta}{\sin\theta + 2\sin 2\theta + \sin 3\theta}$$

$$= \frac{Sin 3\theta + \sin\theta - 2\sin 2\theta}{\sin 3\theta + \sin\theta + 2\sin 2\theta}$$

$$= \frac{2\sin\left(\frac{3\theta+\theta}{2}\right)\cos\left(\frac{3\theta-\theta}{2}\right) - 2\sin 2\theta}{2\sin\left(\frac{3\theta+\theta}{2}\right)\cos\left(\frac{3\theta-\theta}{2}\right) + 2\sin 2\theta}$$

$$= \frac{2\sin 2\theta\cos\theta - 2\sin 2\theta}{2\sin 2\theta\cos\theta + 2\sin 2\theta}$$

$$= \frac{2\sin 2\theta(\cos\theta - 1)}{2\sin 2\theta(\cos\theta + 1)}$$

$$= \frac{\cos\theta - 1}{\cos + 1} = -\frac{-(1-\cos\theta)}{1+\cos\theta}$$

But $\cos\theta = 1 - 2\sin^2 \frac{\theta}{2}$

$\cos\theta = 2\sin^2 \frac{\theta}{2} - 1$

$$-\frac{-(1-\cos\theta)}{1+\cos\theta} = \frac{-(1-(1-2\sin^2 \frac{\theta}{2}))}{1-2\sin^2 \frac{\theta}{2}}$$

$$= \frac{-2\sin^2 \frac{\theta}{2}}{2\cos^2 \frac{\theta}{2}}$$

$$= -\tan^2 \frac{\theta}{2}$$

Example VII (UNEB Question)

Show that $\frac{\sin 3\theta \sin 6\theta + \sin\theta \sin 2\theta}{\sin 3\theta \cos 6\theta + \sin\theta \cos 2\theta} = \tan 5\theta$.

Solution

$$= \frac{\sin 6\theta \sin 3\theta + \sin 2\theta \sin\theta}{\cos 6\theta \sin 3\theta + \cos 2\theta \sin\theta}$$

$\cos A - \cos B = -\sin \frac{A+B}{2} \sin \frac{A-B}{2}$

$\sin \frac{A+B}{2} \sin \frac{A-B}{2} = \frac{-1}{2}(\cos A - \cos B)$

$\sin 6\theta \sin 3\theta = \frac{-1}{2}(\cos A - \cos B)$

$\frac{A+B}{2} = 6\theta$

$A + B = 6\theta$ …………………….. (i)

$\frac{A-B}{2} = 3\theta$

$A - B = 6\theta$ ……………………… (ii)

Solving Eqn (i) and Eqn (ii) simultaneously;

$A = 9\theta$, $B = 3\theta$

$$\sin 6\theta \sin 3\theta = \frac{-1}{2}(\cos 9\theta - \cos 3\theta)$$

$$\Rightarrow \frac{\sin 3\theta \sin 6\theta + \sin\theta \sin 2\theta}{\sin 3\theta \cos 6\theta + \sin\theta \cos 2\theta}$$

$$= \frac{\frac{-1}{2}(\cos 9\theta - \cos 3\theta) + \frac{-1}{2}(\cos 3\theta - \cos\theta)}{\frac{1}{2}(\sin 9\theta - \sin 3\theta) + \frac{1}{2}(\sin 3\theta - \sin\theta)}$$

$$= \frac{\frac{1}{2}(\cos\theta - \cos 9\theta)}{\frac{1}{2}(\sin 9\theta - \sin\theta)}$$

$$\frac{-2\sin 5\theta \sin(-4\theta)}{2\cos 5\theta \sin 4\theta}$$

$$\Rightarrow \frac{2\sin 5\theta \sin(4\theta)}{2\cos 5\theta \sin 4\theta} = \tan 5\theta$$

Example VIII (UNEB Question)

If *A*, *B*, *C* are angles of the triangle, show that
cos 2*A* + cos 2*B* + cos 2*C* = -1 − 4 cos*A* cos *B* cos *C*.

Solution

cos 2A + cos 2B + cos 2C

2cos(A + B) cos(A – B) + 2cos^{2}C – 1

= -1 + 2cos(A + B) cos (A – B) + 2cos^{2}C

A + B + C = 180

A + B = (180 – C)

cos(A + B) = cos(180 – C)

cos (A + B) = cos 180 cos C + sin 180 sin C

= -cos C

$\Rightarrow$ -1 + 2cos(A + B) cos (A – B) + 2cos^{2}A

= -1 – 2cos C cos(A – B) + 2cos^{2}C

= -1 – 2cos C[cos(A – B) − cos C]

= -1 – 2cos C[cos(A – B) – cos C]

cos *C* = -cos(*A* + *B*)

= -1 – 2cos C[cos(A – B) + cos (A + B)

= -1 – 4cos A cos B cos C.

cos 2*A* + cos 2*B* + cos 2*C* = -1 − 4 cos*A* cos *B* cos *C*.

Example IX (UNEB Question)

Use the factor formula to show that

$$\frac{\sin(A+2B)+\sin A}{\cos(A+2B)+\cos A} = \tan(A+B)$$

Solution

$$\frac{\sin(A+2B)+\sin A}{\cos(A+2B)+\cos A}$$

$$= \frac{2\sin(A+B)\cos B}{2\cos(A+B)\cos B}$$

$$= \frac{\sin(A+B)}{\cos(A+B)}$$

= tan(A + B)

$$\Rightarrow \frac{\sin(A+2B)+\sin A}{\cos(A+2B)+\cos A} = \tan(A+B)$$

UNEB 2008

(i) Prove that $\frac{\cos A + \cos B}{\sin A + \sin B} = \cot\frac{A+B}{2}$

(ii) Deduce that $\frac{\cos A + \cos B}{\sin A + \sin B} = \tan\frac{C}{2}$ where A, B and C are

solution

(i) $$\frac{\cos A + \cos B}{\sin A + \sin B} = \frac{2\cos\frac{A+B}{2}\cos\frac{A+B}{2}}{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}$$

$$= \frac{2\cos\frac{A+B}{2}}{2\sin\frac{A+B}{2}}$$

$$= \cot\frac{A+B}{2}$$

(ii) A + B + C = 180^0

A + B =180 – C

$$\frac{A+B}{2} = 90 - \frac{C}{2}$$

$$\cot\frac{A+B}{2} = \frac{\cos\left(90-\frac{c}{2}\right)}{\sin\left(90-\frac{c}{2}\right)}$$

$$= \frac{\cos 90\cos\frac{c}{2} + \sin 90\sin\frac{c}{2}}{\sin 90\cos\frac{c}{2} - \cos 90\sin\frac{c}{2}}$$

$$= \frac{\sin\frac{c}{2}}{\cos\frac{c}{2}}$$

$$= \tan\frac{c}{2}$$

$$\Rightarrow \frac{\cos A + \cos B}{\sin A + \sin B} = \tan\frac{C}{2}$$

Example X (UNEB Question)

Solve sin*x* – sin 4*x* = sin2*x* – sin3*x* for $-\pi \le x \le \pi$

Solution

sin *x* – sin 4*x* = sin 2*x* – sin 3*x*

sin 3*x* + sin *x* = sin 4*x* + sin 2*x*

$$2\sin\left(\tfrac{3x+x}{2}\right)\cos\left(\tfrac{3x-x}{2}\right) = 2\sin\left(\tfrac{4x+2x}{2}\right)\cos\left(\tfrac{4x-2x}{2}\right)$$

$$2\sin(2x)\cos x = 2\sin 3x\cos x$$

$$\sin 2x\cos x - \sin 3x\cos x = 0$$

$$\cos x(\sin 2x - \sin 3x) = 0$$

Taking cos = 0

$$x = \cos^{-1}(0)$$

$$x = \frac{-\pi}{2}, \frac{\pi}{2}$$

Taking sin 2*x* – sin 3*x* = 0

sin 3x – sin 2*x* = 0

$$2\cos\left(\frac{3x+2x}{2}\right)\sin\left(\frac{3x-2x}{2}\right) = 0$$

$$\cos\left(\tfrac{5}{2}x\right)\sin\left(\tfrac{1}{2}x\right) = 0$$

Either $\cos\left(\frac{5}{2}x\right) = 0$

$$\frac{5}{2}x = \cos^{-1}(0)$$

$$\frac{5}{2}x = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi$$

$$x = \pm\frac{\pi}{5}, \ \pm\frac{3\pi}{5}$$

Or $\quad \text{Sin}\left(\tfrac{1}{2}x\right) = 0$

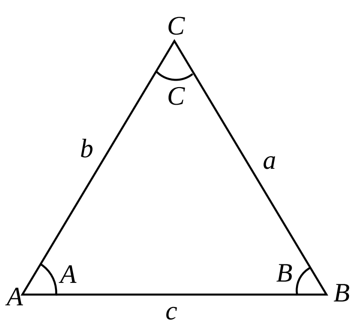
$$\frac{1}{2}x = \sin^{-1}(0) = 0, \pm\pi$$

$$x = 0^0$$

$\Rightarrow x = 0, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{-\pi}{5}, \frac{\pi}{5}, \frac{3\pi}{5}, \frac{-3\pi}{5}$ are the solutions to the equation

Relationship between sides of a triangle

In a triangle ABC with angles A, B and C, we denote the side opposite these angles by their corresponding small letters a, b, and c respectively as shown in the figure below.



The sine rule

Let O be the centre of the circle circumscribing the triangle ABC with radius, R.

Figure I

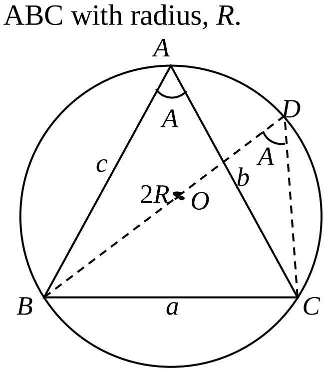


Figure II

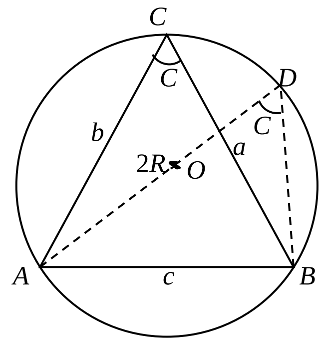
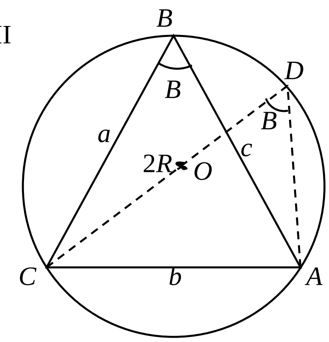


Figure III



From figure I, $\angle$BCD = 90°

Since this angle is subtended by the diameter,

$$\Rightarrow \sin A = \frac{a}{2R} \qquad \text{from figure I.}$$

$$\Rightarrow 2R = \frac{a}{\sin A} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

From figure II;

$$\sin C = \frac{c}{2R}$$

$$2R = \frac{c}{\sin C} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

From figure III;

$$\sin B = \frac{b}{2R}$$

$$2R = \frac{b}{\sin B} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

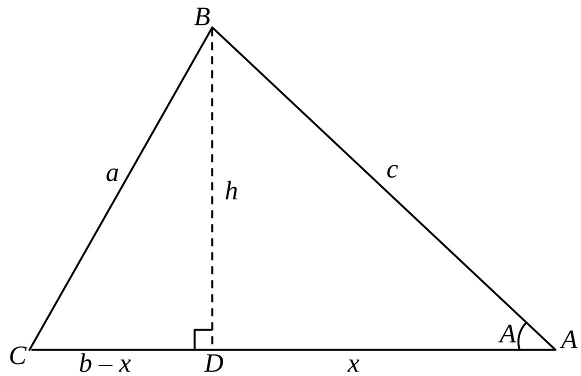
Equating equations (i), (ii), and (iii)

$$\Rightarrow \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

This is the sine rule

The Cosine rule

Consider a triangle ABC. Assume angle A is acute.



Considering the right-angled triangle BDA,

$x^2 + h^2 = c^2$ ……………………..…. (i)

from the right-angled triangle BCD,

$a^2 = (b - x)^2 + h^2$

$a^2 = b^2 - 2bx + x^2 + h^2$ ………….. (ii)

From Eqn (i);

$h^2 = c^2 - x^2$ ……………………..…. (iii)

Substituting Eqn (iii) in Eqn (ii)

$a^2 = b^2 - 2bx + x^2 + c^2 - x^2$

$a^2 = b^2 - 2bx + c^2$ ……………….. (iv)

From triangle ABD;

$$\cos A = \frac{x}{c}$$

$x = c \cos A$ …………..………….. (v)

Substituting Eqn (v) into (iv)

$\Rightarrow a^2 = b^2 - 2bc \cos A + c^2$

$a^2 = b^2 + c^2 - 2bc \cos A$

Application of cosine and sine rules

Example I

Prove that in a triangle ABC, $\dfrac{a^2 - b^2}{c^2} = \dfrac{\sin(A-B)}{\sin(A+B)}$

Solution

From the sine rule; $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

$a = 2R \sin A$, $b = 2R \sin B$ and $c = 2R \sin C$

$$\frac{4R^2 \sin^2 A - 4R^2 \sin^2 B}{4R^2 \sin^2 C}$$

$$\frac{\sin^2 A - \sin^2 B}{\sin^2 C}$$

$$\frac{(\sin A + \sin B)(\sin(A - \sin B)}{\sin C \sin C}$$

$A + B + C = 180$

$C = 180 - (A + B)$

$\sin C = \sin(180 - (A + B))$

$\sin C = \sin 180 \cos (A+B) - \cos 180 \sin (A+B)$

$= \sin (A + B)$

$$\frac{(\sin A + \sin B)(\sin A - \sin B)}{[\sin(A+B)]^2}$$

$$= \frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}\cdot 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{\left[2\sin\frac{A+B}{2}\cos\frac{A+B}{2}\right]\left[\sin(A+B)\right]}$$

$$= \frac{2\sin\frac{A-B}{2}\cos\frac{A-B}{2}}{\sin(A+B)}$$

$$= \frac{\sin(A-B)}{\sin(A+B)}$$

Example II

Prove that in any triangle ABC, $\dfrac{a^2 + b^2 - c^2}{a^2 - b^2 + c^2} = \tan B \cot C$

Solution

From the cosine rule;

$a^2 = b^2 + c^2 - 2bc \cos A$ ……………… (i)

$b^2 = a^2 + c^2 - 2ac \cos B$ ……………… (ii)

$c^2 = a^2 + b^2 - 2ab \cos C$ ……………… (iii)

From Eqn (i);

$2ac \cos B = a^2 + c^2 - b^2$

From Eqn (iii);

$2ab \cos C = a^2 + b^2 - c^2$

$$\frac{a^2 + b^2 - c^2}{a^2 - b^2 + c^2} = \frac{2ab\cos C}{2ac\cos B}$$

$$= \frac{b\cos C}{c\cos B}$$

But from the sine rule;

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$a = 2R \sin A$, $b = 2R \sin B$, and $c = 2R \sin C$

$$\Rightarrow \frac{b\cos C}{c\cos B} = \frac{2R\sin B\cos C}{2R\sin C\cos B}$$

$$= \frac{\sin B}{\cos B} \times \frac{\cos C}{\sin C}$$

$= \tan B \times \cot C$

$$\Rightarrow \frac{a^2 + b^2 + c^2}{a^2 - b^2 + c^2} = \tan B \cot C$$

Example III

Prove that in any triangle ABC, $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$

Solution

$\frac{b-c}{b+c}\cot\frac{A}{2}$

From the sine rule; $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

$a = 2R \sin A$, $b = 2R \sin B$, $c = 2R \sin C$

$$\Rightarrow \frac{2R\sin B - 2R\sin C}{2R\sin B + 2R\sin C}\cot\frac{A}{2} = \frac{\sin B - \sin C}{\sin B + \sin C}\cot\frac{A}{2}$$

$$= \frac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}\cos\frac{A}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}\sin\frac{A}{2}}$$

But $A + B + C = 180$

$A = 180 - (B + C)$

$$\frac{A}{2} = 90 - \frac{B+C}{2}$$

$\cos\frac{A}{2} = \cos(90 - \frac{B+C}{2})$

$= \cos 90 \cos\frac{B+C}{2} + \sin 90 \sin\frac{B+C}{2}$

$= \sin\frac{B+C}{2}$

$\sin\frac{A}{2} = \sin(90 - \frac{B+C}{2})$

$= \sin 90 \cos\frac{B+C}{2} - \cos 90 \sin\frac{B+C}{2}$

$= \cos\frac{B+C}{2}$

$$\Rightarrow \frac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}\sin\frac{B+C}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}\cos\frac{B+C}{2}} = \tan\frac{B-C}{2}$$

$$\Rightarrow \tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$$

Example IV

Prove that in any triangle *ABC*,

$$\frac{bc}{ab+ac} = \frac{\text{cosec }(B+C)}{\text{cosec}B + \text{cosec }C}$$

Solution

From the sine rule, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

$a = 2R\sin A$, $b = 2R\sin B$, and $c = 2R\sin C$

$$\frac{bc}{ab+ac} = \frac{2R\sin B \cdot 2R\sin C}{(2R\sin A)(2R\sin B) + (2R\sin A)(2R\sin C)}$$

$$= \frac{4R^2\sin B\sin C}{4R^2\sin A\sin B + 4R^2\sin A\sin C}$$

$$= \frac{\sin B\sin C}{\sin A\sin B + \sin A\sin C}$$

$$= \frac{\frac{\sin B\sin C}{\sin B\sin C}}{\frac{\sin A\sin B}{\sin B\sin C} + \frac{\sin A\sin C}{\sin B\sin C}}$$

$$= \frac{1}{\frac{\sin A}{\sin C} + \frac{\sin A}{\sin B}}$$

$$= \frac{1}{\sin A(\frac{1}{\sin C} + \frac{1}{\sin B})}$$

$$= \frac{1}{\sin A(\text{cosec}B + \text{cosec }C)}$$

$$= \frac{1}{\sin A} \times \frac{1}{(\text{cosec }B + \text{cosec }C)}$$

From triangle ABC;

$A + B + C = 180$

$A = 180 - (B + C)$

$\sin A = \sin(180 - (B + C))$

$= \sin 180 \cos B + C - \cos 180 \sin(B + C)$

$= \sin (B + C)$

$$\Rightarrow \frac{1}{\sin(B+C)} \times \frac{1}{(\text{cosec }B + \text{cosec }C)} = \frac{\text{cosec }(B+C)}{\text{cosec}B + \text{cosec }C}$$

$$\Rightarrow \frac{bc}{ab+ac} = \frac{\text{cosec }(B+C)}{\text{cosec}B + \text{cosec }C}$$

Area of a triangle

Let D denote the area of a triangle ABC, then

$D = \frac{1}{2}bc\sin A$

$\Rightarrow D = \frac{1}{2}bc \cdot 2\sin\frac{A}{2}\cos\frac{A}{2}$

$D = bc\sin\frac{A}{2}\cos\frac{A}{2}$

$S = \frac{a+b+c}{2}$

Where S is the semi perimeter.

From the cosine rule, $a^2 = b^2 + c^2 - 2bc\cos A$

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$\cos A = 1 - 2\sin^2\frac{A}{2}$

$$\sin^2\frac{A}{2} = \frac{1-\cos A}{2}$$

$$\sin^2\frac{A}{2} = \frac{1}{2}\left(1 - \frac{b^2+c^2-a^2}{2bc}\right)$$

$$\sin^2\frac{A}{2} = \frac{1}{2}\left(\frac{a^2-b^2-c^2+2bc}{2bc}\right)$$

$$\sin^2\frac{A}{2} = = \frac{1}{2}\left(\frac{a^2-(b-c)^2}{2bc}\right)$$

$$\sin\frac{A}{2} = \sqrt{\frac{(a+b-c)(a+c-b)}{4bc}}$$

$a + b + c = 2s$

$a + b - c = a + b + c - 2c$

$= 2s - 2c$

$= 2(s - c)$

$a + c - b = a + b + c - 2b$

$= 2s - 2b$

$= 2(s - b)$

$$\sin\frac{A}{2} = \sqrt{\frac{2(s-c)2(s-b)}{4bc}}$$

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

From the cosine rule, $a^2 = b^2 + c^2 - 2bc\cos A$

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$\cos A = 2\cos^2 \tfrac{A}{2} - 1$$

$$\cos^2 \tfrac{A}{2} = \frac{1}{2}(1 + \cos A)$$

$$\cos^2 \tfrac{A}{2} = \frac{1}{2}\left(\frac{2bc + b^2 + c^2 - a^2}{2bc}\right)$$

$$\cos^2 \tfrac{A}{2} = \frac{1}{2}\left(\frac{(b+c)^2 - a^2}{2bc}\right)$$

$$\cos^2 \tfrac{A}{2} = \frac{(b+c+a)(b+c-a)}{4bc}$$

$$\cos \tfrac{A}{2} = \sqrt{\frac{(b+c+a)(b+c-a)}{4bc}}$$

$a + b + c = 2s$

$b + c - a = a + b + c - 2a$

$= 2s - 2a$

$= 2(s - a)$

$$\cos \tfrac{A}{2} = \sqrt{\frac{2s \cdot 2(s-a)}{4bc}} = \sqrt{\frac{s(s-a)}{bc}}$$

From the area of a triangle D;

$D = bc \sin \tfrac{A}{2} \cos \tfrac{A}{2}$

$$\cos \tfrac{A}{2} = bc\sqrt{\frac{(S-b)(S-c}{bc}} \cdot \sqrt{\frac{S(S-a)}{bc}}$$

$$= bc\frac{\sqrt{S(S-a)(s-b)(S-c)}}{bc}$$

$$= \sqrt{S(S-a)(S-b)(S-c)}$$

The area of a triangle is $\sqrt{S(S-a)(S-b)(S-c)}$

This is called the Heron formula named after the Greek Mathematician Heron

Differentiation and integration of trigonometric functions

Function	**Differentiate**	**Integrate**
Sin x	Cos x	$-\cos x$
cos x	-sinx	sin x
Sin ax	a cos ax	$\frac{-1}{a}\cos ax$
cos ax	$-a \sin ax$	$\frac{1}{a}\sin ax$
sin $3x$	$3\cos 3x$	$\frac{-1}{3}\cos 3x$
cos $3x$	$-3\sin 3x$	$\frac{1}{3}\sin 3x$

Differentiation of trigonometric functions

Example I

Differentiate the following

(a) $\sin 6x$

(b) $-3\cos 5x$

(c) $-4\sin \tfrac{3}{2}x$

(d) $\sin x^2$

(e) $2\sin \tfrac{1}{2}(x + 1)$

Solutions

(a) $y = \sin 6x$

$$\frac{dy}{dx} = 6\cos 6x$$

(b) $-3\cos 5x$

$y = -3\cos 5x$

$$\frac{dy}{dx} = -3[5(-\sin 5x)]$$

$$\frac{dy}{dx} = 15\sin 5x$$

(c) $-4\sin \tfrac{3}{2}x$

$y = -4\sin \tfrac{3}{2}x$

$$\frac{dy}{dx} = -4 \times \frac{3}{2}\cos \tfrac{3x}{2}$$

$$= -6\cos \tfrac{3x}{2}$$

(d) $\sin x^2$

$y = \sin x^2$

$$\frac{dy}{dx} = 2x\cos x^2$$

(e) $2\sin \tfrac{1}{2}(x + 1)$

$y = 2\sin \tfrac{1}{2}(x + 1)$

$$\frac{dy}{dx} = 2 \times \frac{1}{2}\cos \tfrac{1}{2}(x+1)$$

$$\frac{dy}{dx} = \cos \tfrac{1}{2}(x+1)$$

Example II

Differentiate the following

(a) $\sin^2 x$
(b) $4\cos^2 x$
(c) $\cos^3 x$
(d) $2\sin^3 x$
(e) $3\sin^4 2x$
(f) $\sqrt{\sin 2x}$

Solutions

(a) $\sin^2 x$

$y = \sin^2 x$

$$\frac{dy}{dx} = 2\sin x(\cos x)$$

(b) $4\cos^2 x$

$y = 4\cos^2 x$

$$\frac{dy}{dx} = 8\cos x(-\sin x)$$
$$= -8\sin x\cos x$$
$$\frac{dy}{dx} = -8\sin x\cos x$$

(c) $\cos^3 x$

$y = \cos^3 x$

$$\frac{dy}{dx} = 3(\cos^2 x)(-\sin x)$$
$$\frac{dy}{dx} = -3\cos^2 x\sin x$$

(d) $2\sin^3 x$

$y = 2\sin^3 x$

$$\frac{dy}{dx} = 6\sin^2 x(\cos x)$$
$$\frac{dy}{dx} = 6\sin^2 x(\cos x)$$

(e) $3\sin^4 2x$

$y = 3\sin^4 2x$

$$\frac{dy}{dx} = 12\sin^3 2x(2\cos 2x)$$
$$\frac{dy}{dx} = 24\sin^3 2x\cos 2x$$

(f) $\sqrt{\sin 2x}$

$$\frac{dy}{dx} = \frac{1}{2}(\sin 2x)^{\frac{-1}{2}} \cdot 2\cos 2x$$
$$= \frac{\cos 2x}{\sqrt{\sin 2x}}$$

Example II

Differentiate the following

(a) $x\cos x$
(b) $\boldsymbol{x}\sin 2x$
(c) $x^2\sin x$
(d) $\dfrac{x}{\sin x}$
(e) $\dfrac{x^2}{\cos x}$
(f) $\dfrac{\cos 2x}{x}$

Solutions

(a) $y = x\cos x$

From $y = uv$;

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$\frac{dy}{dx} = \boldsymbol{x}(-\sin x) + \cos x$$
$$\frac{dy}{dx} = -x\sin x + \cos x$$

(b) $\boldsymbol{x}\sin 2x$

$y = x\sin 2x$

$$\frac{dy}{dx} = x\ .2\cos 2x + \sin 2x.1$$
$$\frac{dy}{dx} = 2x\cos 2x + \sin 2x$$

(c) $x^2\sin x$

$y = x^2\sin x$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$\frac{dy}{dx} = x^2\cos x + (\sin x)\,2x$$
$$\frac{dy}{dx} = x^2\cos x + 2x\sin x$$

(d) $\dfrac{x}{\sin x}$

$$y = \frac{x}{\sin x}$$
$$y = \frac{u}{v}$$
$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$y = \frac{x}{\sin x}$$
$$\frac{dy}{dx} = \frac{\sin x \cdot 1 - x\cos x}{(\sin x)^2}$$

$$\frac{dy}{dx} = \frac{\sin x - x\cos x}{(\sin x)^2}$$

(e) $\dfrac{x^2}{\cos x}$

$$\boldsymbol{y} = \frac{x^2}{\cos x}$$

From $y = \dfrac{u}{v}$;

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$y = \frac{x^2}{\cos x}$$

$$\frac{dy}{dx} = \frac{\cos x \cdot 2x - x^2(-\sin x)}{(\cos x)^2}$$

$$\frac{dy}{dx} = \frac{2x\cos x + x^2 \sin x}{\cos^2 x^2}$$

(f) $\dfrac{\cos 2x}{x}$

$$y = \frac{\cos 2x}{x}$$

From $y = \dfrac{u}{v}$;

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{x \cdot {}^{-}2\sin 2x - \cos 2x}{x^2}$$

$$\frac{dy}{dx} = \frac{{}^{-}2x\sin 2x - \cos 2x}{x^2}$$

Derivatives of tan x, cot x, sec x, and cosec x

(i) $\dfrac{d}{dx}(\tan x) = \sec^2 x$

$\dfrac{d}{dx}(\sec x) = \sec x \tan x$

$\dfrac{d}{dx}(\cot x) = -\text{cosec}^2\, x)$

$\dfrac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$

Proofs

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\cos x(\cos x) - \sin x(-\sin x)}{(\cos x)^2}$$

$$= \frac{\cos^2 + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$\Rightarrow \frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$$

$$= \frac{\sin x(-\sin x) - \cos x(\cos x)}{(\sin x)^2}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= -\text{cosec}^2\, x$$

$$\frac{d}{dx} = -\text{cosec}^2\, x$$

(iii) $\dfrac{d}{dx}$ **(sec x)**

$$= \frac{d}{dx}\left(\frac{1}{\cos x}\right)$$

$$\frac{dy}{dx} = \frac{\cos x \cdot 0 - 1({}^{-}\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x}$$

$$= \frac{1}{\cos x} \times \frac{\sin x}{\cos x}$$

$$= \sec x \tan x$$

$$\Rightarrow \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\Rightarrow \frac{d}{dx}\text{cosec}\, x = \frac{d}{dx}\left(\frac{1}{\sin x}\right)$$

$$= \frac{\sin x \cdot 0 - 1 \cdot \cos x)}{(\sin x)^2}$$

$$\frac{-\cos x}{\sin^2 x}$$

$$= \frac{-\cos x}{\sin x} \cdot \frac{1}{\sin x}$$

$$= -\cot x \,\text{cosec}\, x$$

Example I

Differentiate the following

(a) tan $2x$

(b) cot $3x$

(c) 2cosec $\frac{1}{2}x$

(d) $-\tan(2x+1)$

(e) $\frac{1}{3}\sec(3x-2)$

(f) $\tan\sqrt{x}$

Solution

(a) $\tan 2x$

$y = \tan 2x$

$\frac{dy}{dx} = 2\sec^2 2x$

(b) $\cot 3x$

$y = \cot 3x$

$\frac{dy}{dx} = 3(-\text{cosec}^2\, 3x)$

$= -3\text{cosec}^2 3x$

(c) $2\text{cosec}\,\tfrac{1}{2}x$

$y = 2\text{cosec}\,\tfrac{1}{2}x$

$\frac{dy}{dx} = 2\cdot\frac{1}{2}\left(-\text{cosec}\,\tfrac{1}{2}x\cot\tfrac{1}{2}x\right)$

$\frac{dy}{dx} = \left(-\text{cosec}\,\tfrac{1}{2}x\cot\tfrac{1}{2}x\right)$

(d) $-\tan(2x+1)$

$\boldsymbol{y =} -\tan(2x+1)$

$\frac{dy}{dx} = -2\sec^2(2x+1)$

(e) $\frac{1}{3}\sec(3x-2)$

$y = \frac{1}{3}\sec(3x-2)$

$\frac{dy}{dx} = \frac{1}{3}\cdot 3\sec(3x-2)\tan(3x-2)$

$\frac{dy}{dx} = \sec(3x-2)\tan(3x-2)$

(f) $\tan\sqrt{x}$

$y = \tan\sqrt{x}$

$\frac{dy}{dx} = \frac{1}{2}x^{\frac{-1}{2}}\sec^2\sqrt{x}$

$\frac{dy}{dx} = \frac{\sec^2\sqrt{x}}{2\sqrt{x}}$

Example II

Differentiate the following:

(a) $x\tan x$

(b) $\sec x\tan x$

(c) $x^2\cot x$

(d) $3x\,\text{cosec}\,x$

(e) $\text{cosec}\,x\cot x$

(f) $\frac{\tan x}{x}$

Solutions

(a) $x\tan x$

$y = x\tan x$

$\frac{dy}{dx} = x\sec^2 x + (\tan x).1$

$\frac{dy}{dx} = x\sec^2 x + \tan x$

(b) $\sec x\tan x$

$y = \sec x\tan x$

$\frac{dy}{dx} = \sec x\sec^2 x + \tan x\cdot(\sec x\tan x)$

$\frac{dy}{dx} = \sec^3 x + \tan^2 x\sec x.$

(d) $3x\,\text{cosec}\,x$

$y = 3x\,\text{cosec}\,x$

$\frac{dy}{dx} = 3x(-\text{cosec}\,x\cot x) + \text{cosec}\cdot 3$

$\frac{dy}{dx} = -3x\,\text{cosec}\,x\cot x + 3\text{cosec}\,x$

(e) $\text{cosec}\,x\cot x$

$y = \text{cosec}\,x\cot x$

$\frac{dy}{dx} = \text{cosec}\,x\cdot -\text{cosec}^2 x + (\cot x)(-\cot x\,\text{cosec}\,x)$

$\frac{dy}{dx} = \text{cosec}^3 x - \cot^2 x\,\text{cosec}\,x$

Example III

Differentiate the following

(a) $\tan^2 x$

(b) $\sec^2 x$

(c) $3\text{cosec}^2 x$

(d) $-\tan^2 2x$

(e) $\frac{1}{2}\cot^2 3x$

(f) $\sqrt{\tan x}$

(g) $-2\text{cosec}^4 x$

Solution

(a) $\tan^2 x$

$y = \tan^2 x$

$$\frac{dy}{dx} = 2\tan x(\sec^2 x)$$
$$\frac{dy}{dx} = 2\sec^2 x \tan x$$

(b) $\sec^2 x$

$$y = \sec^2 x$$
$$\frac{dy}{dx} = 2\sec x(\sec x \tan x)$$
$$\frac{dy}{dx} = 2\sec^2 x \tan x$$

(c) $3\text{cosec}^2 x$

$$y = 3\text{cosec}^2 x$$
$$\frac{dy}{dx} = 3 \times 2\text{cosec } x(-\text{cosec } x \cot x)$$
$$\frac{dy}{dx} = -6\text{cosec}^2 x \cot x$$

(d) $-\tan^2 2x$

$$y = -\tan^2 2x$$
$$\frac{dy}{dx} = -2(\tan 2x)(2\sec^2 2x)$$
$$\frac{dy}{dx} = -4\sec^2 2x \tan 2x$$

(e) $\frac{1}{2}\cot^2 3x$

$$y = \frac{1}{2}\cot^2 3x$$
$$\frac{dy}{dx} = \frac{1}{2} \times 2\cot 3x(-3\text{cosec}^2 3x)$$
$$= -3\text{cosec}^2 3x \cot 3x$$

(f) $\sqrt{\tan x}$

$$\frac{dy}{dx} = \frac{1}{2}(\tan x)^{\frac{-1}{2}} \cdot \sec^2 x$$
$$\frac{dy}{dx} = \frac{\sec^2 x}{2\sqrt{\tan x}}$$

(g) $-2\text{cosec}^4 x$

$$y = -2\text{cosec}^4 x$$
$$\frac{dy}{dx} = -8\text{cosec}^3 x(-\text{cosec } x \cot x)$$
$$\frac{dy}{dx} = 8\text{cosec}^4 x \cot x$$

Integration of Trigonometric functions

Integration is the process of obtaining a function from its derivative

Note: $\int \cos ax \, dx = \frac{1}{a}\sin(ax) + c$

$\int \sin ax \, dx = \frac{-1}{a}\cos(ax) + c$

Example I

Integrate the following

(a) $\cos 3x$
(b) $\sin 3x$
(c) $\cos(3x - 1)$
(d) $\sin(2x + 1)$
(e) $6\cos 4x$

Solution

(a) $\cos 3x$

$$y = \cos 3x$$
$$\int y \, dx = \int \cos 3x \, dx$$
$$= \frac{1}{3}\sin 3x + c$$
$$\int \cos 3x \, dx = \frac{1}{3}\sin 3x + c$$

(b) $\int \sin 3x \, dx = \frac{1}{3}\sin 3x + c$

$$= \frac{-1}{3}\cos 3x + c$$

(c) $\int \cos(3x - 1) \, dx = \frac{1}{3}\sin(3x - 1) + c$

(d) $\int \sin(2x + 1) \, dx = \frac{-1}{2}\cos(2x + 1) + c$

(e) $\int 6\cos 4x \, dx = 6\int \cos 4x \, dx$

$$= 6\left[\frac{1}{4}\sin 4x\right] + c$$
$$= \frac{3}{2}\sin 4x + c$$

Example

Integrate the following

(a) $\sec^2 2x$
(b) $3\sec x \tan x$
(c) $-\text{cosec}^2 \frac{1}{2}x$
(d) $\frac{1}{3}\text{cosec } 3x \cot 3x$
(e) $2\sec^2 x \tan x$

(f) $\dfrac{\sin x}{\cos^2 x}$

(g) $\dfrac{1}{\sin^2 2x}$

(h) $\dfrac{\cos 2x}{\sin^2 2x}$

Solution

Note:

$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\Rightarrow \int \sec^2 x\, dx = \tan x + c$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$
$$\Rightarrow \int \sec x \tan x\, dx = \sec x + c$$
$$\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$$
$$\Rightarrow \int \text{cosec}^2 x\, dx = -(\cot x) + c$$
$$\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$$
$$\Rightarrow \int \text{cosec}\, x \cot x\, dx = -\text{cosec}\, x + c$$

(a) $\int \sec^2 2x\, dx$

Let $u = 2x$

$du = 2dx$

$dx = \dfrac{du}{2}$

$$\int \sec^2 2x\, dx = \int \sec^2 u \cdot \frac{du}{2}$$
$$= \frac{1}{2}\int \sec^2 u\, du$$
$$= \frac{1}{2}\tan u + c$$
$$= \frac{1}{2}\tan(2x) + c$$

(b) $\int 3\sec x \tan x\, dx$

$$= 3\int \sec x \tan x\, dx$$
$$= 3\sec x + c$$

(c) $\int {}^{-}\text{cosec}^2 \tfrac{1}{2}x\, dx$

Let $u = \dfrac{1}{2}x$

$\dfrac{du}{dx} = \dfrac{1}{2}$

$dx = 2\, du$

$$\int -\text{cosec}^2 \tfrac{1}{2}x\, dx = \int -\text{cosec}^2 u(2du)$$
$$= 2\int -\text{cosec}^2 u$$
$$= 2[\cot u] + c$$
$$= 2\cot \tfrac{1}{2}x + c$$

(d) $\int \tfrac{1}{3}\text{cosec}\, 3x \cot 3x\, dx$

Let $u = 3x$

$du = 3\, dx$

$dx = \dfrac{du}{3}$

$$\int \tfrac{1}{3}\text{cosec}\, 3x \cot 3x\, dx = \frac{1}{3}\int \text{cosec}\, 3x \cot 3x\, dx$$
$$= \frac{1}{3}\int \text{cosec}\, u \cot u \cdot \frac{du}{3}$$
$$= \frac{1}{9}\int \text{cosec}\, u \cot u\, du$$
$$= \frac{1}{9}(-\text{cosec}\, u) + c$$
$$= \frac{-1}{9}\text{cosec}\, 3x + c$$

(e) $\int 2\sec^2 x \tan x\, dx$

Consider $\dfrac{d}{dx}(\sec^2 x) = 2\sec x(\sec x \tan x)$

$= 2\sec^2 x \tan x$

$$\Rightarrow \int 2\sec^2 x \tan x\, dx = \sec^2 x + c$$

(f) $\int \dfrac{\sin x}{\cos^2 x}dx$

$$= \int \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}dx$$
$$= \int \tan x \sec x\, dx$$
$$= \sec x + c$$

(g) $\dfrac{1}{\sin^2 2x} = \int \text{cosec}^2 2x\, dx$

Let $u = 2x$

$du = 2\, dx$

$dx = \dfrac{du}{2}$

$$\int \operatorname{cosec}^2 2x\ dx = \int \operatorname{cosec}^2 u \cdot \frac{du}{2}$$
$$= \frac{1}{2}\int \operatorname{cosec}^2 u$$
$$= \frac{-1}{2}\cot u + c$$
$$= \frac{-1}{2}\cot 2x + c$$

(h) $\dfrac{\cos 2x}{\sin^2 2x}$

$$= \int \frac{\cos 2x}{\sin 2x}\cdot\frac{1}{\sin 2x}\ dx$$
$$= \int \cot 2x \operatorname{cosec}\ 2x\ dx$$

Let $u = 2x$

$du = 2\ dx$

$$dx = \frac{du}{2}$$

$$\int \cot u \operatorname{cosec}\ u \cdot \frac{du}{2}$$
$$= \frac{1}{2}\int \operatorname{cosec} u \cot u\ \ du$$
$$= \frac{1}{2}(-\operatorname{cosec} u) + c$$
$$= \frac{-1}{2}\operatorname{cosec} 2x + c$$

Example III

Evaluate the following

(a) $\int_0^{\frac{\pi}{2}} \sin 2x\ \ dx$

(b) $\int_{\frac{-\pi}{3}}^{\frac{\pi}{6}} \sec^2 x\ \ dx$

(c) $\int_0^{\pi} \sin^2 x\ dx$

Solution

(a) $\int_0^{\frac{\pi}{2}} \sin 2x\ \ dx$

$$= \left[\frac{-1}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$$
$$= \frac{-1}{2}\cos 2\left(\tfrac{\pi}{2}\right) - \frac{-1}{2}\cos 0$$
$$= \frac{-1}{2}(-1) + \frac{1}{2}$$
$$= 1$$

(b) $\int_{\frac{-\pi}{3}}^{\frac{\pi}{6}} \sec^2 x\ \ dx$

$$= [\tan x]_{\frac{-\pi}{3}}^{\frac{\pi}{6}}$$
$$= \tan\left(\tfrac{\pi}{6}\right) - \tan\left(\tfrac{-\pi}{3}\right)$$
$$= \frac{1}{\sqrt{3}} - \left(-\sqrt{3}\right)$$
$$= \frac{1}{\sqrt{3}} + \sqrt{3}$$
$$= \frac{\sqrt{3}}{3} + \sqrt{3} = \frac{4\sqrt{3}}{3}$$

(c) $\int_0^{\pi} \sin^2 x\ dx$

From $\cos 2x = 1 - 2\sin^2 x$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\int_0^{\pi} \sin^2 x\ dx = \int_0^{\pi} \frac{1}{2}(1 - \cos 2x)\, dx$$
$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi}$$
$$= \frac{1}{2}\left[\left(\pi - \frac{1}{2}\sin 2\pi\right) - (0 - 0)\right]$$
$$= \frac{1}{2}[\pi]$$
$$= \frac{1}{2}\pi$$

Example

A particle moves in a straight line such that its velocity in m/s after passing through a fixed point O is $3\cos t - 2\sin t$. Find:

(a) Its distance from O after $\frac{1}{2}\pi$ s

(b) Its acceleration after π s

(c) The time when its velocity is first zero.

Solution

$V = 3\cos t - 2\sin t$

$$\frac{dS}{dt} = 3\cos t - 2\sin t$$

$dS = (3\cos t - 2\sin t)\ dt$

$S = 3\sin t + 2\cos t + c$

When $t = 0$, $S = 0$

$0 = 3\sin(0) + 2\cos(0) + c$

$-2 = c$

$\Rightarrow S = 3\sin t + 2\cos t - 2.$

When $t = \frac{1}{2}\pi$,

$S = 3\sin\frac{\pi}{2} + 2\cos\frac{\pi}{2} - 2$
$S = 3 - 2$
$S = 1$ m

$V = 3\cos t - 2\sin t$

$a = \frac{dV}{dt} = -3\sin t - 2\cos t$

$a = \left.\frac{dV}{dt}\right|_{\pi} = -3\sin\pi - 2\cos\pi$

$= 2$ m/s^2

$\Rightarrow a = 2$ m/s^2

$V = 3\cos t - 2\sin t$

$3\cos t - 2\sin t = 0.$

$R = \cos(t + \alpha) = 0$

$R = \sqrt{3^2 + 2^2} = \sqrt{13}$

$\sqrt{13}\cos(t + \alpha) = 0$

$\sqrt{13}\cos(t + 33.7) = 0$

$\cos(t + 33.7) = 0$

$t + 33.7 = \cos^{-1}0$

$t + 33.7 = 90$

$t = 56.3$

$t = \frac{56.3\pi}{180}$

$t = 0.983$ s

Revision Exercise

1. Solve the following for all values of x from 0° to 360°.

   (a) $\sin x = \frac{1}{2}$
   (b) $\cos x = \frac{-1}{2}$
   (c) $\tan x = 1$
   (d) $\tan x = -1$
   (e) $\sin x = \frac{-\sqrt{3}}{2}$
   (f) $\cos x = \frac{1}{\sqrt{2}}$

2. Solve the following equations for values of x from -180° to 180°

   (a) $\sin x = \frac{-1}{2}$
   (b) $\cos x = \frac{1}{2}$
   (c) $\sin x = \frac{\sqrt{3}}{2}$
   (d) $\tan x = \sqrt{3}$
   (e) $\cos x = \frac{-1}{\sqrt{2}}$
   (f) $\cos x = \frac{-\sqrt{3}}{2}$

3. Solve the following equations for all values of x from 0° to 360°

   (a) $\sin x = \frac{-1}{2}$
   (b) $\cos x = -0.7$
   (c) $\tan x$ -0.75
   (d) $\cos^2 x = \frac{1}{4}$
   (e) $\sin x = 2\cos x$
   (f) $2\sin x - 3\cos x = 0$
   (g) $\sin 2x = \frac{-\sqrt{3}}{2}$
   (h) $\cos 2x = \frac{1}{2}$
   (i) $\sin(x + 20) = \frac{-\sqrt{3}}{2}$
   (j) $\tan(x - 30) = 1$
   (k) $3(\cos x - 1) = -1$
   (l) $\sin x\,(1 - 2\cos x) = 0$
   (m) $\cos x(2\sin x + \cos x) = 0$
   (n) $2\sin x\cos x + \sin x = 0$
   (o) $4\sin x\cos x = 3\cos x$
   (p) $4\cos^2 x + \cos x = 0$
   (q) $\tan x = 4\sin x$
   (r) $(2\sin x - 1)(\sin x + 1) = 0$
   (s) $2\sin^2 x - \sin x - 1 = 0$
   (t) $2\tan^2 x - \tan x - 6 = 0$
   (u) $2\tan x - \frac{1}{\tan x} = 1$

Solve the following equations for all values of x from -180° to 180°

1. $\cos^2 x = \frac{3}{4}$
2. $\sin 2x = 2\cos 2x$
3. $\cos(x - 20) = \frac{-1}{\sqrt{2}}$
4. $\cos x(\sin x - 1) = 0$
5. $3\sin^2 x = 2\sin x\cos x$
6. $2\cos^2 x - 5\cos x + 2 = 0$

7. Factorise the expression $6\sin\theta\cos\theta + 3\cos\theta + 4\sin\theta + 2$. Hence solve $6\sin\theta\cos\theta + 3\cos\theta + 4\sin\theta + 2 = 0$ for $-180° \leq 180°$

8. Factorise the equation $3\sin\theta\cos\theta - 3\sin\theta + 2\cos\theta - 2$. Hence solve $3\sin\theta\cos\theta - 3\sin\theta + 2\cos\theta = 2$.

9. Without using tables or calculator, find the values of:
   (a) sec 45° (g) cosec 330°
   (b) cot 45° (h) sec 240°
   (c) cosec 30 (i) cot -135°
   (d) sec 60° (j) sec -60°
   (e) cosec 135° (k) sec(-120°)
   (f) sec 120° (l) cosec 315°

10. Simplify the following expression:
   (a) $\sqrt{(1-\sin A)(1+\sin A)}$
   (b) $\operatorname{cosec}\theta\tan\theta$
   (c) $\frac{1}{\sin^2\theta}+\frac{1}{\cos^2\theta}$
   (d) $\cot\theta\sqrt{1-\cos^2\theta}$

11. Prove the following identities
   (a) $\sin\theta\tan\theta + \cos\theta = \sec\theta$
   (b) $\operatorname{cosec}\theta - \sin\theta = \cot\theta\cos\theta$
   (c) $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2\theta = 2$
   (d) $(\sin\theta + \operatorname{cosec}\theta)^2 = \sin^2\theta + \cot^2\theta + 3\theta$
   (e) $\cot^4\theta + \cot^2\theta = \operatorname{cosec}^4\theta - \operatorname{cosec}^2\theta$
   (f) $\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \operatorname{cosec}\theta - \cot\theta$
   (g) $\frac{\sin\theta}{1+\cos\theta}+\frac{1+\cos\theta}{\sin\theta} = 2\operatorname{cosec}\theta$
   (h) $\frac{\operatorname{cosec}\theta}{\cos\theta+\tan\theta} = \cot\theta$
   (i) $\frac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$
   (j) $\frac{\sin\theta}{1-\cos\theta}+\frac{\sin\theta}{1+\cos\theta} = 2\operatorname{cosec}\theta$
   (k) $\cos^4 x - \sin^4 x = \cos^2 x$
   (l) $\cos A + \cos(B + C) = 0$
   (m) $\frac{\cos^2\theta}{1+\cot^2\theta} = 2\cos\theta$

12. Prove the following identities:
   (a) $2\operatorname{cosec} 2\theta = \operatorname{cosec}\theta\sec\theta$
   (b) $\tan A + \cot A = 2\text{coesec } 2A$
   (c) $\frac{1+\tan^2 A}{2-\tan^2 A} = \sec 2A$
   (d) $\cot 2A = \operatorname{cosec} 2A - \tan A$
   (e) $\frac{\sin 2\theta}{1-\cos 2\theta} = \cot\theta$
   (f) $\tan\theta - \cot\theta = -2\cot 2\theta$

13. Prove the following identities:
   (a) $\frac{1+\cos 2\theta}{1+\cos 2\theta} = \tan^2\theta$
   (b) $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$
   (c) $\frac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta} = \tan\theta$

14. Eliminate θ from each of the following pairs of relationships
   (a) $x = \sin\theta$, $y = \cos\theta$
   (b) $x = 3\sin\theta$, $y = \operatorname{cosec}\theta$
   (c) $5x = \sin\theta$, $y = 2\cos\theta$
   (d) $x = 3 + \sin\theta$, $y = \cos\theta$
   (e) $x = 2 + \sin$, $\cos\theta = 1 + y$.

15. Solve the following equations for all values of θ from -180° to 180°.
   (a) $4 - \sin\theta = 4\cos^2\theta$
   (b) $\sin^2\theta + \cos\theta + 1 = 0$
   (c) $5 - 5\cos\theta = 3\sin^2\theta$
   (d) $8\tan\theta = 3\cos\theta$
   (e) $\sin^2\theta + 5\cos^2\theta = 0$
   (f) $1 - \cos^2\theta = -2\sin\theta\cos\theta$

16. Solve the following equations from 0° to 360°
   (a) $\sec\theta = 2$
   (b) $\cot 2\theta = \frac{-2}{5}$
   (c) $3\cot\theta = \tan\theta$
   (d) $2\sin\theta = -3\cot\theta$
   (e) $2\sec^2\theta - 3 + \tan\theta = 0$

17. If $A + B + C = 180°$, prove that
$$\cos 2A + \cos 2B + \cos 2C = 1 - \cos A\cos B\cos C$$

18. Prove that sin 3A = 4sinAsin(60 + A)sin(60 – A)

19. Show that in a triangle ABC, if $2S = a + b + c$, then
$$1 - \tan\tfrac{A}{2} + \tan\tfrac{B}{2} = \frac{C}{A}$$

20. Prove that in any triangle ABC,
$$(a + b + c)(\tan\tfrac{A}{2} + \tan\tfrac{B}{2}) = 2c\cot\tfrac{C}{2}.$$

21. Prove that in any triangle
$$ABC,\ \frac{a+b-c}{a+b+c} = \tan\tfrac{A}{2}\tan\tfrac{B}{2}$$

22. From a point A, a light wind due to north of A has an elevation α from a point B, due west of A. The angle

of elevation is β. Prove that the angle of elevation from the midpoint of AB. is $\tan^{-1}\left(\dfrac{2}{\sqrt{3\cot^2\alpha+\cot^2\beta}}\right)$

23. Solve: $4\cos\alpha - 3\sin\alpha = 2$
24. Solve the equation $15\cos 2\theta + 20\sin 2\theta + 7 = 0$
25. Find all the possible values of x that satisfy $\tan^{-1}3x+\tan^{-1}x=\dfrac{\pi}{4}$
26. Prove that $\dfrac{\sin 8\theta\cos\theta-\sin 6\theta\cos 3\theta}{\cos 2\theta\cos\theta-\sin 3\theta\sin 4\theta}=\tan 2\theta$
27. Solve the equation $2\cos^2(x-\frac{\pi}{2})-3\cos(x-\frac{\pi}{2}) = 0$ for $0 \le x \le 2\pi$.
28. Solve $\cos^4 x + \sin^4 x = \dfrac{7}{8}$ for $0 \le x \le \dfrac{\pi}{2}$.
29. Find the value of x for $3\cos^2 x - 8\cos x + 4 = 0$
30. Show that $\left(\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}\right)^2=\dfrac{1-\cos\theta}{1+\cos\theta}$
31. Prove that $\dfrac{\tan\theta}{1-\cot\theta}+\dfrac{\cot\theta}{1-\tan\theta}=1+\tan\theta+\cot\theta$
32. Solve the equation $\cos x - \cos 4x = \cos 2x - \cos 3x$ for $-\pi \le x \le \pi$.
33. Given that $y = 4\cos x - 6\sin x$. Express y in the form $R\cos(x+\alpha)$, where R is a constant. Find the maximum and minimum value of y.
34. Express $(45° + x)$ in terms of $\tan x$. Hence or otherwise express tam 75° in the form $a + b\sqrt{3}$.
35. Given $\sin x = \dfrac{-4}{5}$, where $180° \le x \le 270$, find without using tables or calculator the value of $\tan 3x$.
36. Show that:
    (a) $\tan^{-1}\dfrac{1}{3}+\sin^{-1}\dfrac{1}{\sqrt{5}}=\dfrac{\pi}{4}$
    (b) $2\tan^{-1}\dfrac{1}{3}+\tan^{-1}\dfrac{1}{7}=\dfrac{\pi}{4}$
    (c) $\cos^{-1}x + \sin^{-1}x = \dfrac{\pi}{2}$
37. Solve the equation
    (a) $\tan^{-1}(2x+1) + \tan^{-1}(2x-1) = \tan^{-1}2$
    (b) $\tan^{-1}(1+x) + \tan^{-1}(1-x) = 32$
    (c) $\cos^{-1}x + \cos^{-1}x\sqrt{8} = \dfrac{\pi}{2}$
    (d) $2\sin\dfrac{x}{2}+\sin^{-1}x\sqrt{2}=\dfrac{\pi}{2}$
38. Without using tables or calculator, evaluate $\tan^{-1}\dfrac{1}{2}+\tan^{-1}\dfrac{1}{5}+\tan^{-1}\dfrac{1}{5}$

SYMBOLS

± plus or minus
= is equal to
≠ is not equal to
≃ is approximately equal to
< is less than
≤ is less than or equal to
> is greater than
≥ is greater than or equal to
√ positive square root
⇒ implies that
: such that
∴ therefore
∝ is proportional to
|A| determinant of a matrix A
|a| modulus of a vector **a**
∑ the sum of
i,j,k unit vectors in direction of x-,y- and z-axes
$\hat{a}$ unit vector in direction of **a**
a•b scalar product of **a** and **b**
→ tends to
δ or Δ small change or small increment
g acceleration due to gravity (taken as $9{\cdot}8\ m\ s^{-2}$ otherwise stated)
ω angular velocity
ln x the natural logarithm of x, $\log_e x$
μ coefficient of friction
λ angle of friction
$\dot{x}$ first derivative with respect to time
$\ddot{x}$ second derivative with respect to time
$\frac{dy}{dx}$ the derivative of y with respect to x
∫ydx the indefinite integral of y with respect to x
$\int_a^b ydx$ the definite integral of y with respect to x
∞ infinity

The Greek Alphabet

Letters		Name	Letters		Name	Letters		Name
Α	α	alpha	Ι	ι	iota	Ρ	ρ	rho
Β	β	beta	Κ	κ	kappa	Σ	σ	sigma
Γ	γ	gamma	Λ	λ	lambda	Τ	τ	tau
Δ	δ	delta	Μ	μ	mu	Υ	υ	upsilon
Ε	ε	epsilon	Ν	ν	nu	Φ	φ	phi
Ζ	ζ	zeta	Ξ	ξ	xi	Χ	χ	chi
Η	η	eta	Ο	ο	omicron	Ψ	ψ	psi
Θ	θ	theta	Π	π	pi	Ω	ω	omega

MECHANICS

Mechanics is the mathematical study of forces and effects of forces. A force may be completely specified by stating its magnitude, its direction and either its point of application or its line of action. Thus force is a vector quantity, associated with a particular point or line in space. Hence vectors will also be considered. The effect of a force is to set a body in motion or to change the speed and direction of its motion. If forces act on a stationary body, then the body is said to be in equilibrium under the action of the forces.

Mechanics is divided into three branches:

1. **Kinematics**: This is the study of motion of a particle or body without reference to the forces involved. It is the study of displacement, velocity and acceleration.

2. **Statics**: This is **the** study of forces that act on a body at rest.

3. **Dynamics**: This is the study of forces that act on a body in motion.

In mechanics a **body** is an object to which a force can be applied.
A **rigid body** is a body whose shape is unaltered by any force applied on it.
A **particle** is a body whose dimensions, except mass, are negligible.
A **lamina** is a flat body having area but negligible thickness.
A **hollow body** is a three-dimensional shell having negligible thickness.

Drawing force diagrams

Drawing a clear force diagram is an essential first step in the solution of any problem in mechanics which is concerned with action of forces on a body.

The following important points should be remembered when drawing force diagrams

1. Make the diagram large enough to show clearly all the forces acting on the body to enable any necessary geometry and trigonometry to be done.
2. Show only forces acting on the body being considered. A common fault is including forces which the body is applying on its surroundings(including other bodies).
3. Weight always acts on a body unless the body

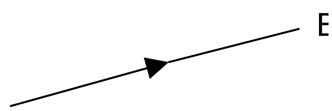
is described as light.

4. Contact with another object or surface gives rise to a normal reaction and sometimes friction.
5. Attachment to an object (by a string, spring, hinge etc) gives rise to a force on the body at the point of attachment.
6. Forces acting on a particle are concurrent. Forces acting on other bodies may act at different points.
7. Check that no forces have been omitted or included more than once.

1. VECTORS

1.1 Introduction

Quantities which have both magnitude and a definite direction in space are called vector quantities. In general, a vector may be described as a number associated with a particular direction in space.



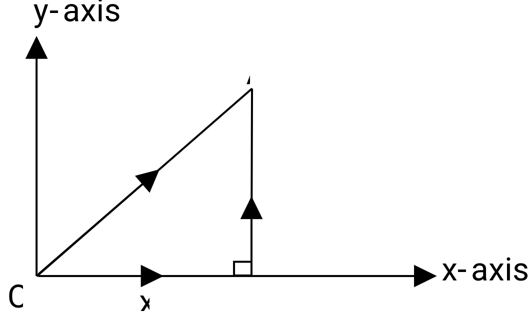
The arrow represents the direction.

The modulus or magnitude of the vector is $|\vec{AB}|$.

We shall consider vectors in one, two or three dimensions.

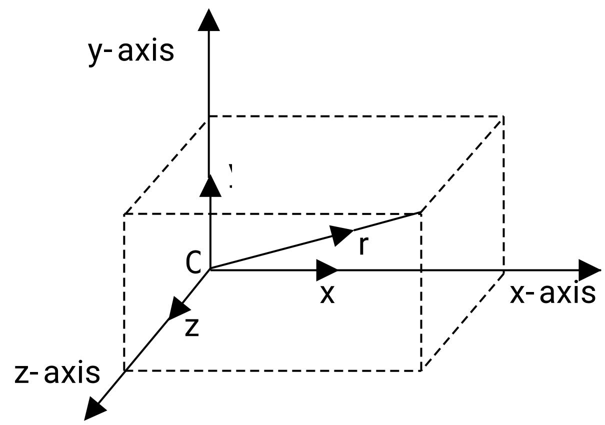
In two dimensions, a vector **a** can be represented by:

$\vec{OA} = \mathbf{a}$, $\mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ or $\mathbf{a} = (x\mathbf{i} + y\mathbf{j})$



In three dimensions, a vector **r** can be represented by:

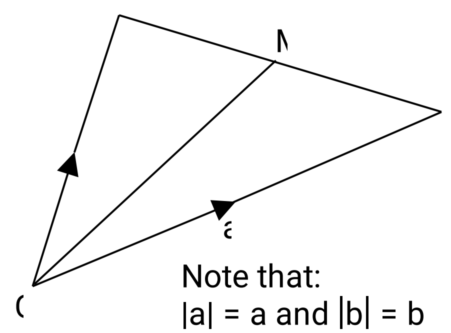
$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ or $\mathbf{r} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$.



1.1.1 Position vectors

Taking a fixed point O as the origin, the position of any point P can be specified by giving the vector OP, which is called the position vector of P.

If A and B are two points with position vectors **a** and **b** respectively, then:



Note that:
$|a| = a$ and $|b| = b$

$\vec{OB} = \mathbf{b}$

$\vec{OA} = \mathbf{a}$

$\vec{AB} = \vec{AO} + \vec{OB}$

$= -\vec{OA} + \vec{OB}$

$= \vec{OB} - \vec{OA}$

$= \mathbf{b} - \mathbf{a}$

Now consider M, the midpoint of $\vec{AB}$

$\vec{AM} = \vec{MB}$

This implies that:

$$\vec{OM} - \vec{OA} = \vec{OB} - \vec{OM}$$
$$2\vec{OM} = \vec{OA} + \vec{OB}$$
$$\vec{OM} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$$

Hence the position vector of the midpoint of $\vec{AB}$ is $\frac{1}{2}(\mathbf{a} + \mathbf{b})$

1.2 Elementary Operations on Vectors

1.2.1 Vector addition

When adding vectors, we add components of vectors in the same direction to obtain the resultant component in the same direction. Given the vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$.

$$\mathbf{a} + \mathbf{b} = (a_1+b_1)\mathbf{i} + (a_2+b_2)\mathbf{j} + (a_3+b_3)\mathbf{k}$$

1.2.2 Vector subtraction

When subtracting vectors, we subtract components of vectors in the same direction to obtain a component in the same direction. Given the vectors: $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $= b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$. Then; $\mathbf{a} - \mathbf{b} = (a_1-b_1)\mathbf{i} + (a_2-b_2)\mathbf{j} + (a_3-b_3)\mathbf{k}$.

1.2.3 Scalar multiplication

When multiplying a vector by a scalar, all components of the vector are multiplied by the scalar. Given the vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and scalars λ and β then:

(i) $\lambda\mathbf{a} = \lambda\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \\ \lambda a_3 \end{pmatrix}$

(ii) $\lambda\mathbf{a} + \beta\mathbf{b} = \lambda\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \beta\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} \lambda a_1+\beta b_1 \\ \lambda a_2+\beta b_2 \\ \lambda a_3+\beta b_3 \end{pmatrix}$

1.2.4 Modulus or magnitude of a vector

This is the length of a vector. Given a vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, the modulus of vector $\mathbf{a}$ is denoted by $|\mathbf{a}|$ and given by $|\mathbf{a}| = \sqrt{a_1^2+a_2^2+a_3^2}$.

If $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, $|\mathbf{b}| = \sqrt{b_1^2+b_2^2}$.

Hence the magnitude of a vector is always positive.

1.2.5 Unit Vectors

A unit vector is a vector of magnitude one unit. If $\mathbf{a}$ is a vector, the unit vector in the direction of vector $\mathbf{a}$ is denoted by $\hat{a}$ and given by $\hat{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$.

If $\mathbf{a} = x\mathbf{i} + y\mathbf{j}$ then: $|\mathbf{a}| = \sqrt{x^2+y^2}$

From $\hat{a} = \frac{\mathbf{a}}{|\mathbf{a}|} \Rightarrow \hat{a} = \frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}$

Alternatively $\mathbf{a}$ and $\hat{a}$ are parallel vectors hence

$\hat{a} = h\mathbf{a}$, where h is a scalar

$$|\hat{a}| = h|\mathbf{a}| \Rightarrow 1 = h|\mathbf{a}| \quad \therefore \quad h = \frac{1}{|\mathbf{a}|}$$

Hence $\hat{a} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}$

Unit vectors are mainly applied when expressing quantities in vector form.

If two vectors $\mathbf{a}$ and $\mathbf{b}$ are in the same direction then $\hat{a} = \hat{b}$, if $\mathbf{a} = x\mathbf{i} + y\mathbf{j}$ and vector $\mathbf{a}$ is in the same direction as an unknown vector $\mathbf{b}$ and $|\mathbf{b}|$ is known then: $\mathbf{b} = |\mathbf{b}|\hat{a} = |\mathbf{b}| \times \frac{(x\mathbf{i}+y\mathbf{j})}{\sqrt{x^2+y^2}}$.

1.2.6 Parallel Vectors

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are said to be parallel if they are in the same direction. Whenever two vectors are parallel, one vector can be obtained by multiplying the other by a scalar. Therefore $\mathbf{b} = h\mathbf{a}$, where is h a scalar.

Example 1

A force of 5 N acts in the direction of the vector $3\mathbf{i} + 4\mathbf{j}$ and a force of 13 N acts in the direction of the vector $5\mathbf{i} - 12\mathbf{j}$. Find the resultant of the forces.

$$|F_1| = 5\text{ N},\ F_1 = h_1(3\mathbf{i}+4\mathbf{j})$$

$|F_1| = |h_1(3i+4j)| \Rightarrow 5 = 5h_1$

$\therefore\ h_1 = 1$

$F_1 = 1(3i+4j) \Rightarrow F_1 = (3i+4j)$ N

$|F_2| = 13$ N, $F_2 = h_2(5i-12j)$

$|F_2| = |h_2(5i-12j)| \Rightarrow 13 = 13h_2 \quad \therefore\ h_2 = 1$

$F_2 = 1(5i-12j) = (5i-12j)$ N

$F = F_1 + F_2 = (3i+4j) + (5i-12j)$

$\Rightarrow F = (8i-8j)$ N

1.3 Scalar Product/Dot Product

The scalar or dot product of vectors **a** and **b** is **a•b** = |**a**||**b**|cos θ, where θ is the angle between vectors **a** and **b**. If **a** and **b** are parallel, that is when θ = 0° then **a•b** = |**a**||**b**|.

If **a** and **b** are perpendicular, that is when θ = 90° then **a• b** = 0.

If **a** = x_1**i** + y_1**j** + z_1**k** and **b** = x_2**i** + y_2**j** + z_2**k**

$\mathbf{a}\bullet\mathbf{b} = (x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k})\bullet(x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k})$

$\mathbf{a}\bullet\mathbf{b} = x_1x_2 + y_1y_2 + z_1z_2$

Since **i•i** = **j•j** = **k•k** = 1 and

i•j = **i•k** = **k•j** = 0

The dot product can be used when finding the angle between two vectors and finding the component of a vector in a given direction.

Example 2

Two particles P and Q are moving with constant velocities 8**i** + **j**−4**k** and 4**i**−2**j**−4**k** respectively. Find the angle between the paths of the particles.

Solution

$v_P = 8\mathbf{i} + \mathbf{j}-4\mathbf{k},\ v_Q = 4\mathbf{i}-2\mathbf{j}-4\mathbf{k}$

$v_P\bullet v_Q = (8i+j-4k)\bullet(4i-2j-4k) = 32-2 + 16$

$= 46$

$v_P\bullet v_Q = |v_P|\,|v_Q| \cos\theta$

$46 = \sqrt{8^2+1^2+(-4)^2}\times\sqrt{4^2+(-2)^2+(-4)^2}\cos\theta$

$46 = 9\times6\cos\theta$

$\theta = 31\cdot6°$

Example 3

(a) Find the values of a for which the vectors a**i**−2**j** + **k** and 2a**i** + a**j**−4**k** are perpendicular.

(b) Given that the vectors 6**i** + 4**j** + 4**k** + s(a**i** + b**j** + 3**k**),-10**i** + 2**j**−3**k** + t(2**j**−c**k**), -8**i** + 10**j**−10**k** + u(-4**i**−4**j**+4**k**) are concurrent (pass through a common point), find the values of a, b and c.

Solution:

(a) $(a\mathbf{i}-2\mathbf{j} + \mathbf{k})\bullet(2a\mathbf{i} + a\mathbf{j}-4\mathbf{k}) = 0$

$2a^2 - 2a - 4 = 0$

$\Rightarrow \left(a-\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - 2 = 0$

$a = \frac{1}{2} \pm \frac{3}{2} \Rightarrow$

either $a = \frac{1}{2} + \frac{3}{2} = 2$ or $a = \frac{1}{2} - \frac{3}{2} = -1$

(b) $6\mathbf{i} + 4\mathbf{j} + 4\mathbf{k} + s(a\mathbf{i} + b\mathbf{j} + 3\mathbf{k}) = -10\mathbf{i} + 2\mathbf{j}-3\mathbf{k} + t(2\mathbf{j}-c\mathbf{k})$

$= -8\mathbf{i} + 10\mathbf{j}-10\mathbf{k} + u(-4\mathbf{i}-4\mathbf{j} + 4\mathbf{k})$

On simplifying:

$(6 + sa)\mathbf{i} + (4 + sb)\mathbf{j} + (4 + 3s)\mathbf{k}$

$= -10\mathbf{i} + (2 + 2t)\mathbf{j}-(3 + ct)\mathbf{k}$

$= (-8-4u)\mathbf{i} + (10-4u)\mathbf{j} + (-10 + 4u)\mathbf{k}$

Comparing coefficients:

For i: $6 + sa = -10 = -8-4u$

$-4u-8 = -10 \Rightarrow u = \frac{1}{2}$

$6 + sa = -10 \Rightarrow sa = -16$................ (i)

For j: $4 + sb = 2 + 2t = 10-4u$

$\Rightarrow 2 + 2t = 10-4\times\frac{1}{2} \Rightarrow t = 3$

$4 + sb = 2 + 2\times3 \Rightarrow sb = 4$ ....(ii)

For **k**: $4 + 3s = -3-ct = -10 + 4u$

$\Rightarrow -3-c\times3 = -10 + 4\times\frac{1}{2} \Rightarrow c = \frac{5}{3}$

$4 + 3s = -10 + 4\times\frac{1}{2} \Rightarrow s = -4$

From equation (i): $-4a = -16 \Rightarrow a = 4$

From equation (ii): $-4b = 4 \Rightarrow b = -1$

$\therefore\ a = 4, b = -1$ and $c = \frac{1}{3}$.

1.3.1 Component of a vector in a given direction:

Given two vectors **b** and **c**, the component of vector **b** in the direction of vector **c**, is given by

$\mathbf{b}\bullet\frac{c}{|c|}$ or $\mathbf{b}\bullet \hat{c}$.

1.4 Vector product/Cross product

The cross product of vectors **a** and **b** is given by $\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{u}$, where θ is the angle between vectors **a** and **b** and $\hat{u}$ is the unit vector perpendicular to both **a** and **b**. Hence the cross product of two vectors is a vector perpendicular to both vectors, that is, if $\mathbf{a}\times\mathbf{b} = \mathbf{c}$ then $\mathbf{a}\bullet\mathbf{c} = \mathbf{b}\bullet\mathbf{c} = 0$.

For vectors in i – j – k notation:

$\mathbf{i}\times\mathbf{j} = (|\mathbf{i}||\mathbf{j}|\cos 90)\,\hat{u}\ ;\ \hat{u} = \mathbf{k}$

$\mathbf{i}\times\mathbf{j} = \mathbf{k}$

$\mathbf{j}\times\mathbf{k} = (|\mathbf{j}||\mathbf{k}|\cos 90)\,\hat{u}\ ;\ \hat{u} = \mathbf{i}$

$\mathbf{j}\times\mathbf{k} = \mathbf{i}$

$\mathbf{k}\times\mathbf{i} = (|\mathbf{k}||\mathbf{i}|\cos 90)\,\hat{u}\ ;\ \hat{u} = \mathbf{j}$

$\mathbf{k}\times\mathbf{i} = \mathbf{j}$

Given the vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ anc $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$.

$$\text{Then } \mathbf{a}\times\mathbf{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \mathbf{j}\begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Note that:

$\mathbf{i}\times\mathbf{j} = \mathbf{k}\ ;\ \mathbf{j}\times\mathbf{k} = \mathbf{i}\ ;\ \mathbf{k}\times\mathbf{i} = \mathbf{j}$

$\mathbf{j}\times\mathbf{i} = -\mathbf{k}\ ;\ \mathbf{k}\times\mathbf{j} = -\mathbf{i}\ ;\ \mathbf{i}\times\mathbf{k} = -\mathbf{j}$

Note that if $\mathbf{c} = \mathbf{a}\times\mathbf{b}$, then **c** is a vector perpendicular to both **a** and **b**.

The unit vector perpendicular to both **a** and **b** is

$$\hat{c} = \frac{c}{|c|} \Rightarrow \hat{c} = \frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}.$$

Example 4

Find the unit vector which is perpendicular to the vectors $2\mathbf{i}-3\mathbf{j}+6\mathbf{k}$ and $-6\mathbf{i}+2\mathbf{j}+3\mathbf{k}$.

Solution:

$$\begin{vmatrix} i & j & k \\ 2 & -3 & 6 \\ -6 & 2 & 3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -3 & 6 \\ 2 & 3 \end{vmatrix} + \mathbf{j}\begin{vmatrix} 6 & 2 \\ 3 & -6 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & -3 \\ -6 & 2 \end{vmatrix}$$

$= \mathbf{i}(-9-12) + \mathbf{j}(-36-6) + \mathbf{k}(4-18)$

$= -21\mathbf{i} - 42\mathbf{j} - 14\mathbf{k}$

Unit vector perpendicular to the two vectors

$$= \frac{-21\mathbf{i}-42\mathbf{j}-14\mathbf{k}}{\sqrt{(-21)^2+(-42)^2+(-14)^2}}$$

$$= -\frac{1}{7}(3\mathbf{i}+6\mathbf{j}+2\mathbf{k})$$

Example 5

The vectors **p**, **q** and **r** are defined as $\mathbf{p} = \mathbf{i}+2\mathbf{j}-\mathbf{k}$, $\mathbf{q} = -3\mathbf{i}+4\mathbf{k}$, $\mathbf{r} = 4\mathbf{k}$.

(i) Calculate $\mathbf{p}\times\mathbf{q}$ and $\mathbf{q}\times\mathbf{r}$.

(ii) Verify that $\mathbf{p}\bullet(\mathbf{q}\times\mathbf{r}) = (\mathbf{p}\times\mathbf{q})\bullet\mathbf{r}$ but $\mathbf{p}\times(\mathbf{q}\times\mathbf{r}) \neq (\mathbf{p}\times\mathbf{q})\times\mathbf{r}$.

(iii) Determine the value of λ if $\lambda\mathbf{p} + \mathbf{r}$ is perpendicular to **q**.

Solution:

$\mathbf{p} = \mathbf{i}+2\mathbf{j}-\mathbf{k}$, $\mathbf{q} = -3\mathbf{i}+4\mathbf{k}$, $\mathbf{r} = 4\mathbf{k}$

(i) $$\mathbf{p}\times\mathbf{q} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -3 & 0 & 4 \end{vmatrix}$$

$= \mathbf{i}(2\times4-0\times-1) - \mathbf{j}(1\times4 - -3\times-1) + \mathbf{k}(1\times0 - -3\times2)$

$= 8\mathbf{i}-\mathbf{j}+6\mathbf{k}$

$$\mathbf{q}\times\mathbf{r} = \begin{vmatrix} i & j & k \\ -3 & [illegible] & 4 \\ 0 & [illegible] & 4 \end{vmatrix}$$

$= \mathbf{i}(0\times4 - 0\times4) - \mathbf{j}(-3$ [illegible] $0\times4) + \mathbf{k}(-3\times0 - 0\times0)$

k

$= 12\mathbf{j}$

(ii) $\mathbf{p}\bullet(\mathbf{q}\times\mathbf{r}) = (\mathbf{i}+2\mathbf{j}-\mathbf{k})\bullet(12\mathbf{j})$

$= 2\times12 = 24$

$(\mathbf{p}\times\mathbf{q})\bullet\mathbf{r} = (8\mathbf{i}-\mathbf{j}+6\mathbf{k})\bullet(4\mathbf{k})$

$= 6\times4 = 24$

Hence $\mathbf{p}\bullet(\mathbf{q}\times\mathbf{r}) = (\mathbf{p}\times\mathbf{q})\bullet\mathbf{r} = 24$

$$\mathbf{p}\times(\mathbf{q}\times\mathbf{r}) = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 0 & 12 & 0 \end{vmatrix}$$

$= \mathbf{i}(2\times0 - 12\times-1) - \mathbf{j}(1\times0 - 0\times-1) + \mathbf{k}$

$(1\times12-0\times12)$
$= 12\mathbf{i} + 12\mathbf{k}$

$$(\mathbf{p}\times\mathbf{q})\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & -1 & 6 \\ 0 & 0 & 4 \end{vmatrix}$$

$= \mathbf{i}(-1\times4 - 0\times6) - \mathbf{j}(8\times4 - 0\times6) + \mathbf{k}(8\times0 - 0\times-1)$

$= -4\mathbf{i} - 32\mathbf{j}$

Hence $\mathbf{p}\times(\mathbf{q}\times\mathbf{r}) \neq (\mathbf{p}\times\mathbf{q})\times\mathbf{r}$

(iii) If $\lambda\mathbf{p} + \mathbf{r}$ is perpendicular to $\mathbf{q}$;

then $(\lambda\mathbf{p}+\mathbf{r})\bullet\mathbf{q} = 0$

$[\lambda(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) + 4\mathbf{k}]\bullet[-3\mathbf{i} + 4\mathbf{k}] = 0$

$[\lambda\mathbf{i} + 2\lambda\mathbf{j} + (4-\lambda)\mathbf{k}]\bullet(-3\mathbf{i} + 4\mathbf{k}) = 0$

$-3\lambda + 4(4-\lambda) = 0$

$-3\lambda + 16 - 4\lambda = 0$

$\lambda = \frac{16}{7}$

Exercise: 1A

1. Express the following in vector form:
   (a) A force of magnitude 70 N acting in the direction of the vector $12\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}$.
   (b) The velocity of a body moving with a speed of 81 m s^{-1} parallel to the vector $4\mathbf{i} - \mathbf{j} + 8\mathbf{k}$.
2. Given that the vectors
   $-\mathbf{i} + 2\mathbf{j} + \mathbf{k} + t(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$ and
   $-3\mathbf{i} + p\mathbf{j} + 7\mathbf{k} + s(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ are concurrent, find the values of t, s and p.

3. A particle acted upon by a force of 15 N in the direction $4\mathbf{i} - 3\mathbf{j}$ causes it to move with constant speed 26 m s^{-1} in the direction $5\mathbf{i} + 12\mathbf{j}$. Find the:
   (a) force vector.
   (b) velocity vector.
   (c) power developed by the force.
4. Given that the force $\mathbf{F} = 8\mathbf{i} + a\mathbf{j}$ has a magnitude of 17 units. Find two possible values of a.

5. (a) (i) Find a unit vector in the direction of the vector $\mathbf{i} + 4\mathbf{j} + 8\mathbf{k}$.
   (ii) A boat is rowed with a speed of 28 m s^{-1} in the direction of the vector $2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$. Find the velocity of the boat.
   (b) Given the vectors $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = 4\mathbf{i} + 4\mathbf{k}$ and $\mathbf{c} = 5\mathbf{i} + 7\mathbf{j} + \mathbf{k}$. Determine:
   (i) the resultant of the vectors
   (ii) $3\mathbf{c} - 2\mathbf{b}$
   (iii) $|2\mathbf{a} + \mathbf{b}|$
   (iv) $(\mathbf{a}\bullet\mathbf{b})\mathbf{c} - (\mathbf{a}\bullet\mathbf{c})\mathbf{b}$

6. The magnitude of the resultant of three forces $3\mathbf{i} + 6\mathbf{j}$, $3\mathbf{i} - 2\mathbf{j}$ and $\lambda\mathbf{j}$ is 10 N. Find two possible values of λ.
7. The resultant of forces $F_1 = 3\mathbf{i} + (a-c)\mathbf{j}$, $F_2 = (2a+3c)\mathbf{i} + 5\mathbf{j}$, $F_3 = 4\mathbf{i} + 6\mathbf{j}$ acting on a particle is $10\mathbf{i} + 12\mathbf{j}$. Find the:
   (a) values of a and c.
   (b) magnitude of force F_2.
8. If $\mathbf{a} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} + 4\mathbf{j}$ and $\mathbf{c} = 3\mathbf{i} + \mathbf{j} + 4\mathbf{k}$, find:
   (a) the resultant of **a** and **b**.
   (b) the resultant of **a**, **b** and **c**.
   (c) $|\mathbf{a}|$
   (d) $|\mathbf{b}|$
   (e) $|\mathbf{c}|$
   (f) $|\mathbf{a} + \mathbf{b} + \mathbf{c}|$
   (g) a vector parallel to **a** and has a magnitude of 28 units.
   (h) a vector that is parallel to (a+b+c) and has a magnitude of 55 units.

9. If $\mathbf{a} = \begin{pmatrix} 2 \\ 7 \\ 7 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 0 \\ -4 \\ -3 \end{pmatrix}$, find:
   (a) the resultant of **a** and **b**.
   (b) the resultant of **a** and **c**.
   (c) $|\mathbf{a}|$
   (d) $|\mathbf{b}|$
   (e) $|\mathbf{c}|$
   (f) $|\mathbf{a} + \mathbf{b} + \mathbf{c}|$
   (g) a vector that is parallel to $(\mathbf{a} + \mathbf{b} + \mathbf{c})$ and has magnitude 50 units.

10. If $\mathbf{a} = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$, find;
    (a) the magnitudes of **a**, **b** and $5\mathbf{a} + 2\mathbf{b}$,
    (b) the unit vector in the direction of $\mathbf{a} + 2\mathbf{b}$.
11. Find the scalar product of each of the following pairs of vectors.
    (a) $3\mathbf{i} + 4\mathbf{j}$ and $5\mathbf{i} - 12\mathbf{j}$
    (b) $2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$ and $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

(c) $\begin{pmatrix}3\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-2\end{pmatrix}$

(d) $\begin{pmatrix}2\\2\\3\end{pmatrix}$ and $\begin{pmatrix}2\\1\\-1\end{pmatrix}$

12. Find the angle between each of the following pairs of vectors:
    (a) 2**i**−3**j** and 6**i** + 4**j**
    (b) **i** + 2**j**−**k** and −**i** + 2**j**−**k**
    (c) $\begin{pmatrix}2\\1\end{pmatrix}$ and $\begin{pmatrix}3\\-2\end{pmatrix}$
    (d) $\begin{pmatrix}0\\1\\-1\end{pmatrix}$ and $\begin{pmatrix}-1\\0\\1\end{pmatrix}$

Exercise: 1B

1. Find the component of force **F** in the direction of the vector **d** if:
    (a) **F** = 3**i**−4**j** + 4**k**, **d** = 6**i**−2**j**−3**k**.
    (b) **F** = **i** + 5**j**−3**k**, **d** = 2**i**−**j** + 2**k**.
    (c) **F** = 7**i**−**j**−4**k**, **d** = 3**i**−5**j** + 4**k**.
    (d) **F** = 3**i** + 5**j**−**k**, **d** = 4**i**−4**j** + 7**k**.
2. A force of magnitude 9 N acts in the direction **i**−2**j** + 2**k**. Find the component of the force in the direction 2**i** + **j** + 2**k**.
3. If the resultant of 3**i** + 4**j** and a**i** + b**j** is 7**i**−**j**, find the values of a and b.
4. If P and Q are points with position vectors **P** = -**i** + 3**j** + 2**k** and **Q** = **i**−7**j** + 4**k** respectively. Find the:
    (a) distance PQ.
    (b) unit vector in the direction of PQ.
    (c) position vector of the midpoint of PQ.
5. Show that the vectors **a** = 4**i**−2**j**−4**k**, **b** = 8**i** + 10**j** + 3**k**, c = -17**i** + 22**j**−28**k** are mutually perpendicular.
6. (a) Find the angle between the vectors **i** + 5**j**−6**k** and -3**i** + 4**k**.
    (b) Given **p** = 2**i**−2**j** + **k**, **q** = 2**i** + **j** + **k** obtain a unit vector perpendicular to **p** and **q**.
7. Given that **a** = 6**i** + (p−10)**j** + (3p−5)**k** and |**a**| = 11, find the possible values of p.
8. Given that **a** = 3**i**−2**j** + **k**, **b** = 2**i** + **j** + 3**k**:
    (a) find **a**×**b**.
    (b) calculate the angle between **a** and **b**.
    (c) show that |**a**−**b**| = |**a**|.
9. (a) Find the angle between the vectors r_1 = **i** + 3**j** + 2**k** and r_2 = 5**i** + **k**.
    (b) The resolved part of the force $\begin{pmatrix}2b\\-2b\\b\end{pmatrix}$N, in the direction of the vector $\begin{pmatrix}1\\-2\\-2\end{pmatrix}$ is 12 N. Find the constant b.
    (c) Find the value of α if the angle between the vectors **i** + 2**j** + 2**k** and **i** + α**k** is $\frac{\pi}{2}$.
10. The angle between the vectors r_1 and r_2 is $\cos^{-1}\left(\frac{4}{21}\right)$. If r_1 = 6**i** + 3**j**−2**k** and r_2 = -2**i**−λ**j**−4**k**. Find the values of λ.
11. Given that v_1 = 4**i** + 5**j**−2**k** and v_2 = -**i** + 2**j** + **k**, determine a unit vector perpendicular to v_1 and v_2.
12. Find the value of λ if λ**i** + 2**j**−**k** and 5**i**−λ**j** + **k** are perpendicular vectors.

Answers to exercises

Exercise: 1A

1. (a) (60i-20j+30k) N (b) (36i-9j+72k) m s^{-1} 2. t = 10;s = 12;p = -6
3. (a) (12i-9j) N (b) (10i+24j) m s^{-1} (c) -96 W 4. ±15
5. (a) (i) $\frac{1}{9}$(i+4j+8k) (ii) (8i+12j-24k) m s^{-1} (b) (i) 11**i** + 10**j** + 2**k** (ii) 7**i** + 21**j** - 5**k** (iii) $2\sqrt{26}$ (iv) -132**i** - 28**j** - 116**k**
6. 4 ; -12 7. (a) a = $\frac{6}{5}$; c = $\frac{1}{5}$ (b) $\sqrt{34}$ 8. (a) 6**i** + **j** + 2**k** (b) 9**i** + 2**j** + 6**k** (c) $\sqrt{14}$ (d) $\sqrt{41}$ (e) $\sqrt{26}$ (f) 11 (g) $2\sqrt{14}$(i-3j+2k) (h) 45**i** + 10**j** + 30**k**
9. (a) $\begin{pmatrix}8\\4\\9\end{pmatrix}$ (b) $\begin{pmatrix}2\\3\\4\end{pmatrix}$ (c) $\sqrt{102}$ units (d) 7 units (e) 5 units

(f) 10 units (g) $\begin{pmatrix} 40 \\ 0 \\ 30 \end{pmatrix}$ 10. (a) $2\sqrt{6}$;

$5\sqrt{2}$; 20 (b) $\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$

11. (a) -33 (b) -5 (c) 1 (d) 3 12.
(a) 90° (b) 48·2°
(c) 60·3° (d) 120°

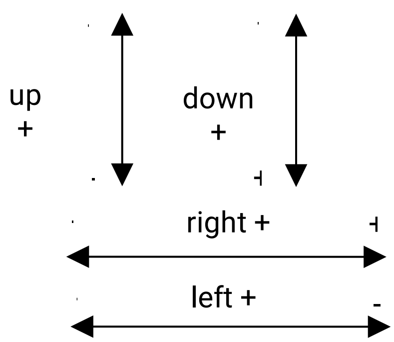
Exercise: 1B

1. (a) 2 (b) -3 (c) $\sqrt{2}$ (d) $-\frac{5}{3}$ 2. 4 N
3. 4 ; -5
4. (a) $6\sqrt{3}$ (b) $\frac{\sqrt{3}}{9}(\text{i-5j+k})$ (c) -2**i** + 3**k** 5.
a•**b** = **a**•**c** = **b**•**c** = 0
6. (a) 133·3° (b) $\frac{\sqrt{5}}{5}(\text{-i+2k})$
7. 1 ; 4 8. (a) 7(-i-j+k)
(b) 60° (c) |a-b| = |a| = $\sqrt{14}$
9. (a) 68·5° (b) 9 (c) $-\frac{1}{2}$
10. -4 ; $\frac{44}{65}$ 11. $\frac{1}{\sqrt{254}}(\text{9i-2j+13k})$ 12. $\frac{1}{3}$

2. LINEAR MOTION

- **Displacement** is the position of a point relative to an origin O. It is a vector.
- **Distance** is the magnitude of displacement. It is a scalar.
- **Velocity** is the rate of change of displacement with respect to time. It is a vector.
- **Speed** is the magnitude of velocity. It is a scalar.
- **Acceleration** is the rate of change of velocity with respect to time. It is a vector.
- **Negative acceleration** is also called retardation.
- **Uniform acceleration** is constant acceleration in a fixed direction.

When a particle moves in one dimension (along a straight line) it has two possible directions in which to move. Positive and negative signs are used to identify the two directions.



Whenever the direction is a positive, the opposite direction is negative.

Motion with constant acceleration:

2.1 Equations for uniformly accelerated motion

Consider a particle with initial velocity u and constant acceleration a. let its displacement from its initial position be s and its velocity be v at time t.

$\frac{dv}{dt} = a$, where a is a constant called the acceleration.

$\int dv = \int a\,dt$

$v = at + c$

When $t = 0$, $v = u \Rightarrow c = u$

Hence $v = u + at$ ………………………… (i)

And $v = \frac{ds}{dt}$

Therefore $\frac{ds}{dt} = u + at$

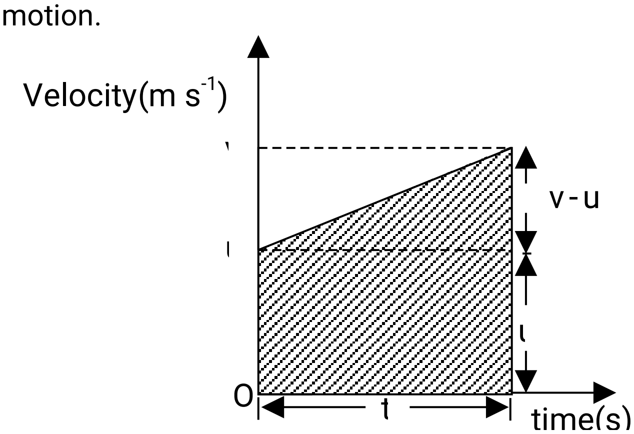
$\int ds = \int (u+at)dt$

$s = ut + \frac{1}{2}at^2 + c$

When $t = 0$, $s = 0 \Rightarrow c = 0$

Hence $s = ut + \frac{1}{2}at^2$…………………………(ii)

Alternatively, the above equations can be obtained from a velocity-time graph for the motion.



Acceleration, a = $\frac{\text{Change in velocity}}{\text{Time taken}}$

$a = \frac{v-u}{t} \Rightarrow v = u + at$…………………(i)

The displacement s is equal to the area under the graph

$s = \frac{1}{2}t(u+v) = \frac{1}{2}(u+u+at)$

Hence $s = ut + \frac{1}{2}at^2$…………………………(ii)

From equation (i): $v = u + at \Rightarrow t = \frac{v-u}{a}$

Substituting in equation (ii)

$s = \frac{u(v-u)}{a} + \frac{1}{2}a\left(\frac{v-u}{a}\right)^2$

$2as = (v-u)(2u + v - u)$

$2as = (v-u)(v + u)$

$v^2 = u^2 + 2as$ …………………………(iii)

Distance covered in the n^{th} second, ($s_{n^{th}}$)

$s_{n^{th}} = s_n - s_{n-1}$

From $s = ut + \frac{1}{2}at^2$

When $t = n$: $s_n = un + \frac{1}{2}an^2$

When $t = n-1$: $s_{n-1} = u(n-1) + \frac{1}{2}a(n-1)^2$

$s_{n^{th}} = s_n - s_{n-1} = un + \frac{1}{2}an^2 - \left\{u(n-1) + \frac{1}{2}a(n-1)^2\right\}$

$s_{n^{th}} = u + an - \frac{1}{2}a$

Example 1

A particle traveling in a straight line with uniform acceleration covers distances x and y in the third and fourth seconds of its motion respectively. Show that its initial velocity is given by $u = \frac{1}{2}(7x - 5y)$.

Solution:

From $s_{n^{th}} = s_n - s_{n-1}$ and $s = ut + \frac{1}{2}at^2$

$s_{3^{rd}} = s_3 - s_2 = x$ and $s_{4^{th}} = s_4 - s_3 = y$

$s_2 = 2u + \frac{1}{2}\times a\times 2^2 = 2u + 2a$

$s_3 = 3u + \frac{1}{2}\times a\times 3^2 = 3u + \frac{9}{2}a$

$s_4 = 4u + \frac{1}{2}\times a\times 4^2 = 4u + 8a$

$x = \left(3u + \frac{9}{2}a\right) - (2u + 2a)$

$\Rightarrow u + \frac{5}{2}a = x$……………(i)

$y = (4u + 8a) - \left(3u + \frac{9}{2}a\right)$

$\Rightarrow u + \frac{7}{2}a = y$……………(ii)

Subtracting equation (i) from equation (ii)

$a = y - x$

From equation (i): $u + \frac{5}{2}(y - x) = x$

$u = \frac{1}{2}(7x - 5y)$

Example 2

A driver of a car traveling at 72 km h^{-1} notices a tree which has fallen across the road, 800 m ahead, and suddenly reduces speed to 36 km h^{-1} by applying the brakes. For how long did the driver apply the brakes?

Solution:

$u = 72 \text{ km h}^{-1} = 72\times\frac{1000}{60\times60} = 20 \text{ m s}^{-1}$

$v = 36 \text{ km h}^{-1} = 36\times\frac{1000}{60\times60} = 10 \text{ m s}^{-1}$

$s = 800$ m

From $v^2 = u^2 + 2as$

$10^2 = 20^2 + 2\times a\times 800 \Rightarrow a = -\frac{3}{16} \text{ m s}^{-2}$

From $v = u + at$

$10 = 20 - \frac{3}{16}t \Rightarrow t = 53\frac{1}{3}$ s

Example 3

An over loaded taxi traveling at a constant speed of 90 km h^{-1} overtakes a stationary traffic police car. Two seconds later the police car sets off in pursuit of the taxi accelerating at 6 m s^{-2}. How far does the traffic car travel before catching up with the taxi?

Solution:

	Taxi	Police car
Initial speed	90 km h^{-1} = 25 m s^{-1}	0 m s^{-1}
Time	(t+2) s	t s
Acceleration	0 m s^{-2}	6 m s^{-2}
Distance	s_T	s_P

From $s = ut + \frac{1}{2}at^2$

$s_T = 25(t + 2)$…………………………(i)

$s_P = \frac{1}{2}\times 6\times t^2 = 3t^2$………………..(ii)

When the police car catches up with the taxi

$s_P = s_T$

$3t^2 = 25(t + 2)$

$3t^2 - 25t - 50 = 0$

$3t^2 - 30t + 5t - 50 = 0$

$3t(t-10) + 5(t-10) = 0$

$(3t + 5)(t-10) = 0$

Either $3t + 5 = 0 \Rightarrow t = -\frac{5}{3}$ s Or $t - 10 = 0$

$\Rightarrow t = 10$ s

Hence $t = 10$ s and $s = 3\times10^2 = 300$ m

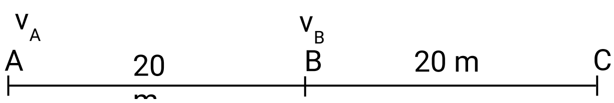
Example 4

A, B and C are points on a straight road such that AB = BC = 20 m. A cyclist moving with uniform acceleration passes A and then notices that it takes him 10 s and 15 s to travel between A and B, and A and C respectively. Find:

(i) his acceleration.

(ii) the velocity with which he passes A.

Solution:



Consider motion between A and B:

$u = v_A$, $s = 20$ m, $t = 10$ s

From $s = ut + \frac{1}{2}at^2$

$$20 = v_A\times10 + \frac{1}{2}\times a\times10^2$$

$$20 = 10v_A + 50a$$

$$\Rightarrow 2 = v_A + 5a \ldots\ldots\ldots\ldots (i)$$

Consider motion between A and C:

$u = v_A$, $s = 40$ m, $t = 15$ s

From $s = ut + \frac{1}{2}at^2$

$$40 = v_A\times15 + \frac{1}{2}\times a\times 15^2 \Rightarrow$$

$$40 = 15v_A + 112\cdot5a \ldots\ldots (ii)$$

From equation (i): $v_A = 2 - 5a$

Substituting in equation (ii)

(i) $40 = 15(2 - 5a) + 112\cdot5a \Rightarrow 10 = 37\cdot5a$

$a = \frac{4}{15}$ m s^{-2}

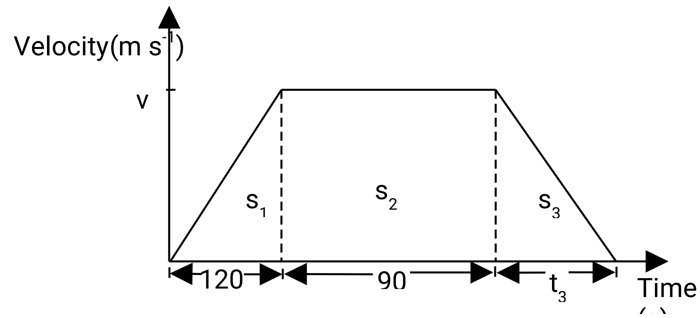
(ii) $v_A = 2 - 5\times\frac{4}{15} \Rightarrow v_A = \frac{2}{3}$ m s^{-1}

Note: When motion takes place and acceleration changes after given time intervals but is constant during each interval, solutions to such problems can be obtained with aid of a velocity-time graph.

Example 5

A train starts from station A with uniform acceleration of $0\cdot2$ m s^{-2} for 2 minutes and attains a maximum speed and moves uniformly for 15 minutes. It is then brought to rest at a constant retardation of $\frac{5}{3}$ m s^{-2} at station B. Find the distance between stations A and B.

Solution:



From $v = u + at \Rightarrow v = 0 + 0\cdot2\times120$

$= 24$ m s^{-1}

Also $0 = 24 - \frac{5}{3}t_3 \Rightarrow t_3 = 14\cdot4$ s

$$s = s_1 + s_2 + s_3$$

$$= \frac{1}{2}\times120\times24 + 900\times24 + \frac{1}{2}\times14\cdot4\times24$$

$$s = 23\,212\cdot8 \text{ m}$$

Example 6

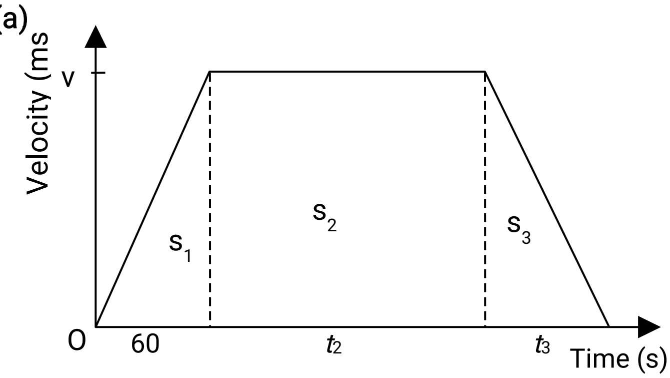
A train starting from rest is uniformly accelerated during the first minute of its journey when it covers 600 m. It then runs at a constant speed until it is brought to rest in a distance of 1 km from the instant the brakes were applied causing a constant retardation.

(a) Given that the total journey time is 5 minutes, calculate the distance covered during the constant speed.

(b) If the magnitude of the retardation instead of being constant is directly proportional to the speed and the train comes to rest from the constant speed in a distance of 500 m, find the magnitude of the retardation when the train's speed is 10 m s^{-1}.

Solution:

(a)



$t_2 + t_3 = 5\times60-60 \Rightarrow t_2 + t_3 = 240$……………… (i)

$s_1 = 600$ m

From $s_1 = \frac{1}{2}\times60\times v = 30v$

$\therefore\ 30v = 600 \Rightarrow v = 20\ m\ s^{-1}$

$s_3 = 1000$ m, $s_3 = \frac{1}{2}\times20\times t_3 = 20t_3$

$\therefore\ 10t_3 = 1000 \Rightarrow t_3 = 100$ s

From equation

(i): $t_2 + 100 = 240 \Rightarrow t_2 = 140$ s

$s_2 = 20\times140 = 2800$ m

Distance covered during constant speed = 2800 m

(b) $\frac{dv}{dt} \propto v \Rightarrow \frac{dv}{dt} = k_1v$, k_1 being a constant.

But $\frac{dv}{dt} = \frac{dv}{dx}\times\frac{dx}{dt} = v\frac{dv}{dx}$

Thus: $\frac{dv}{dx} = k_1 \Rightarrow \int dv = \int k_1 dx$

$\therefore\ v = k_1x + k_2$, where k_2 is a constant.

When x = 0, v = 20

$20 = k_2 \Rightarrow k_2 = 20\ \therefore\ v = k_1x + 20$

When x = 500, v = 0

$0 = 500k_1 + 20 \Rightarrow k_1 = -\frac{1}{25}$

$\therefore\ \frac{dv}{dt} = -\frac{v}{25}$

When $v = 10\ m\ s^{-1}$

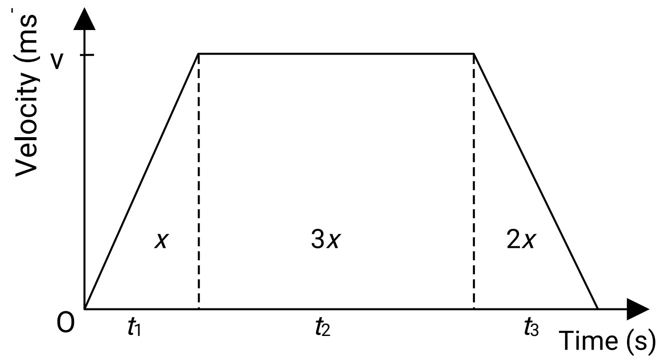
$\frac{dv}{dt} = -\frac{10}{25} = -\frac{2}{5} m\ s^{-2}$

Hence magnitude of deceleration/retardation is $\frac{2}{5} m\ s^{-2}$

Example 7

Two stations A and B are a distance of 6x m apart along a straight track. A train starts from rest at A and accelerates uniformly to a speed v $m\ s^{-1}$, covering a distance x m. The train maintains this speed until it has traveled a further 3x m, it then retards uniformly to rest at B. Sketch a velocity-time graph for the motion and show that if T is the time taken for the train to travel from A to B, then $T = \frac{9x}{v}$ seconds.

Solution



$\frac{1}{2}vt_1 = x \Rightarrow t_1 = \frac{2x}{v}$

$vt_2 = 3x \Rightarrow t_2 = \frac{3x}{v}$

$\frac{1}{2}vt_3 = 2x \Rightarrow t_3 = \frac{4x}{v}$

$T = t_1 + t_2 + t_3$

$T = \frac{2x}{v} + \frac{3x}{v} + \frac{4x}{v}$

$T = \frac{9x}{v}$ seconds

Example 8

A car travels along a straight horizontal road, passing two points A and B. The car passes A at u $m\ s^{-1}$ and maintains this speed for 60 s, during which it travels 900 m. Approaching a junction, the car then slows at a uniform rate of a $m\ s^{-2}$ over the next 125 m to reach a speed of 10 $m\ s^{-1}$, at which instant, with the road clear, the car accelerates uniformly at 0·75 $m\ s^{-2}$. This acceleration is maintained for 20 s by which time the car has reached a speed of v $m\ s^{-1}$ which is then maintained. The car passes B, 45 seconds after its speed reaches v $m\ s^{-1}$.

(i) Calculate the values of u, a and v.

(ii) Sketch a velocity-time graph for the motion of the car between A and B.

(iii) Find the distance between points A and B and the time taken by the car to travel this distance.

Solution

(i) From $s = ut + \frac{1}{2}at^2$; $a = 0$

$900 = 60u \Rightarrow u = 15 \text{ m s}^{-1}$

From $v^2 = u^2 + 2as$

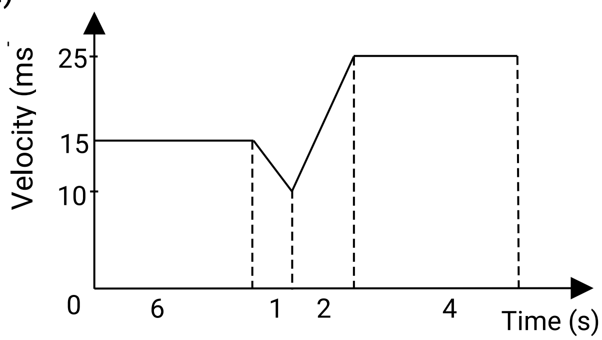
$10^2 = 15^2 + 2\times a\times 125 \Rightarrow a = -0.5 \text{ m s}^{-2}$

From $v = u + at$

$10 = 15 - 0\cdot 5t \Rightarrow t = 10 \text{ s}$

Also $v = 10 + 0\cdot 75\times 20 \Rightarrow v = 25 \text{ m s}^{-1}$

(ii)



(iii) distance between A and B:

$s = 900 + 125 + \frac{1}{2}\times 20(10+25) + 25\times 45 = 2\,500 \text{ m}$

Total time = 60 + 10 + 20 + 45

= 135 seconds

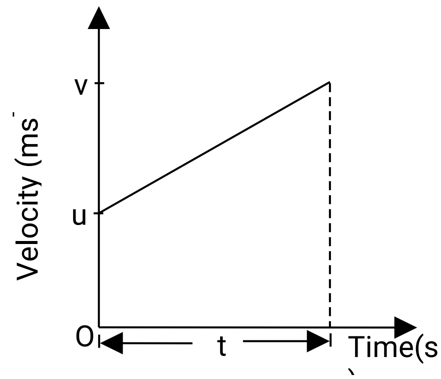
Example 9

(a) A particle moves in a straight line with uniform acceleration, a and has initial velocity, u. Show that the distance, s, traveled is given by $s = ut + \frac{1}{2}at^2$, t being the time taken.

(b) A particle is uniformly accelerated along a straight line *PQRS*. If it covers successive distances PQ = p, QR = q, RS = r, in equal intervals of time. Prove that $r = 2q - p$, and that the ratio of the speed at S to that at P is $(5q-3p):(3p-q)$.

Solution

(a)

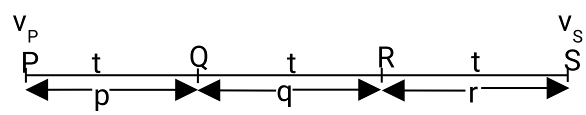


If the particle attains a velocity v after time t. The distance, s covered during this time is equal to the area under the velocity-time graph.

$s = \frac{1}{2}t(u + v)$, but from $v = u + at$

$s = \frac{1}{2}t(u + u + at) \Rightarrow s = ut + \frac{1}{2}at^2$

(b)



From $s = ut + \frac{1}{2}at^2$, $a = f$

Between P and Q

$p = v_Pt + \frac{1}{2}ft^2$……………………….(i)

Between P and R

$p + q = v_P\times 2t + \frac{1}{2}\times f\times(2t)^2$

$\Rightarrow p + q = 2v_Pt + 2ft^2$…………(ii)

Between P and S

$p + q + r = v_P\times 3t + \frac{1}{2}\times f\times(3t)^2 = 3v_Pt + \frac{9}{2}ft^2$…………(iii)

Eqn (ii) – 4× Eqn (i)

$q - 3p = -2v_Pt$

$\Rightarrow v_Pt = \frac{1}{2}(3p - q)$………………… (iv)

Equation (iii) – 9× equation (i)

$q + r - 8p = -6v_Pt$

$\Rightarrow v_Pt = \frac{1}{6}(8p - q - r)$ …………… (v)

Dividing equation (iv) by equation (v)

$1 = \frac{3(3p-q)}{8p-q-r} \Rightarrow r = 2q - p$

From $v = u + at$, $a = f \Rightarrow v_S = v_P + 3ft$

From equation(iii):

$$p + q + (2q - p) = 3v_P t + \frac{9}{2}ft^2$$

$$\Rightarrow q = v_P t + \frac{3}{2}ft^2 \text{........(vi)}$$

Subtracting Ean (i) from Ean (vi)

$$ft^2 = q - p \text{ .... (vii)}$$

Dividing equation (vii) by equation (iv)

$$\frac{ft}{v_P} = \frac{2(q-p)}{3p-q} \Rightarrow ft = \frac{2v_P(q-p)}{3p-q}$$

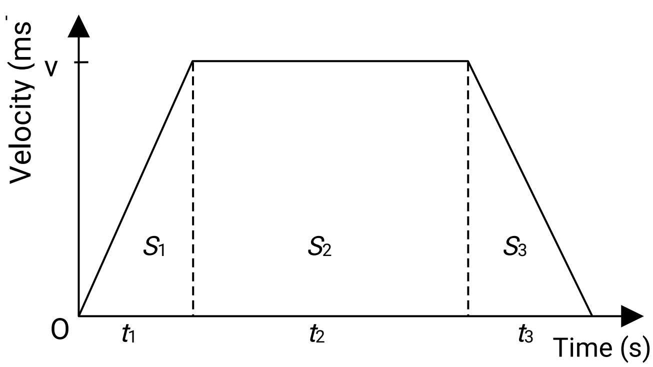
$$\therefore v_S = v_P + \frac{3\times 2v_P(q-p)}{3p-q} \Rightarrow \frac{v_S}{v_P} = \frac{5q-3p}{3p-q}$$

$$v_S : v_P = (5q - 3p) : (3p - q)$$

Example 10

A lift ascends with constant acceleration a then with constant velocity and then brought to rest under constant retardation a. If the total distance covered is s and the total time taken is t, show that the time taken when the lift moves with constant velocity is $\left(t^2 - \frac{4s}{a}\right)^{\frac{1}{2}}$.

Solution



$$s = s_1 + s_2 + s_3 \text{ .... (i)}$$

$$t = t_1 + t_2 + t_3 \text{ .... (ii)}$$

From v = u + at

$$v = at_1 \text{ also } 0 = v - at_3 \Rightarrow v = at_3$$

Thus $at_1 = at_3 \Rightarrow t_1 = t_3$

From equation (ii) $t = 2t_1 + t_2$

$$\Rightarrow t_2 = t - 2t_1 \Rightarrow t_1 = \frac{1}{2}(t - t_2)$$

$$s_1 = \frac{1}{2}vt_1 = \frac{1}{2}at_1^2 = \frac{1}{2}a\times\frac{1}{4}(t-t_2)^2 = \frac{1}{8}a(t-t_2)^2$$

$$s_2 = vt_2 = at_1t_2 = a\times\frac{1}{2}(t-t_2)t_2$$

$$= \frac{1}{2}at_2(t - t_2)$$

$$s_3 = \frac{1}{2}vt_3 = \frac{1}{2}\times at_1^2$$

$$= \frac{1}{2}a\times\frac{1}{4}(t-t_2)^2 = \frac{1}{8}a(t-t_2)^2$$

But from equation (i)

$$\frac{1}{8}a(t-t_2)^2 + \frac{1}{2}at_2(t-t_2) + \frac{1}{8}a(t-t_2)^2 = s$$

$$(t - t_2)(t + t_2) = \frac{4s}{a} \Rightarrow t^2 - t_2^2 = \frac{4s}{a}$$

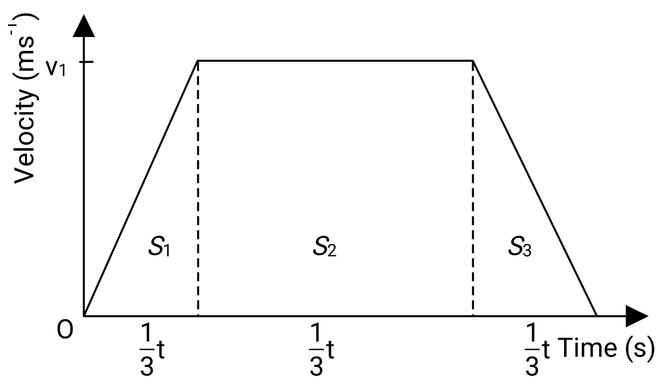
Hence $t_2 = \left(t^2 - \frac{4s}{a}\right)^{\frac{1}{2}}$

Example 11

Two trains P and Q travel by the same route from rest at station A to rest at station B. Train P has a constant acceleration f for the first third of the time, constant speed for the second third and constant retardation f for the last third of the time. Train Q has constant acceleration f for the first third of the distance, constant speed for the second third and constant retardation f for the last third of the distance. Show that the times taken by the two trains are in the ratio $3\sqrt{3}:5$.

Solution

Train P:

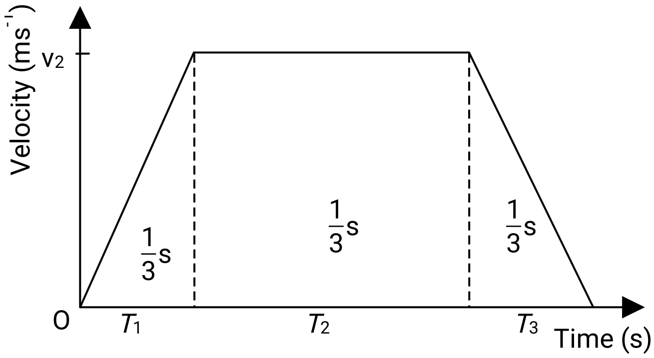


From v = u + at $\Rightarrow v_1 = \frac{1}{3}ft$

$$s = \frac{1}{2}v_1\left(\frac{1}{3}t + t\right) = \frac{1}{2}\times\frac{1}{3}ft\times\frac{4}{3}t$$

$$\therefore \quad s = \frac{2}{9}ft^2 \Rightarrow t = 3\sqrt{\frac{s}{2f}}$$

Train Q:



From $v = u + at$ and $T = T_1 + T_2 + T_3$

$v_2 = fT_1$, $0 = v_2 - fT_3 \Rightarrow v_2 = fT_3$

$\therefore \quad fT_1 = fT_3 \Rightarrow T_1 = T_3 \Rightarrow T = 2T_1 + T_2$

$$\frac{1}{3}s = \frac{1}{2}\times T_1 \times fT_1 \Rightarrow \frac{1}{3}s = \frac{1}{2}fT_1^2 \Rightarrow T_1 = T_3 = \sqrt{\frac{2s}{3f}}$$

$$T = 2\sqrt{\frac{2s}{3f}} + T_2 \Rightarrow T_2 = T - 2\sqrt{\frac{2s}{3f}}$$

Also $\frac{1}{3}s = v_2 T_2 \Rightarrow \frac{1}{3}s = f\sqrt{\frac{2s}{3f}}\left(T - 2\sqrt{\frac{2s}{3f}}\right)$

$$\frac{1}{3}s = fT\sqrt{\frac{2s}{3f}} - \frac{4}{3}s \Rightarrow fT\sqrt{\frac{2s}{3f}} = \frac{5}{3}s$$

$$\therefore \quad T = 5\sqrt{\frac{s}{6f}}$$

Ratio of time:

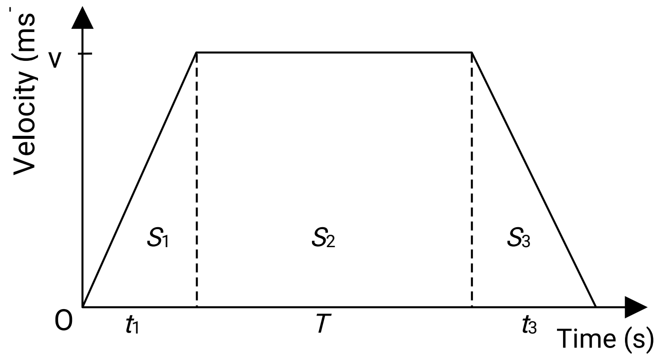
$$\frac{t}{T} = 3\sqrt{\frac{s}{2f}} \div 5\sqrt{\frac{s}{6f}} = \frac{3\sqrt{3}}{5}$$

$$t:T = 3\sqrt{3}:5$$

Example 12

Two points P and Q are x metres apart. A particle starts from rest at P and moves directly towards Q with an acceleration of a m s^{-2} until it acquires a speed v m s^{-1}. It maintains this speed for T seconds and is then brought to rest at Q under a retardation of a m s^{-2}. Prove that $T = \frac{x}{v} - \frac{v}{a}$.

Solution



From $v = u + at$

$$v = at_1 \Rightarrow t_1 = \frac{v}{a}$$

Also $0 = v - at_3 \Rightarrow t_3 = \frac{v}{a}$

$$s = x = s_1 + s_2 + s_3$$

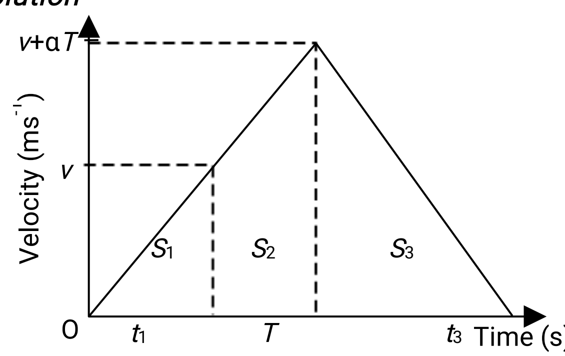
$$x = \frac{1}{2}\times t_1 \times v + v\times T + \frac{1}{2}\times t_3 \times v = \frac{v^2}{a} + v\times T$$

$$x = \frac{v^2}{a} + vT \Rightarrow T = \frac{x}{v} - \frac{v}{a}$$

Example 13

A car starts from rest and moves with uniform acceleration α. The driver notices that a tree has fallen ahead when the car is moving at speed v. If he takes T seconds to react from the instant the speed was v and retards uniformly at retardation β to come to rest in a distance x from the instant the speed was v, prove that $2\beta x = v^2 + T(\alpha + \beta)(2v + \alpha T)$.

Solution



From $v = u + at$

$v = \alpha t_1 \qquad 0 = (v + \alpha T) - \beta t_3$

$t_1 = \frac{v}{\alpha} \qquad t_3 = \left(\frac{v + \alpha T}{\beta}\right)$

$$s_2 = \frac{1}{2}T(v + v + \alpha T) = \frac{1}{2}T(2v + \alpha T)$$

$$s_3 = \frac{1}{2}\times\frac{(v+\alpha T)}{\beta}\times(v + \alpha T)$$

$$= \frac{1}{2\beta}(v^2 + 2v\alpha T + \alpha^2T^2)$$

$x = s_2 + s_3$

$$\Rightarrow x = \frac{2vT+\alpha T^2}{2} + \frac{v^2+2v\alpha T+\alpha^2T^2}{2\beta}$$

$2\beta x = 2\beta vT + \alpha\beta T^2 + v^2 + 2v\alpha T + \alpha^2T^2$

$2\beta x = v^2 + T(\alpha + \beta)(2v + \alpha T)$

2.2 Motion under gravity

The equations of linear motion can be modified as follows for motion under gravity:

1. **Upward motion under gravity**

   For upward motion under gravity, $a = -g$

   $v = u - gt$ ....(i)

   $s = ut - \frac{1}{2}gt^2$ .... (ii)

   $v^2 = u^2 - 2gs$......................................(iii)

2. **Downward motion under gravity**

   For downward motion under gravity $a = g$

   $v = u + gt$ .... (i)

   $s = ut + \frac{1}{2}gt^2$ .... (ii)

   $v^2 = u^2 + 2gs$ .... (iii)

3. **For a particle released from rest**

   $u = 0, a = g$

   $v = gt$................................. (i)

   $s = \frac{1}{2}gt^2$ .... (ii)

   $v^2 = 2gs$ .... (iii)

Example 14

A particle is projected vertically upwards from the ground with speed u m s^{-1} and clears the top of a vertical pole of height H in t s and returns to the top of the pole $\frac{1}{2}$t s later, show that:

(i) $12u^2 = 25gH$.

(ii) the speed at the top of the pole is $\frac{u}{5}$.

Solution

(i) From $s = ut - \frac{1}{2}gt^2$

$$H = ut - \frac{1}{2}gt^2 \text{ .... (i)}$$

$$H = u\times\frac{3}{2}t - \frac{1}{2}g\left(\frac{3}{2}t\right)^2$$

$$\Rightarrow H = \frac{3}{2}ut - \frac{9}{8}gt^2 \text{.................... (ii)}$$

From equation (i) and equation (ii)

$$ut - \frac{1}{2}gt^2 = \frac{3}{2}ut - \frac{9}{8}gt^2 \Rightarrow t = \frac{4u}{5g}$$

Substituting in equation (i)

$$H = u\times\frac{4u}{5g} - \frac{1}{2}\times g\left(\frac{4u}{5g}\right)^2$$

$$\Rightarrow H = \frac{12u^2}{25g} \Rightarrow 12u^2 = 25gH$$

(ii) From $v = u - gt \Rightarrow v = u - g\times\frac{4u}{5g} \Rightarrow v = \frac{u}{5}$

Example 15

A particle is projected vertically upwards with velocity u m s^{-1} and after t seconds another particle is projected vertically upwards from the same point with the same initial velocity. Prove that they will meet at a height of $\frac{4u^2-g^2t^2}{8g}$.

Solution

	1st Particle	2nd Particle
Displacement	s_1	s_2
Initial speed	u	u
Time	T + t	T

From $s = ut + \frac{1}{2}at^2$

$$s_1 = u(T + t) - \frac{1}{2}g(T+t)^2$$

$$s_2 = uT - \frac{1}{2}gT^2$$

When the particles meet, $s_1 = s_2$

$$\Rightarrow u(T + t) - \frac{1}{2}g(T+t)^2 = uT - \frac{1}{2}gT^2$$

$$\Rightarrow T = \left(\frac{u}{g} - \frac{t}{2}\right)$$

From $s_2 = uT - \frac{1}{2}gT^2$

$$s_2 = s = u\left(\frac{u}{g} - \frac{t}{2}\right) - \frac{1}{2}g\left(\frac{u}{g} - \frac{t}{2}\right)^2$$

$$\Rightarrow s = \left(\frac{2u-gt}{2g}\right)\left(\frac{2u+gt}{4}\right)$$

Hence $s = \frac{4u^2-g^2t^2}{8g}$

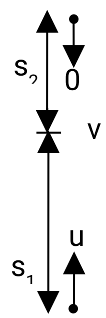
Example 16

(a) A particle is projected vertically upwards and at the same instant another is left to fall to meet it. Show that if the particles have the same speed when they collide, one of them has traveled thrice the distance traveled by the other.

(b) A particle starts from rest and travels 20 m with uniform acceleration. It then maintains this speed for a time after which it is uniformly retarded to rest, the magnitude of retardation being twice that of the acceleration. If the total distance traveled is 90 m and the total time taken is 6 s, find the greatest speed and the distance covered at this speed.

Solution

(a)



From $v = u + at$

For 1st particle $v = u - gt$

For 2nd particle; $v = gt \Rightarrow gt = u - gt$.

$$\Rightarrow t = \frac{u}{2g}$$

For 1st particle:

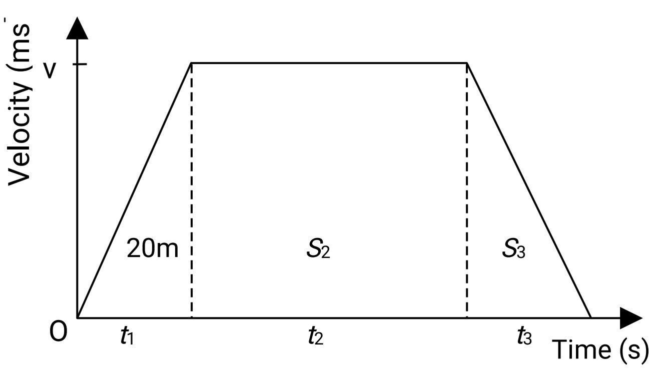
$$s_1 = ut - \frac{1}{2}gt^2$$

$$= \frac{u^2}{2g} - \frac{1}{2}g\times\left(\frac{u}{2g}\right)^2 = \frac{3u^2}{8g}$$

$$\text{and } s_2 = \frac{u^2}{8g}$$

$$\frac{s_1}{s_2} = \frac{3}{1} \Rightarrow s_1 = 3s_2$$

(b)



$$s = 90 \text{ m}, t = 6 \text{ s}$$

From $v = u + at$

$$v = at \ldots\ldots\ldots\ldots(i)$$

Also $0 = v - 2at_3 \Rightarrow v = 2at_3$ ……………..(ii)

From equation (i) and equation (ii)

$$at_1 = 2at_3 \Rightarrow t_1 = 2t_3$$

But $\quad t_1 + t_2 + t_3 = 6 \Rightarrow t_2 + 3t_3 = 6$ ………(iii)

$$\frac{1}{2}\times v\times t_1 = 20 \Rightarrow v = \frac{40}{t_1}\frac{1}{2}\times v\left(6+t_2\right)$$

$$= 90 \Rightarrow \frac{1}{2}\times\frac{40}{t_1}(6 + t_2) = 90$$

$$12 + 2t_2 = 9t_1 \Rightarrow 6 + t_2 = 9t_3 \ldots\ldots\ldots\ldots(iv)$$

Subtracting equation (iv) from equation (iii)

$$3t_3 - 6 = 6 - 9t_3 \Rightarrow t_3 = 1$$

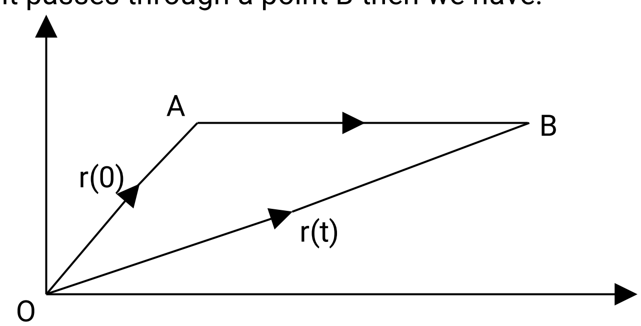
$$t_1 = 2,\ t_2 + 3\times1 = 6 \Rightarrow t_2 = 3$$

Greatest speed, $v = \frac{40}{2} = 20 \text{ m s}^{-1}$

Distance covered at this speed = 20×3 = 60 m

2.3 Linear motion: (i – j or i – j – k) spaces

Consider a particle which has initial velocity **u** from a point A with position vector **r**(0) and moves with constant acceleration **a**. If after time t, it passes through a point B then we have:



$$\mathbf{r}(t) = \mathbf{r}(0) + \vec{AB}$$

$$\mathbf{r}(t) = \mathbf{r}(0) + \mathbf{s},\ \mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

Example17

A particle initially passes through the point (3, -2) with velocity $-4\mathbf{i} + 2\mathbf{j}$ m s^{-1} while accelerating uniformly and in 4 seconds it covers a distance of $8\sqrt{5}$ m in the direction $\mathbf{i} + 2\mathbf{j}$. Find the:

(i) acceleration.

(ii) position vector after the 4 s.

Solution

$$\mathbf{r}(0) = (3\mathbf{i}-2\mathbf{j}),\ \mathbf{v} = (-4\mathbf{i}+2\mathbf{j})\ \text{m s}^{-1}$$

$$t = 4\text{ s};\ |\mathbf{s}| = 8\sqrt{5}\text{ m}$$

$$\mathbf{s} = 8\sqrt{5}\times\frac{1}{\sqrt{5}}(\mathbf{i} + 2\mathbf{j}) = (8\mathbf{i} + 16\mathbf{j})$$

(i) From $\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

$$(8\mathbf{i}+16\mathbf{j}) = (-4\mathbf{i}+2\mathbf{j})\times 4 + \frac{1}{2}\times\mathbf{a}\times 4^2$$

$$\Rightarrow \mathbf{a} = (3\mathbf{i}+\mathbf{j})\text{ m s}^{-2}$$

(ii) $\mathbf{r}(4) = \mathbf{r}(0) + \mathbf{s}$

$$= (3\mathbf{i}-2\mathbf{j}) + (8\mathbf{i} + 16\mathbf{j} = 11\mathbf{i} + 14\mathbf{j}$$

Example 18

A particle with position vector $(10\mathbf{i}+3\mathbf{j}+5\mathbf{k})$ m moves with constant speed of 6 m s^{-1} in the direction $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$. Find its distance from the origin after 5 seconds.

Solution

$$\mathbf{r}(0) = (10\mathbf{i}+3\mathbf{j}+5\mathbf{k})\text{ m}$$

$$\mathbf{v} = 6\times\frac{(\mathbf{i}+2\mathbf{j}+2\mathbf{k})}{\sqrt{1^2+2^2+2^2}}$$

$$= \frac{6}{3}(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = (2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k})\text{ m s}^{-1}$$

$$\mathbf{r}(t) = \mathbf{r}(0) + \mathbf{s},\ \mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

$$\mathbf{r}(t) = (10\mathbf{i}+3\mathbf{j}+5\mathbf{k}) + (2\mathbf{i}+5\mathbf{j}+4\mathbf{k})t$$

$$= (10+2t)\mathbf{i} + (3+4t)\mathbf{j} + (5+4t)\mathbf{k}$$

When t = 5 s

$$\mathbf{r}(5) = (10+2\times5)\mathbf{i} + (3+4\times5)\mathbf{j} + (5+4\times5)\mathbf{k}$$

$$\mathbf{r}(5) = 20\mathbf{i} + 23\mathbf{j} + 25\mathbf{k}$$

Distance from origin after 5 seconds = $|\mathbf{r}(5)|$

$$|\mathbf{r}(5)| = \sqrt{20^2+23^2+25^2} = 39.4208\text{ m}$$

Example 19

A particle P leaves the origin and moves with constant velocity $(2\mathbf{i}+6\mathbf{j})$ m s^{-1} while particle Q starts from the origin with initial velocity $(4\mathbf{i}-8\mathbf{j})$ m s^{-1} and moves with acceleration $(\mathbf{i}+\mathbf{j})$ m s^{-2}. Find the:

(a) position vectors of P and Q after 2 seconds.

(b) time at which the velocities of the two particles are perpendicular to each other.

(c) time at which the point R with position vector $17\mathbf{i} + 25\mathbf{j}$ lies on the line joining the positions of the particles P and Q.

Solution

	Particle P	Particle Q
Initial velocity	$2\mathbf{i} + 6\mathbf{j}$	$4\mathbf{i}-8\mathbf{j}$
Acceleration	$\mathbf{0}$	$\mathbf{i} + \mathbf{j}$
Position vectors	$\mathbf{r}_P$	$\mathbf{r}_Q$

(a) $\mathbf{r}_P(t) = \mathbf{r}_P(0) + \mathbf{s}_P$

$$\Rightarrow \mathbf{r}_P(t) = (2\mathbf{i}+6\mathbf{j})t = (2t\mathbf{i}+6t\mathbf{j})$$

When t = 2 $\Rightarrow \mathbf{r}_P(2) = (2\times2)\mathbf{i} + (6\times2)\mathbf{j}$

$$= 4\mathbf{i} + 12\mathbf{j}$$

$$\mathbf{r}_Q(t) = \mathbf{r}_Q(0) + \mathbf{s}_Q$$

$$\Rightarrow \mathbf{r}_Q(t) = (4\mathbf{i}-8\mathbf{j})t + \frac{1}{2}(\mathbf{i}+\mathbf{j})t^2$$

$$\mathbf{r}_Q(t) = \left(4t+\frac{1}{2}t^2\right)\mathbf{i} + \left(\frac{1}{2}t^2-8t\right)\mathbf{j}$$

When t = 2

$$\Rightarrow \mathbf{r}_Q(2) = \left(4\times2+\frac{1}{2}\times2^2\right)\mathbf{i} + \left(\frac{1}{2}\times2^2-8\times2\right)\mathbf{j}$$

Hence $\mathbf{r}_Q(2) = 10\mathbf{i}-14\mathbf{j}$

(b) From $\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$$\mathbf{v}_P = 2\mathbf{i} + 6\mathbf{j}$$

$$\mathbf{v}_Q = (4\mathbf{i}-8\mathbf{j}) + (\mathbf{i} + \mathbf{j})t$$

$$= (4 + t)\mathbf{i} + (t-8)\mathbf{j}$$

When the velocities of the two particles are perpendicular:

$$\mathbf{v}_P\bullet\mathbf{v}_Q = 0$$

$$\Rightarrow (2\mathbf{i} + 6\mathbf{j})\bullet[(4 + t)\mathbf{i} + (t-8)\mathbf{j}] = 0$$

$$2(4+t) + 6(t-8) = 0$$

$$\Rightarrow 8 + 2t + 6t-48 = 0 \Rightarrow t = 5\text{ s}$$

(c) If R lies on PQ, then $\vec{PQ} = h\vec{RQ}$, h being a constant

$$\vec{PQ} = \mathbf{r}_Q(t)-\mathbf{r}_P(t)$$

$$= \left[\left(4t + \frac{1}{2}t^2\right)\mathbf{i} + \left(\frac{1}{2}t^2-8t\right)\mathbf{j}-(2t\mathbf{i} + 6t\mathbf{j})\right]$$

$$= \left(2t + \frac{1}{2}t^2\right)\mathbf{i} + \left(\frac{1}{2}t^2-14t\right)\mathbf{j}$$

$$\vec{RQ} = \mathbf{r}_Q-\vec{OR}$$

$$= \left[\left(4t+\frac{1}{2}t^2\right)\mathbf{i}+\left(\frac{1}{2}t^2-8t\right)\mathbf{j}\right]-[17\mathbf{i} + 25\mathbf{j}]$$

$$= \left(\frac{1}{2}t^2+4t-17\right)\mathbf{i} + \left(\frac{1}{2}t^2-8t-25\right)\mathbf{j}$$

From $\vec{PQ} = h\vec{RQ}$

$$\left(2t+\frac{1}{2}t^2\right)\mathbf{i} + \left(\frac{1}{2}t^2-14t\right)\mathbf{j}$$

$$= h\left[\left(\frac{1}{2}t^2+4t-17\right)i+\left(\frac{1}{2}t^2-8t-25\right)j\right]$$

For i: $2t + \frac{1}{2}t^2 = h(\frac{1}{2}t^2 + 4t-17)$…………(i)

For j: $\frac{1}{2}t^2-14t = h(\frac{1}{2}t^2-8t-25)$…………(ii)

Dividing Eqn (i) by Eqn (ii) and simplifying:

$$t^2 + 18t-144 = 0 \Rightarrow (t+9)^2 = 225$$

$$\Rightarrow t = -9 \pm 15$$

Either t = 6 or t = -24

Hence t = 6 s

Example 20

The acceleration of a particle is -10**j**. If the particle starts at (0, 80) with a velocity 15**i**.

(i) Find the velocity at time t = 5.

(ii) Given that at time T, the particle is at (x,0). Calculate the values of x and T.

Solution:

$$\mathbf{a} = -10\mathbf{j},\ \mathbf{r}(0) = 80\mathbf{j},\ \mathbf{u} = 15\mathbf{i}$$

(i) From $\mathbf{v} = \mathbf{u} + \mathbf{a}t$

When t = 5, $\mathbf{v} = 15\mathbf{i} + (-10\mathbf{j})\times 5$

$$\Rightarrow \mathbf{v} = (15\mathbf{i}-50\mathbf{j})$$

(ii) $\mathbf{r}(t) = \mathbf{r}(0) + \mathbf{s}$

$$\Rightarrow \mathbf{r}(t) = 80\mathbf{j} + (15\mathbf{i})t + \frac{1}{2}\times(-10\mathbf{j})t^2$$

$\therefore\ \mathbf{r}(t) = 15t\mathbf{i} + (80-5t^2)\mathbf{j}$

When t = T, $\mathbf{r}(T) = x\mathbf{i} + 0\mathbf{j}$

$\therefore\ \mathbf{r}(T) = 15T\mathbf{i} + (80-5T^2)\mathbf{j} = x\mathbf{i} + 0\mathbf{j}$

$\Rightarrow 80-5T^2 = 0 \Rightarrow T = 4$ s

$\therefore\ x = 5T \Rightarrow x = 5\times 4 = 20$ m

Exercises

Exercise: 2A

1. A motor cyclist decelerated uniformly from 20 km h^{-1} to 8 km h^{-1} covering 896 m. Find the deceleration in m s^{-2}.
2. A boy and a man start to run from the same point at the same instant and in the same direction. The man runs with a constant speed of 10 m s^{-1}, the boy's initial speed is 2 m s^{-1} and accelerates uniformly at 2 m s^{-2}. Calculate the:
   (i) greatest distance by which the man gets ahead of the boy.
   (ii) speed with which the boy overtakes the man.
3. A car moves from Kampala to Jinja and back. Its average speed on the return journey is 4 km h^{-1} greater than that of the outward journey and it takes 12 minutes less. Given that Kampala and Jinja are 80 km apart, find the average speed on the outward journey.
4. A particle P starts from a point A with an initial velocity of 2 m s^{-1} and travels along a straight line with a constant acceleration of 2 m s^{-2}. Two seconds later a second particle Q starts from rest at A and travels along the same line with an acceleration of 6 m s^{-2}. Find how far from A, Q overtakes P.
5. A particle moving in a straight line with constant acceleration passes in succession through A, B, C. The time taken from A to B is t_1 and from B to C is t_2, AB = a, BC = b. Prove that the acceleration of the particle is $\frac{2(bt_1-at_2)}{t_1t_2(t_1+t_2)}$.
6. A particle traveling in a straight line with constant acceleration covers 4•5 m and 6 m in the third and fourth seconds of its motion respectively. Find its acceleration and initial velocity.
7. P, Q and R are points on a straight road such that PQ = 20 m and QR = 55 m. A cyclist moving with uniform acceleration passes P and then notices that it takes him 10 s and 15 s to travel between (P and Q) and (Q and R) respectively. Find his uniform acceleration.
8. After passing a police car at 60 km h^{-1}, a sports car retarded immediately until its speed was 40 km h^{-1} and it maintained this speed until it was caught by the police car 1 km from the check point. Given that the police car started immediately from rest at the instant it was passed by the sports car with a constant acceleration and that the time taken by the sports car in slowing from 60 km h^{-1} to 40 km h^{-1} was equal to the time it traveled at a constant speed until it was caught by the police car, find the:
   (i) time taken by the police car to reach the sports car.
   (ii) speed of the police car at the instant it caught the sports car.
   (iii) time when the speeds of the cars were

equal measured from the check point.

9. A stone is dropped from the top of a tower. After one second another stone is thrown vertically downwards from the same point at a speed of 15 m s^{-1}. If the stones reach the ground simultaneously, find the height of the tower.

10. In a motor rally, car A is 1 km from the finishing point and has an initial speed of 35 m s^{-1}, moving with an acceleration of 0•4 m s^{-2}. At the same instant a second car B is 200 m behind A and is traveling at 44 m s^{-1} and accelerates uniformly at 0•5 m s^{-2}. Find the distance car B passes car A from the finishing point.

11. A sports car moving with a uniform speed of 80 km h^{-1} passes a police patrol car at rest. After 5 seconds, the police patrol car takes off in pursuit accelerating uniformly at 10 m s^{-2}. Find:
(a) when the police patrol car catches up with the sports car.
(b) where the catch up takes place from the original police car station.

12. A train was timed between three points A, B and C. It took 100 seconds to move from A to B and 150 seconds to move from B to C. Given that $AB = BC = 2$ km, find how far beyond C the train stops.

13. A, B and C are three points which lie in that order on a straight road with AB = 95 m and BC = 80 m. A car travels along the road in the direction ABC with constant acceleration a m s^{-2}. The car passes through A with speed u m s^{-2}, reaches B five seconds later and C two seconds after that. Find the values of u and a.

14. A stone is projected vertically upwards from ground level at a speed of 24•5 m s^{-1}. Find how long after projection the stone is at a height of 19•6 m above the ground:
(a) for the first time.
(b) for the second time.
(c) For how long is the stone at least 19•6 m above ground level.

15. A particle moving in a straight line with uniform acceleration is found to cover 25•4 m in its first two seconds. During the 5^{th} second it covers a distance of 28•9 m, find the initial speed and the acceleration.

Exercise: 2B

1. (a) A train starts from rest and accelerates uniformly at 0•5 m s^{-2} for 5 minutes, it moves uniformly for the next 20 minutes and is retarded uniformly at 2 m s^{-2} to rest. Find the total distance covered by the train and its average speed.
(b) Points P, Q and R lie in that order on a straight road such that $PQ = QR = 40$m. A cyclist moving with uniform acceleration passes P and then notices that it takes him 10 seconds and 15 seconds to travel between P and Q, and P and R respectively. Find:
(i) his acceleration.
(ii) the velocity with which he passes P.

2. A train is traveling at 108 km h^{-1} when the brakes are applied, producing a retardation 3f m s^{-2}. When the speed has been reduced to 54 km h^{-1}, the train is uniformly accelerated at f m s^{-2} until a speed of 108 km h^{-1} is again reached. From the instant the brakes are applied to the instant when the speed again reaches 108 km h^{-1}, the time taken is 6 minutes and in this time the train travels 8 km.
(a) Sketch a velocity-time graph for the motion.
(b) Find the:
(i) value of f
(ii) distance traveled at 54 km h^{-1}

3. A train stops at two stations which are 24 km apart. It takes 3 minutes to accelerate uniformly to a speed of 40 m s^{-1} then maintains this speed until it comes to rest with uniform retardation in a distance of 1200 m. Find the time taken for the journey.

4. The cage of a pit performs the first part of its descent with uniform acceleration f and the remainder with uniform retardation 2f. Prove that if h is the depth of the shaft and t is the time of descent, $h = \frac{1}{3}ft^2$.

5. A train travels between stations A and B in time T. It accelerates uniformly from rest for a time αT and retards uniformly to rest at the end of the journey for a time βT. During the intermediate time, it travels with speed v.

Prove that the average speed for the whole journey is $\frac{1}{2}v(2-\alpha-\beta)$.

6. A motor cyclist passes a point A with a speed, v. Ten seconds later he brakes and the motor cycle decelerates uniformly for 5 s reducing its speed to 30 m s^{-1}. If he covers 600 m in the 15 s, find the:
   (i) value of v.
   (ii) deceleration of the motor cycle.
7. A vehicle traveling on a straight horizontal track joining two points A and B accelerates at a constant rate of 0•25 m s^{-2} and decelerates at a constant rate of 1 m s^{-2}. It covers a distance of 2 km from A to B by accelerating from rest to a speed of v m s^{-1} and traveling at that speed until it starts to decelerate to rest. Express in terms of v the time taken for the acceleration and deceleration. Given that the total time for the journey is 2•5 minutes, find a quadratic equation for v and determine v, explaining clearly the reason for your choice of the value of v.

Exercise: 2C

1. A particle with position vector 40**i** + 10**j** + 20**k** moves with constant speed 5 m s^{-1} in the direction of the vector 4**i** + 7**j** + 4**k**. Find its distance from the origin after 9 seconds.
2. The forces 4**i**−7**j**, 6**i**−11**j**, -2**i** + **j** and 4**i** + 13**j** start to act on a particle of mass $\frac{1}{2}$ kg, initially at (2,−4) and moving with velocity $(3i-4j)$ m s^{-1}. Find:
   (i) the acceleration and hence velocity after 4 seconds.
   (ii) its displacement after 4 seconds.
   (iii) the work done by the force in 4 seconds.
3. A particle starts from a point with position vector $(i+7j)$ m with initial velocity of $(4i+3j)$ m s^{-1} and accelerates uniformly at $(i+2j)$ m s^{-2}. Find its:
   (i) velocity at any time t.
   (ii) position vector at any time t.
4. A particle starts with velocity -4**i** + 5**j** m s^{-1} and accelerates at 6**i**−10**j** m s^{-2}. Find after how long the speed of the particle is 17 m s^{-1}.
5. A force of 4**i**−2**j** + 6**k** newtons acts on a particle of mass 2 kg resting at a point whose position vector is 4**i** + 3**j**−10**k**. Find the position vector of the particle after 5 seconds.
6. An object of mass 5 kg is initially at rest at a point whose position vector is -2**i** + **j**. If it is acted on by a force, **F** = 2**i** + 3**j**−4**k**, find its:
   (i) acceleration.
   (ii) velocity after 3 s.
   (iii) distance from the origin after 3 s.
7. Two forces $F_1 = (3i-j+k)$ N and $F_2 = (2i+5j-5k)$ N act on a particle of mass 2 kg initially at rest. Calculate the magnitude of the particle's acceleration.
8. A particle has an initial position vector of $(7i+5j)$ m. The particle moves with constant velocity of $(ai+bj)$ m s^{-1} and after 3 seconds has a position vector of $(10i-j)$ m. Find the values of a and b.
9. A particle has an initial position vector of $(4i+3j+9k)$ m. The particle moves with constant velocity of $(3i-2j-5k)$ m s^{-1}. Find the:
   (a) position vector of the particle at time t.
   (b) position vector of the particle after 5 seconds.
   How far is the particle from the origin after 5 seconds.
10. A particle has an initial position vector of $(ai+bj+ck)$ m. The particle moves with constant velocity of $(3i+j+4k)$ m s^{-1} and after 2 seconds has a position vector $(7i+j+11k)$ m.
    (a) Find the values of a,b and c.
    (b) How far is the particle from the origin after 3 seconds.
11. At time t = 0 two particles A and B have position vectors $(2i+3j-4k)$ m and $(8i+6k)$ m respectively. A moves with a constant velocity $(-i+3j+5k)$ m s^{-1} and B with constant velocity v m s^{-1}. Given that when t = 5 seconds B passes through the point that A passed through one second earlier, find v.
12. The initial velocity of a particle moving with constant acceleration is $(3i-5j)$ m s^{-1}. After 2 seconds the velocity of the particle is of magnitude $6\sqrt{2}$ m s^{-1} parallel to (**i** + **j**). Find the acceleration of the particle.
13. A particle starts from the origin with initial

velocity $4\mathbf{i}-8\mathbf{j}$ and moves with an acceleration of $\mathbf{i}+\mathbf{j}$. Show that the particle will pass through the point $(24,-24)$ and find the time when it passes through this point.

Answers to exercises

Exercise: 2A

1. $0{\cdot}0145$ m s^{-2} 2. (i) 16 m (ii) 18 m s^{-1} 3. $38{\cdot}05$ km h^{-1} 4. 48 m

5. 6. $\frac{3}{2}$ m s^{-2} ; $\frac{3}{4}$ m s^{-1} 7. $\frac{2}{15}$ m s^{-2}

8. (i) 80 s (ii) 90 km h^{-1} (iii) $\frac{2}{195}$ hours

9. $18{\cdot}4855$ m 10. 220 m 11. (a) $7{\cdot}4338$ s

(b) $276{\cdot}3069$ m 12. $816\frac{2}{3}$ m 13. 4 ; 6 14.

(a) 1 s (b) 4 s (c) 3 s

15. $8{\cdot}0714$ m s^{-1} ; $4{\cdot}6286$ m s^{-2}

Exercise: 2B

1. (a) 208 125 m ; 132.143 m s^{-1} (b)(i) $\frac{8}{15}$m s^{-2}

(ii) $\frac{4}{3}$m s^{-1}

2. (a) (b) (i) $\frac{3}{52}$ (ii) 200m **3.** 12 min. **4.** 5.

6. (i) 42 (ii) 2.4 m s^{-2} 7. $t_1 = 4v$; $t_3 = v$;

$v^2 - 60v + 800 = 0$; $v = 20$

Since v = 40 gives negative value of time for constant speed.

Exercise: 2C

1. 85 m 2. (i) $(24\mathrm{i}-8\mathrm{j})$ m s^{-2};$(99\mathrm{i}-36\mathrm{j})$ m s^{-1} (ii) $(204\mathrm{i}-80\mathrm{j})$ m (iii) 2768 J

3. (i) $[(4+t)\mathrm{i}+(3+2t)\mathrm{j}]$ m s^{-1} (ii) $\left[\left(1+4t+\frac{1}{2}t^2\right)\mathrm{i}+\left(7+3t+t^2\right)\mathrm{j}\right]$m

4. 2 s 5. $29\mathbf{i}-9{\cdot}5\mathbf{j}+27{\cdot}5\mathbf{k}$

6. (i) $\frac{1}{5}(2\mathrm{i}+3\mathrm{j}-4\mathrm{k})$ m s^{-2}

(ii) $\frac{1}{5}(6\mathrm{i}+9\mathrm{j}-12\mathrm{k})$ m s^{-1} (iii) $5{\cdot}1662$ m 7. $3{\cdot}775$ m s^{-2} 8. 1 ; -2

9. (a) $(4+3t)\mathbf{i} + (3-2t)\mathbf{j} + (9-5t)\mathbf{k}$ (b) $(19\mathrm{i}-7\mathrm{j}-16\mathrm{k})$ m ; $3\sqrt{74}$

10. (a) 1 ; -1 ; 3 (b) $\sqrt{329}$ m

11. $(-2\mathrm{i}+3\mathrm{j}+2\mathrm{k})$ 12. $\frac{1}{2}(3\mathrm{i}+11\mathrm{j})$ m s^{-2}

13. $t = 4$

3. FORCE AND NEWTON'S LAWS

3.1 Force

A force is necessary to cause a body to accelerate. More than one force may act on a body. If forces on a body are in equilibrium, then the body may be at rest or moving in a straight line at constant speed. A resultant force on a body causes it to accelerate.

3.2 Newton's Laws of Motion

Newton's three laws of motion are the fundamental basis of study of A-level mechanics.

1st Law: Every body will remain at rest or continue to move in a straight line at constant speed unless an external force acts on it.

Consequences:

(i) If a body has an acceleration, then there must be a force acting on it.
(ii) If a body has no acceleration, then the forces acting on it must be in equilibrium.

2nd Law: The rate of change of momentum of a moving body is proportional to the resultant force acting on it and takes place in the direction of the force.

So when an external force acts on a body of constant mass, the force produces an acceleration which is directly proportional to the force.

Consequences:

(i) The basic equation of motion for a constant mass is:

Force = Mass × Acceleration

$F = ma$

(ii) The force and acceleration of a body are both in the same direction.
(iii) A constant force acting on a constant mass gives it a constant acceleration.

3rd Law: If a body A exerts a force on a body B, then B exerts an equal and opposite force on A.

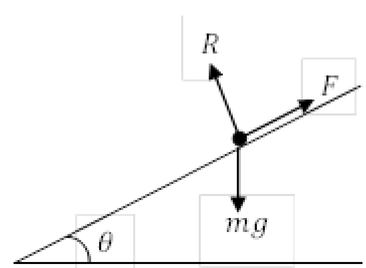
Consequence:
These forces between bodies are often called reactions. In a rigid body the internal forces occur as equal and opposite pairs and the net effect is zero. So only external forces need to be considered.

Problem Solving:
The following are important points to remember when solving problems using Newton's laws of motion:

1. Draw a clear force diagram.
2. If there is no acceleration, that is, when the body is either at rest or moving with uniform velocity, then the forces balance in each direction.
3. If there is an acceleration:
   (i) Mark it on the diagram using $\xrightarrow{a}$
   (ii) Write down, if possible, an expression for the resultant force.
   (iii) Use Newton's 2nd law, that is, write down an equation of motion:
   Force = mass × acceleration
   $F = ma$

Some common cases:

1. A body at rest on a rough inclined plane.

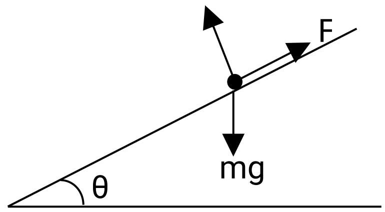


Since there is no acceleration the forces balance:

Resolving along plane:
$F = mg\sin\theta$

Resolving normal to plane:
$R = mg\cos\theta$

2. A body sliding down a rough plane at constant speed.



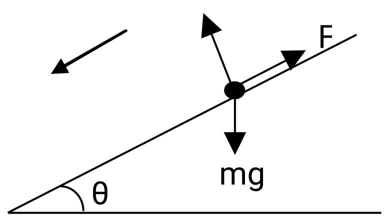
Since there is no acceleration the forces balance:
Resolving along plane:

$$F = mg\sin\theta$$

Resolving normal to plane:

$$R = mg\cos\theta$$

3. **A body sliding down a rough plane with acceleration.**



Resolving normal to the plane:

$$R = mg\cos\theta$$

Applying Newton's 2nd law to motion down the plane:

$$mg\sin\theta - F = ma$$

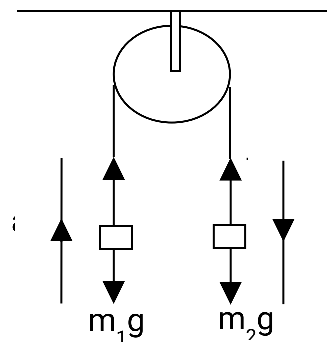
3.3 Connected particles

Two particles connected by a light inextensible string which passes over a frictionless pulley are called connected particles. The tension in the string is the same throughout its length, so each particle is acted upon by the same tension.

Problems concerned with connected particles usually involve finding acceleration of the system and the tension in the string.
To solve problems of this kind:

1. Draw a clear diagram showing the forces on each particle and the common acceleration.
2. Write the equation of motion for each particle using $F = ma$.
3. Solve the two equations to find the common acceleration a and tensing T in the string.

Example 1
Two particles of mass m_1 and m_2, with $m_2 > m_1$ are connected by a light inextensible string which passes over a fixed smooth pulley. Find the common acceleration a and the tension T in the string when the system is moving freely.



For m_1: $T - m_1g = m_1a$ ………………… (i)

For m_2: $m_2g - T = m_2a$ ……………….. (ii)

Adding equation (i) and equation (ii)

$$m_2g - m_1g = m_1a + m_2a$$

$$a = \left(\frac{m_2 - m_1}{m_1 + m_2}\right)g$$

From equation (i):

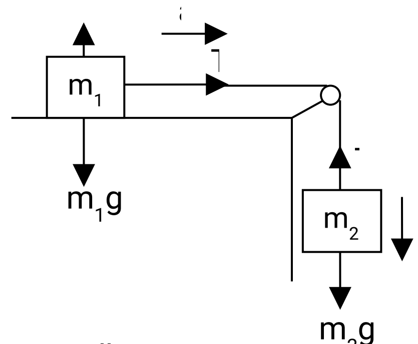
$$T = m_1(a + g) = m_1g\left(\frac{m_2 - m_1}{m_1 + m_2} + 1\right)$$

$$T = \frac{2m_1m_2g}{(m_1 + m_2)}$$

Common situations:

There are other situations involving motion of connected particles, the above case being the simplest.

1. **One particle on a smooth horizontal table.**



For m_1:

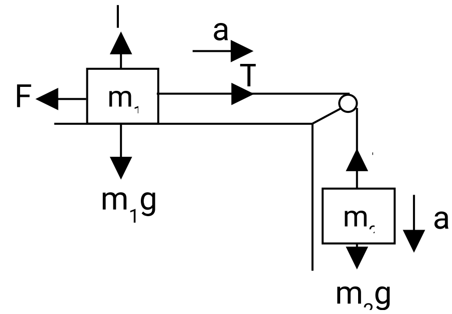
Resolving vertically: $R = m_1g$

Equation of motion: $T = m_1a$

For m_2:

Equation of motion: $m_2g - T = m_2a$

2. **One particle on a rough horizontal table.**



For m_1:

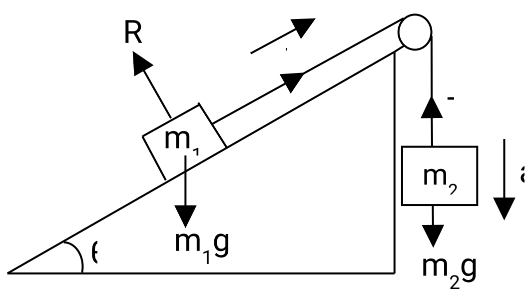
Resolving vertically: $R = m_1g$

Equation of motion: $T - F = m_1a$

For m_2:

Equation of motion: $m_2g - T = m_2a$

3. **One particle on a smooth inclined plane:**



For m_1:

Resolving perpendicular to plane:

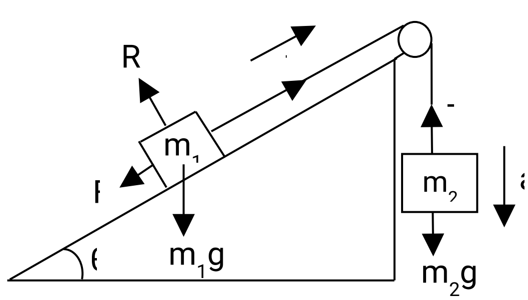
$R = m_1g\cos\theta$

Equation of motion: $T - m_1g\sin\theta = m_1a$

For m_2:

Equation of motion: $m_2g - T = m_2a$

4. **One particle on a rough inclined plane:**



For m_1:

Resolving perpendicular to plane: $R = m_1g\cos\theta$

Equation of motion: $T - F - m_1g\sin\theta = m_1a$

For m_2:

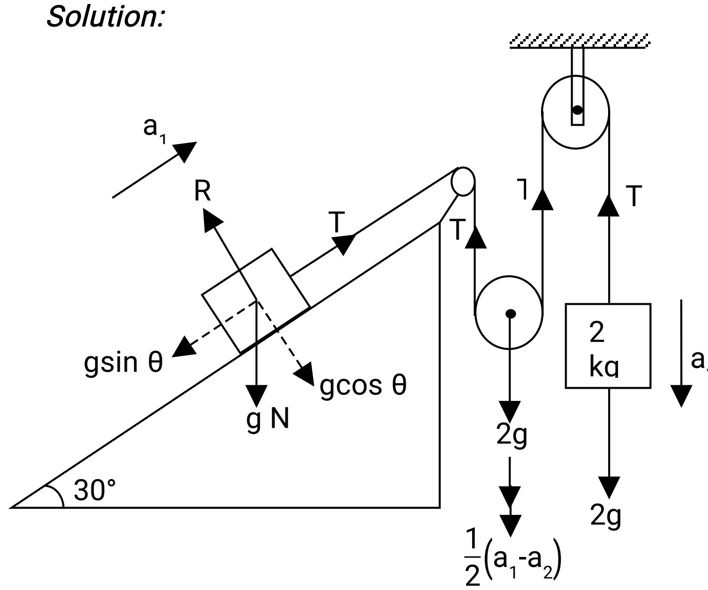
Equation of motion: $m_2g - T = m_2a$

Example 2

To one end of a light inextensible string is attached a mass of 1 kg which rests on a smooth wedge of inclination 30°. The string passes over a smooth fixed pulley at the edge of the wedge, under a second smooth movable pulley of mass 2 kg and over a third smooth fixed pulley and has a mass of 2 kg attached to the other end. Find the accelerations of the masses and the movable pulley and the tension in the string.

(Assume the portions of the string lie in the vertical plane.)

Solution:



For 1 kg mass:

$T - g\sin 30 = 1\times a_1$

$\Rightarrow T - \frac{1}{2}g = a_1$ .... (i)

For movable pulley:

$2g - 2T = 2\times\frac{1}{2}(a_1 - a_2$

$) \Rightarrow 2g - 2T = a_1 - a_2$ .... (ii)

For 2 kg mass: $2g - T = 2a_2$ .... (iii)

Adding equation (i) and equation (iii)

$\frac{3}{2}g = a_1 + 2a_2$ .... (iv)

Adding 2× equation (i) and equation (ii)

$g = 3a_1 - a_2$ .... (v)

2× equation (v)+ equation (iv)

$\frac{7}{2}g = 7a_1$

$a_1 = \frac{1}{2}g = \frac{1}{2}\times 9\cdot 8 = 4\cdot 9\ m\ s^{-2}$

From equation (iv) $\frac{3}{2}g = \frac{1}{2}\times g + 2a_2$

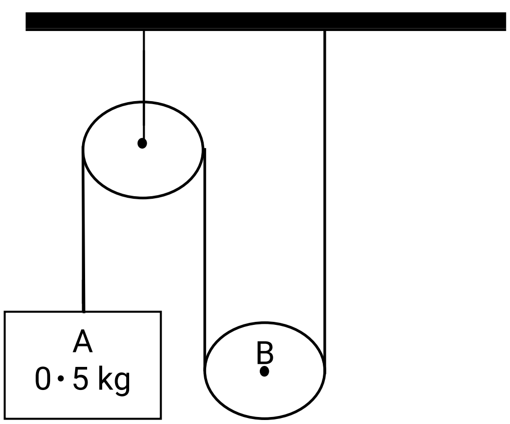
$$\Rightarrow a_2 = \frac{1}{2}g = \frac{1}{2}\times 9\cdot 8 = 4\cdot 9 \text{ m s}^{-2}$$

Accelerations:
1 kg mass : $4\cdot 9$ m s^{-2}
Movable pulley: $\frac{1}{2}(4\cdot 9 - 4\cdot 9) = 0$ m s^{-2}
2 kg mass : $4\cdot 9$ m s^{-2}

From (i):
$T - \frac{1}{2}g = \frac{1}{2}g \quad \therefore \quad T = g \Rightarrow T = 9\cdot 8$ N

Example 3

The diagram shows a particle A of mass $0\cdot 5$ kg attached to the end of a light inextensible string passing over a fixed light pulley and under a light movable pulley B. The other end of the string is fixed as shown below.

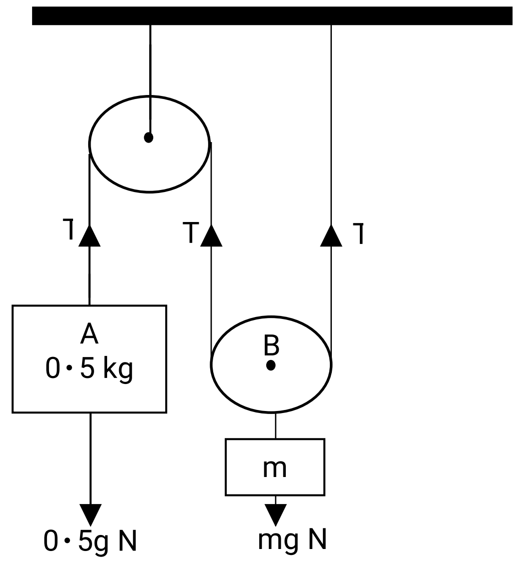


(i) What mass should be attached to B for the system to be in equilibrium?

(ii) If B is $0\cdot 8$ kg, what are the accelerations of particle A and pulley B?

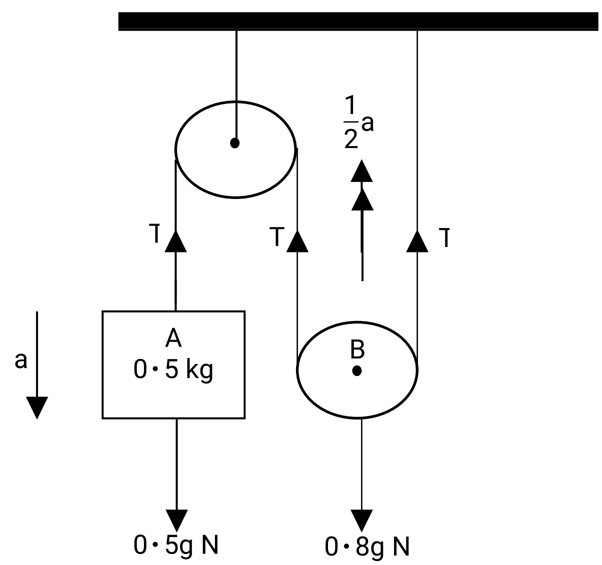
Solution:

(i)



For A: $T = 0\cdot 5g = 0\cdot 5\times 9\cdot 8 = 4\cdot 9$ N
For B: $2T = mg \Rightarrow 2\times 4\cdot 9 = m\times 9\cdot 8$
$\Rightarrow m = 1$ kg

(ii)



For A: $0\cdot 5g - T = 0\cdot 5a$……………………(iii)

For B: $2T - 0\cdot 8g = 0\cdot 8\times\frac{1}{2}a$

$\Rightarrow T - 0\cdot 4g = 0\cdot 2a$……………(iv)

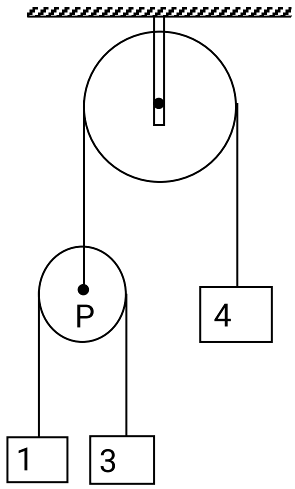
Adding equation (iii) and equation (iv)

$0\cdot 1g = 0\cdot 7a \Rightarrow a = \frac{1}{7}g = \frac{1}{7}\times 9\cdot 8 = 1\cdot 4$ m s^{-2}

Accelerations: A:$1\cdot 4$ m s^{-2} ; B: $0\cdot 7$ m s^{-2}

Example 4

The diagram shows a fixed pulley carrying a string which has a mass of 4 kg attached at one end and a light pulley, P attached at the other end. Another string passes over pulley P and carries a mass of 3 kg at one end and a mass of 1 kg at the other end.



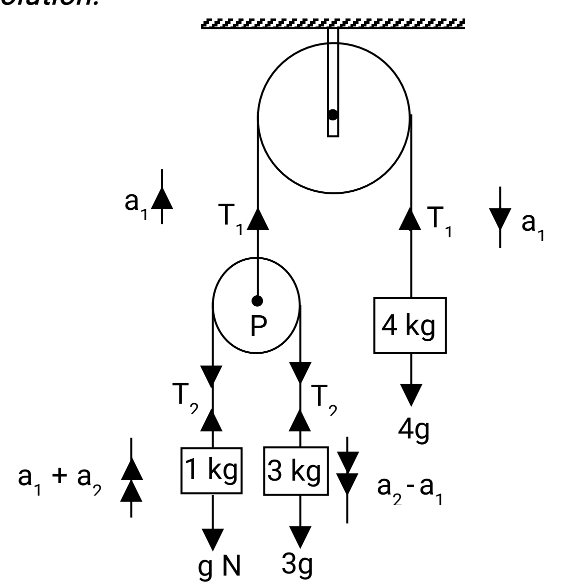
Find the:

(a) acceleration of pulley P.

(b) acceleration of the 1 kg, 3 kg and 4 kg masses.

(c) tensions in the strings.

Solution:



For the 4 kg mass:

$$4g - T_1 = 4a_1 \text{............(i)}$$

For pulley P:

$T_1 = 2T_2$

Substituting in equation (i)

$4g - 2T_2 = 4a_1 \Rightarrow 2g - T_2 = 2a_1$............(ii)

For the 1 kg mass:

$$T_2 - g = a_1 + a_2 \text{............(iii)}$$

For the 3 kg mass:

$$3g - T_2 = 3(a_2 - a_1) \text{............(iv)}$$

Adding equation (ii) and equation (iii)

$3a_1 + a_2 = g \Rightarrow a_2 = g - 3a_1$............(v)

Adding equation (iii) and equation (iv)

$2g = 4a_2 - 2a_1 \Rightarrow g = 2a_2 - a_1$............(vi)

From equation (v) and equation (vi)

$g = 2(g - 3a_1) - a_1 \Rightarrow 7a_1 = g \Rightarrow a_1 = 1{\cdot}4 \text{ m s}^{-2}$

From equation (v)

$a_2 = 9{\cdot}8 - 3 \times 1{\cdot}4 \Rightarrow a_2 = 5{\cdot}6 \text{ m s}^{-2}$

(a) acceleration of pulley P is $1{\cdot}4 \text{ m s}^{-2}$ (upwards).

(b) accelerations:

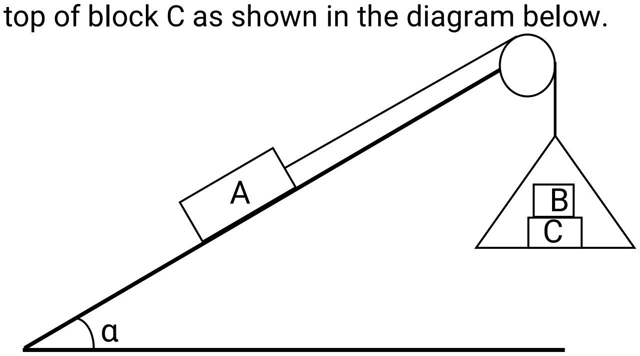
1 kg mass: 7 m s^{-2}(upwards); 3 kg mass: $4{\cdot}2 \text{ m s}^{-2}$(downwards);
4 kg mass: $1{\cdot}4 \text{ m s}^{-2}$(downwards).

(c) Tensions:

From equation (i):
$T_1 = 4(9{\cdot}8 - 1{\cdot}4) \Rightarrow T_1 = 33{\cdot}6 \text{ N}$
$T_1 = 2T_2 \Rightarrow 33{\cdot}6 = 2T_2 \Rightarrow T_2 = 16{\cdot}8 \text{ N}$

Example 5

One end of a light inextensible string is attached to a block A of mass 5 kg. The block A is held at rest on a smooth fixed plane which is inclined at an angle α to the horizontal, where $\sin\alpha = \frac{3}{5}$. The string is parallel to the line of greatest slope of the plane and passes over a smooth light pulley which is fixed at the top of the plane. The other end of the string is attached to a light scale pan which carries two blocks B and C, with block B on top of block C as shown in the diagram below.



The mass of B is 5 kg and the mass of C is 10 kg. The scale pan hangs at rest and the system is released from rest. Find the:

(a) (i) acceleration of the system.

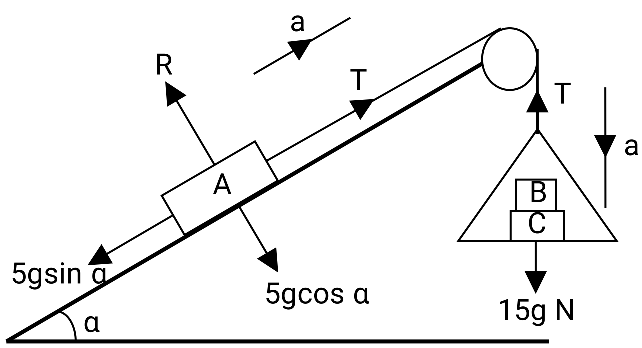
(ii) tension in the string.

(b) (i) magnitude of the force exerted on block B by block C.

(ii) magnitude of the force exerted on block C by the scale pan.

(c) magnitude of the force exerted on the pulley by the string.

Solution



(a) For scale pan:

$$15g - T = 15a \ldots\ldots\ldots\ldots\text{(i)}$$

For block A:

$$T - 5g\sin\alpha = 5a \Rightarrow T - 5g\times\frac{3}{5} = 5a$$

$$\Rightarrow T - 3g = 5a \ldots\ldots\ldots\ldots\text{(ii)}$$

(i) Adding equation (i) and equation (ii)

$$12g = 20a \Rightarrow a = \frac{3}{5}\times 9{\cdot}8$$

$$\Rightarrow a = 5{\cdot}88 \text{ m s}^{-2}$$

(ii) From equation (i)

$$T = 15(9{\cdot}8 - 5{\cdot}88) \Rightarrow T = 58{\cdot}8 \text{ N}$$

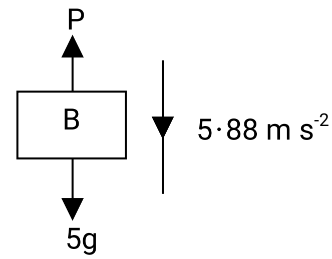
(b) (i) Let P be the magnitude of the reaction between B and C:

For B:

$5g - P = 5\times 5{\cdot}88$

$5\times 9{\cdot}8 - P = 29{\cdot}4$

$P = 19{\cdot}6$ N



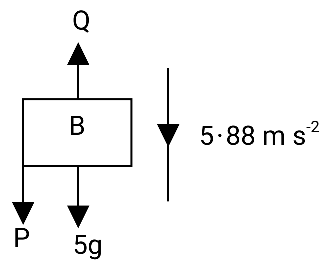
(ii) Let Q be the magnitude of the reaction between Q and the scale pan:

For C:

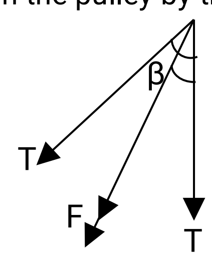
$10g + P - Q = 10\times 5{\cdot}88$

$10\times 9{\cdot}8 + 19{\cdot}6 - Q = 58{\cdot}8$

$Q = 58{\cdot}8$ N



(c) Let F be the magnitude of the force exerted on the pulley by the string



Where $\beta = 90° - \alpha$

$\sin\alpha = \frac{3}{5} \Rightarrow \alpha = 36{\cdot}9°$

$F = 2T\cos\left(\frac{90-\alpha}{2}\right)$

$F = 2\times 58{\cdot}8\cos 26{\cdot}565$

$F = 105{\cdot}2$ N

Exercises

Exercise: 3A

1. A car of mass M kg is towing a trailer of mass λM kg along a straight horizontal road. The tow-bar connecting the car and the trailer is of negligible mass and horizontal.

The resistive forces acting on the car and trailer are constant and of magnitude 300 N and 200 N respectively. At the instant when the car has an acceleration of magnitude 0·3 m s^{-2}, the tractive force has magnitude 2000 N.
Show that M(λ + 1) = 5000. Given that the tension in the tow-bar is 500 N at this same instant, find the value of M and the value of λ.

2. A car of mass 800 kg tows a trailer of mass 400 kg. The resistance to motion for both car and trailer is 0·2 N per kg. Find the tractive force exerted by the car engine and the tension in the tow bar when they are traveling with:
   (a) uniform velocity.
   (b) an acceleration of 2 m s^{-2}.

3. A light inextensible string attached to the ceiling passes under a smooth movable pulley of mass 6 kg and then over a smooth fixed pulley. A particle of mass 1 kg hangs freely from the end of the string. If the system is released from rest, find the acceleration of the particle and the tension in the string.

4. A body of mass 16 kg is released from rest on an inclined plane of inclination 1 in 40, the resistance to the motion being $\frac{1}{8}$N per kg.
   Calculate the:
   (i) acceleration of the body.
   (ii) speed of the body 6 seconds after release.

5. A vehicle of mass 2500 kg is drawn up a slope of 1 in 10 from rest with an acceleration of 1·2 m s^{-2} against a constant frictional resistance of $\frac{1}{100}$ of the weight of the vehicle, using a cable. Find the tension in the cable.

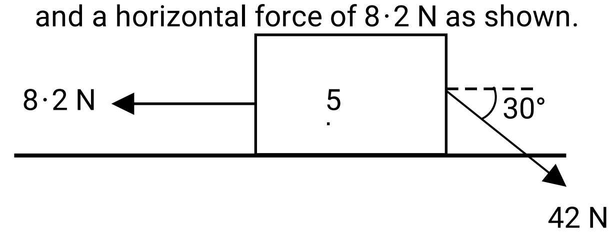
6. When a man of mass m uses a lift to go up and down in a building at a uniform acceleration, the reactions on the floor of the lift are 1185 N and 285 N respectively. Find the uniform acceleration and the mass of the man.

7. Two particles of mass 0·5 kg and 0·7 kg are connected by a light inextensible string passing over a smooth fixed pulley. Initially both parts of the string are taut and vertical, and the 0·5 kg mass is moving vertically downwards with a speed of 12·6 km h^{-1}. Find the distance it covers before coming to instantaneous rest. (Use g = 10 m s^{-2})

8. The engine of a train exerts a force of 35 000 N on a train of mass 240 tonnes and draw it up a slope of 1 in 120 against resistance totaling 60 N per tonne. Find the acceleration of the train.

9. A car of mass 2000 kg tows a truck of mass 1000 kg up a hill of 1 in 20 by a rope. The resistance due to friction on each vehicle is proportional to the mass of the vehicle. The engine of the car exerts a tractive force of 3600 N when traveling up the hill at a steady speed of 18 km h^{-1}.
   (i) Show that the tension in the rope is 1200 N.
   (ii) If the rope breaks and the two vehicles continue to move up the hill, calculate how far the truck travels before coming momentarily to rest.

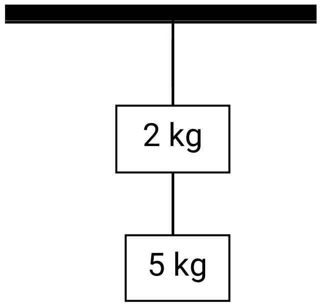
10. A body of mass 5 kg initially resting on a smooth horizontal surface is acted upon by two forces 42 N at 30° below the horizontal and a horizontal force of 8·2 N as shown.



Calculate the:
   (i) normal reaction on the body.
   (ii) acceleration of the body along the surface.

11. A train of mass 6 tonnes is traveling at 40 m s^{-1} when the brakes are applied. If the resultant braking force is 40 kN, find the distance the train travels before coming to rest.

12. The diagram shows a body of mass 5 kg hanging at rest at the end of a light vertical string. The other end of the string is attached to a mass of 2 kg which in turn hangs at the end of another light vertical string. Find the tension in each string.



Exercise: 3B

1. A mass of 3 kg is at rest on a smooth horizontal table. It is attached by a light inextensible string passing over a smooth fixed pulley at the edge of the table to another mass of 2 kg which is hanging freely. The system is released from rest. Determine the resulting acceleration and the tension in the string.

2. A particle of mass 2 kg moves under the action of a constant force $(2i+4j)$ N. At time t = 0 the particle is at rest at a point with position vector $(2i+5j)$ m. Find the position vector of the particle at t = 3 s.

3. Forces $(i-2j)$ N and $(3i+4j)$ N are applied to a body of mass 2 kg.
   (a) Find in vector form the:
      (i) resultant force acting on the body.
      (ii) acceleration of the body.
   (b) Initially the position vector of the body is $(2i-j)$ m and its initial velocity is $(4i+3j)$ m s^{-1}.
      (i) Show that after t seconds, the position vector of the body is
      $$\left[(t^2+4t+2)i+\left(\frac{1}{2}t^2+3t-1\right)j\right]\text{metres.}$$
      (ii) Find the value of t when the body's position vector is in the same direction as its acceleration.

4. A lift of mass 950 kg is carrying a boy of mass 50 kg.
   (a) The lift is ascending at a uniform speed. Calculate the:
      (i) tension in the lift cable.
      (ii) reaction of the boy on the floor of the lift.
   (b) If the lift ascends with retardation of 2 m s^{-2}, calculate the:
      (i) tension in the lift cable.
      (ii) reaction of the boy on the floor of the lift.

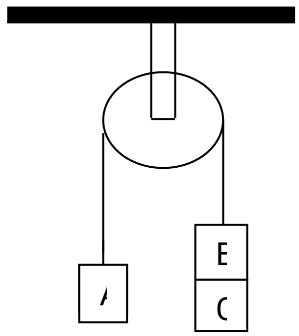
5. Two particles A and B of masses 4 kg and 5 kg respectively, are connected by a light inextensible string passing over a smooth fixed pulley. Initially B is 1·5 m above the ground. If the system is released from rest, find the:
   (i) acceleration of the masses.
   (ii) speed of each mass when the 5 kg mass hits the ground.
   (iii) further time and distance during which the 4 kg mass continues to rise, assuming it does not reach the pulley.

6. Find in vector form the acceleration produced in a body of mass 2 kg subjected to forces of $(2i-3j+4k)$ N and $(i+5j+2k)$ N.

7. A mass of 10 kg resting on a smooth horizontal table is connected by a light string passing over a smooth pulley at the edge of the table, to a mass of 7 kg hanging freely. Find the common acceleration, the tension in the string and force on the pulley.

8. A particle of mass m_1 is drawn up a smooth inclined plane of height h and length d by a string passing over the top of the plane, and supporting at the other end a mass m_2. If m_1 starts from rest at the bottom of the plane and m_2 is detached after m_1 has moved a distance x show that m_1 will just reach the top of the plane if $x = \frac{(m_1+m_2)}{m_2(d+h)}dh$.

9. The diagram shows a light inextensible string passing over a smooth fixed pulley, and carrying a particle A at one end and particles B and C at the other. The masses of A, B and C are 2m, m and 2m respectively.



Find the acceleration of the system when released from rest. After C has traveled 0·5 m it falls off and the system continues without it. Find:

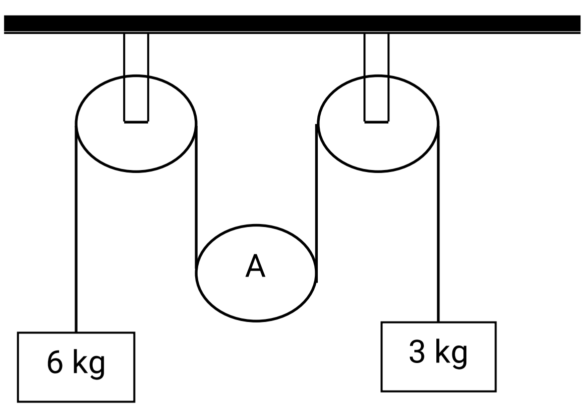
(a) the velocity of B at the instant C falls.

(b) how much further B travels down before it starts to rise.

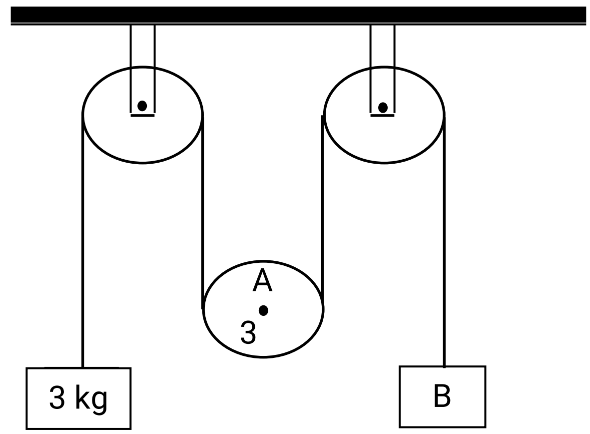
10. Three coplanar forces $(9i-2j)$ N, $(-3i+10j)$ N and $(ai-bj)$ N act on a mass of 5 kg causing it to accelerate at $(3i+j)$ m s^{-2}. Find the values of a and b.

Exercise: 3C

1. In the pulley system shown in the diagram, A is a heavy pulley which is free to move. Find the mass of pulley A if it does not move upwards or downwards when the system is released from rest.

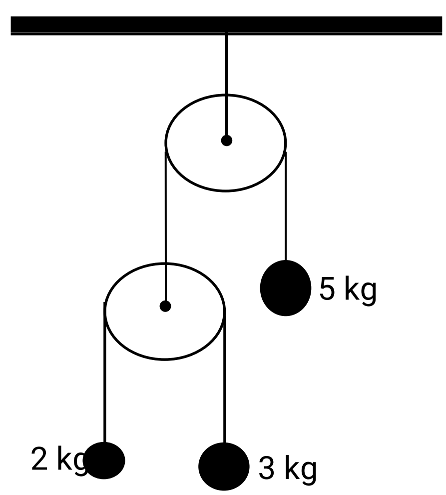


2. In the pulley system shown in the diagram, the pulley A is free to move. Find the mass of the load B if when the system is from rest, pulley A does not move upwards or downwards.



3. A light inextensible string which passes over a smooth fixed pulley P carries at one end a particle of mass 2 kg and at the other end a smooth light pulley Q. Particles A and B of masses 4 kg and 5 kg respectively are connected by a light inextensible string passing over pulley Q. Find the accelerations of particles A and B and the 2 kg mass when the system is moving freely. Find also the tensions in the two strings.

4. The figure shows a light inextensible string which passes over a smooth fixed pulley A and carries at one end a mass of 5 kg and at the other end a smooth light pulley B. A light inextensible string passes over pulley B and carries masses 3 kg and 2 kg at its ends. When the system is released from rest, find the acceleration of pulley B, and the masses. Find also the tension in each string.



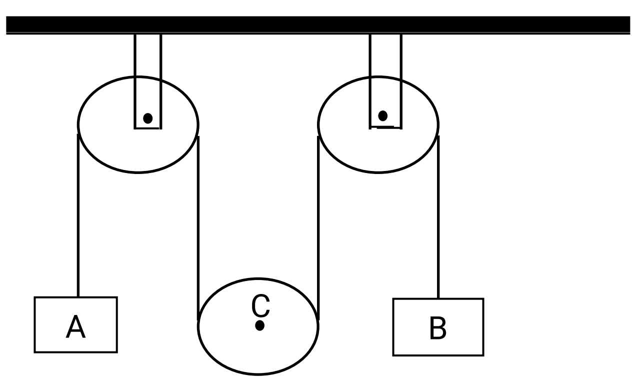
5. A light string passes over one fixed pulley, under a movable pulley of mass M and over

a second fixed pulley. A mass m_1 is attached to one end of the string, a mass m_2 to the other end. Assuming the parts of the string not in contact with the pulley to be vertical, prove that the tension in the string is $\frac{4m_1m_2Mg}{M(m_1+m_2)+4m_1m_2}$.

6. A mass of 2 kg resting on a smooth incline of 45° is connected by a light inextensible string passing over a smooth fixed pulley at the top of the incline to a smooth movable pulley of mass 0·5 kg. A second light inextensible string passing over the second smooth movable pulley has masses of 0·5 kg and 1 kg hanging at its free ends. Find the acceleration of the 1 kg, 2 kg masses and the tension in each string when the system is released from rest.

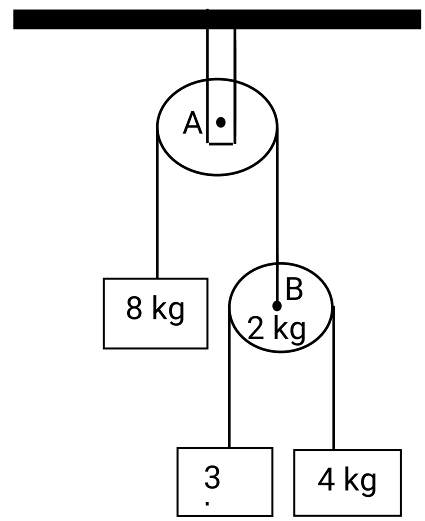
7. A particle of mass 6 kg is connected by a light inextensible string passing over a smooth fixed pulley to a light smooth movable pulley A. Two particles of masses 2 kg and 1 kg are connected by a light inextensible string passing over pulley A. When the system is moving freely, find the:
   (i) acceleration of each mass and the movable pulley.
   (ii) tension in each string. (Leave g in your answers)

8. The diagram shows particles A and B of masses 3 kg and 5 kg connected by a light inextensible string passing over two smooth fixed pulleys and under a smooth movable pulley C of mass 6 kg.



If the system is released from rest find the:
(a) accelerations of A, B and C.
(b) tension in the string.

9. The figure shows a light inextensible string passing over a smooth fixed pulley A, to one end of which is attached a mass of 8 kg and to the other end is attached pulley B of mass 2 kg. Over B passes a second light inextensible string which carries masses of 3 kg and 4 kg at its free ends.



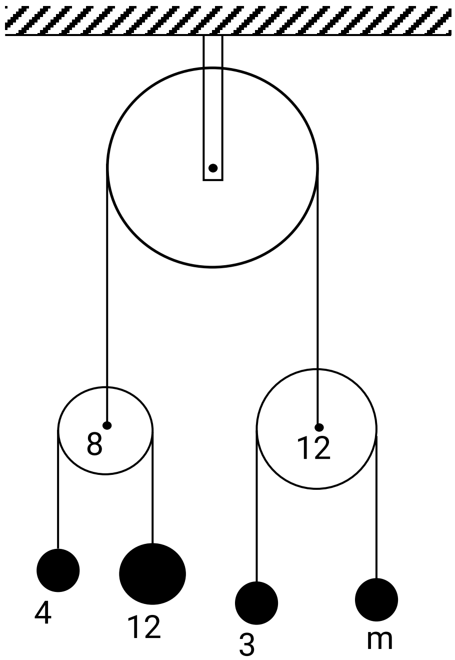
The system is released from rest. Determine the:
(a) acceleration of the movable pulley, 3 kg mass and 4 kg mass.
(b) tensions in the strings.

10. At one end of an inelastic string is attached a mass of 4 kg. The string then passes over a smooth fixed pulley and then under a smooth movable pulley of mass M. The other end of the string is fixed to a ceiling. Given that the 4 kg mass accelerates downwards at $\frac{1}{4}g$ m s^{-2}, where g is the acceleration due to gravity, find the:
   (i) value of M.
   (ii) tension in the string.

11. To the end of a light inextensible string is attached a mass A of 4 kg which rests on a smooth horizontal table. The string passes over a smooth fixed pulley P, at the edge of the table and the other end carries a smooth movable light pulley Q. A light

inextensible string passes over pulley Q and carries at its ends masses B of 2 kg and C of 1 kg. Find the acceleration of the masses and the movable pulley and tensions in the strings.

12. The diagram below shows two pulleys of masses 8 kg and 12 kg connected by a light inextensible string hanging over a fixed pulley.



The accelerations of the 4 kg and 12 kg masses are $\frac{g}{2}$ upwards and $\frac{g}{2}$ downwards respectively. The accelerations of the 3 kg and m kg masses are $\frac{g}{3}$ upwards and $\frac{g}{3}$ downwards respectively. The hanging portions of the strings are vertical. Given that the string of the fixed pulley remains stationary, find the:

(a) tensions in the strings.

(b) value of m.

Answers to exercises

Exercise: 3A

1. 4000 ; $\frac{1}{4}$ 2. (a) 240 N (b) 2640 N 3. $7{\cdot}84\ \text{m s}^{-2}$(upwards) ; $17{\cdot}64$ N

4. (i) $0{\cdot}12\ \text{m s}^{-2}$ (ii) $0{\cdot}72\ \text{m s}^{-1}$ 5. 5695 N

6. $6\ \text{m s}^{-2}$; 75 kg 7. $3{\cdot}675$ m 8. $0{\cdot}0042\ \text{m s}^{-2}$ 9. (i) (ii) $10{\cdot}4167$ m

10. (i) 70 N (ii) $5{\cdot}6346\ \text{m s}^{-2}$ 11. 120 m

12. Upper string: $68{\cdot}6$ N ;
Lower string: 49 N

Exercise: 3B

1. $3{\cdot}92\ \text{m s}^{-2}$; $11{\cdot}76$ N 2. $(6{\cdot}5i+14j)$ m 3. (a) (i) $(4i+2j)$ N (ii) $(2i+j)\ \text{m s}^{-2}$
(b) (i) (ii) 2 4. (a) (i) 9800 N (ii) 490 N (upwards)
(b) (i) 7800 N (ii) 390 N (upwards)
5. (i) $1{\cdot}0889\ \text{m s}^{-2}$ (ii) $1{\cdot}8074\ \text{m s}^{-2}$ (iii) $0{\cdot}1667$ m 6. $(1{\cdot}5i+j+3k)\ \text{m s}^{-2}$

7. $4{\cdot}0353\ \text{m s}^{-2}$; $40{\cdot}353$ N ; $57{\cdot}0678$ N at 45° to horizontal 8. 9. $1{\cdot}96\ \text{m s}^{-2}$
(a) $1{\cdot}4\ \text{m s}^{-1}$ (b) $0{\cdot}3$ m 10. a = 9 ; b = 3

Exercise: 3C

1. 8 kg **2.** 1 kg **3.** A: $5{\cdot}8\ \text{m s}^{-2}$ (downwards); B:$6{\cdot}6\ \text{m s}^{-2}$ (downwards); 2 kg mass: $6{\cdot}2\ \text{m s}^{-2}$ (upwards); upper string: 32 N ; lower string: 16 N
4. B: $0{\cdot}2\ \text{m s}^{-2}$ (upwards); 2 kg mass : $2{\cdot}2\ \text{m s}^{-2}$ (upwards); 3 kg mass: $1{\cdot}8\ \text{m s}^{-2}$ (downwards); 5 kg mass: $0{\cdot}2\ \text{m s}^{-2}$ (downwards); upper string: 48 N
Lower string: 24 N **5.** **6.** 1 kg mass: $3{\cdot}981\ \text{m s}^{-2}$ (downwards); 2 kg mass: $1{\cdot}0715\ \text{m s}^{-2}$ (up the plane); upper string: $16{\cdot}002$ N ; lower string: $5{\cdot}819$ N **7.** (i) 6 kg mass: $\frac{5}{13}g\ \text{m s}^{-2}$ (downwards); 2 kg mass: $\frac{1}{13}g\ \text{m s}^{-2}$ (downwards); 1 kg mass: $\frac{11}{13}g\ \text{m s}^{-2}$ (upwards); movable pulley:

$\frac{5}{13}$g m s^{-2} (upwards) (ii) upper string: $\frac{48}{13}$g N ; lower string: $\frac{24}{13}$g N **8.** (a) A : 1•0889 m s^{-2} (upwards); B : 3•2667 m s^{-2} (downwards); C : 1•089 m s^{-2} (upwards) (b) 32 • 667 N

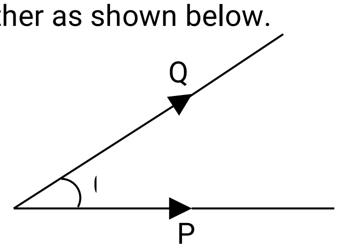
9. (a) 0.4938 ms^{-2} (downwards); 0.8305ms^{-2} (upwards); 1.8271 ms^{-2} (downwards) **(b)** Upper string: 82.3864N; lower string: 31 • 8915 N

10. (i) $5\frac{1}{3}$ kg (ii) 29 • 4 N 11. 4 kg mass: 3•92 m s^{-2} ; Q :3•92 m s^{-2} (downwards)
1 kg mass: 1•96 m s^{-2} (downwards); 2 kg mass: 5 • 88 m s^{-2}(downwards) ; 15 • 68 N ; 7 • 84 N 12.
(a) String over 8 kg pulley: 58 • 8 N ; String over fixed pulley: 196 N ; String over 12 kg pulley: 39 • 2 N (b) 6 kg

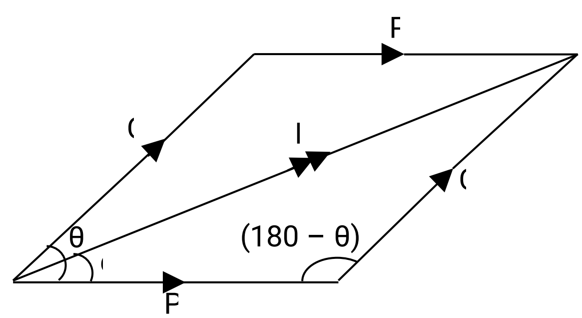
4. RESULTANTS & COMPONENTS OF FORCES

4.1 Resultant of two forces

The resultant of two forces can be obtained by using a parallelogram of forces. Consider two forces of magnitudes P and Q acting at an angle θ to each other as shown below.



Their resultant can be obtained from the parallelogram of forces shown.



The magnitude of the resultant is found using the cosine rule.

$$R^2 = P^2 + Q^2 - 2PQ\cos(180 - \theta)$$

$$R^2 = P^2 + Q^2 + 2PQ\cos\theta$$

The direction α is obtained using the sine rule:

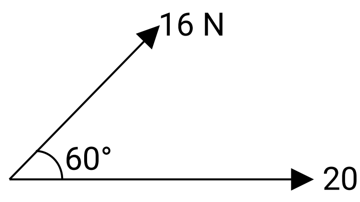
$$\frac{Q}{\sin\alpha} = \frac{R}{\sin(180\text{-}\theta)}$$

$$\frac{Q}{\sin\alpha} = \frac{R}{\sin\theta}$$

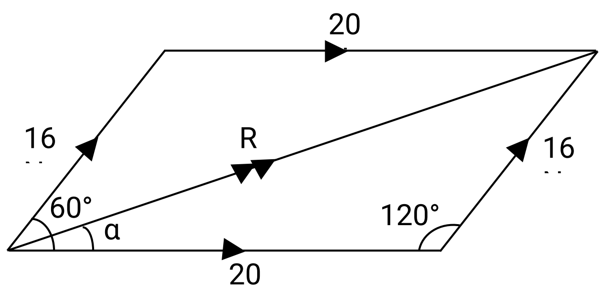
$$\alpha = \sin^{-1}\left(\frac{Q\sin\theta}{R}\right)$$

Example 1

Find the resultant of the forces shown below and the angle it makes with the 20 N force.



Solution



$R^2 = 20^2 + 16^2 \text{-} 2{\times}20{\times}16\cos 120$

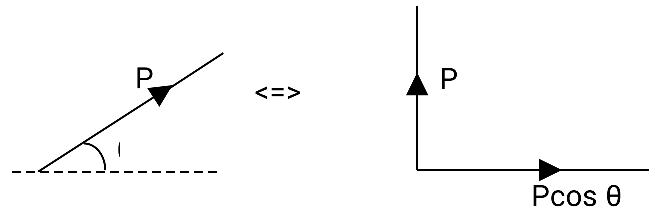
$R = 31{\cdot}24N$

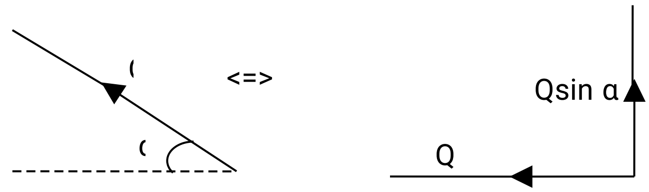
$\frac{R}{\sin 120} = \frac{16}{\sin\alpha} \Rightarrow \frac{31{\cdot}24}{\sin 120} = \frac{16}{\sin\alpha} \Rightarrow \alpha = 26{\cdot}3°$

4.2 Components of Forces

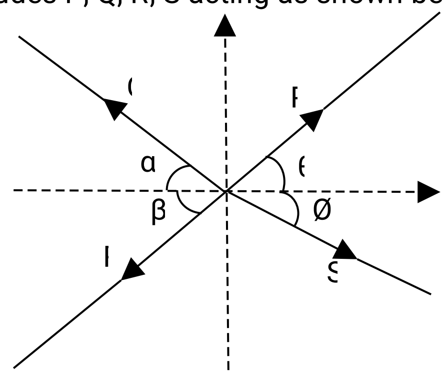
Any given force can be expressed in terms of components in two perpendicular directions. These are called the resolved components of the force.

Forces and their resolved components





More generally consider four forces of magnitudes P, Q, R, S acting as shown below:



The following are components of each of the forces:

$$P = \begin{pmatrix} P\cos\theta \\ P\sin\theta \end{pmatrix}, Q = \begin{pmatrix} -Q\cos\alpha \\ Q\sin\alpha \end{pmatrix},$$

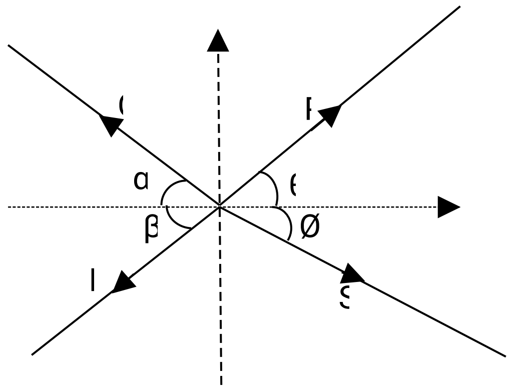
$$R = \begin{pmatrix} -R\cos\beta \\ -R\sin\beta \end{pmatrix}, S = \begin{pmatrix} S\cos Ø \\ -S\sin Ø \end{pmatrix}$$

4.3 Resultant of any given number of forces

Given forces F_1, F_2, F_3, their resultant **F** is the vector sum, **F** = $F_1 + F_2 + F_3$.

However, if the forces are given in terms of magnitude and directions are given in form of angles. We first express the forces in terms of components in two perpendicular directions and find the vector sum. From this we obtain the magnitude and direction of the resultant.

Consider forces of magnitudes P, Q, R, S acting as shown below:

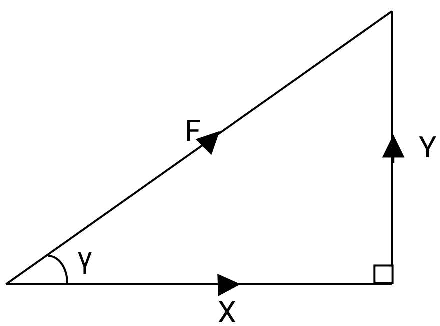


$$\mathbf{P} = \begin{pmatrix} P\cos\theta \\ P\sin\theta \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} -Q\cos\alpha \\ Q\sin\alpha \end{pmatrix},$$

$$\mathbf{R} = \begin{pmatrix} -R\cos\beta \\ -R\cos\beta \end{pmatrix}, \mathbf{S} = \begin{pmatrix} S\cos Ø \\ -S\sin Ø \end{pmatrix}$$

The resultant **F** = **P** + **Q** + **R** + **S**.

$$\mathbf{F} = \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} P\cos\theta - Q\cos\alpha - R\cos\beta + S\cos Ø \\ P\sin\theta + Q\sin\alpha - R\sin\beta - S\sin Ø \end{pmatrix}$$



The magnitude of the resultant is obtained from:

$$|\mathbf{F}| = \sqrt{X^2+Y^2}$$

The direction of the resultant γ can be obtained from: $\gamma = \tan^{-1}\left(\frac{Y}{X}\right)$.

Example 2

A particle of mass 2 kg moves under the action of forces F_1, F_2 and F_3. At time t;

$$F_1 = \left(\frac{1}{4}t-1\right)i + (t-3)j \text{ N},$$

$$F_2 = \left(\frac{1}{2}t+2\right)i + \left(\frac{1}{2}t-4\right)j \text{ N and}$$

$$F_3 = \left(\frac{1}{4}t-4\right)i + \left(\frac{3}{2}t+1\right)j \text{ N}.$$

Find the acceleration of the particle when t = 2 s.

Solution:

When t = 2 s;

$$F_1 = \left(\frac{1}{4}\times 2-1\right)i + (2-3)j = -\frac{1}{2}i - j \text{ N}$$

$$F_2 = \left(\frac{1}{2}\times 2+2\right)i + \left(\frac{1}{2}\times 2-4\right)j = 3i - 3j \text{ N}$$

$$F_3 = \left(\frac{1}{4}\times 2-4\right)i + \left(\frac{3}{2}\times 2+1\right)j = -\frac{7}{2}i + 4j \text{ N}$$

$$\mathbf{F} = F_1 + F_2 + F_3$$

$$= \left(-\frac{1}{2}i-j\right) + (3i-3j) + \left(-\frac{7}{2}i+4j\right)$$

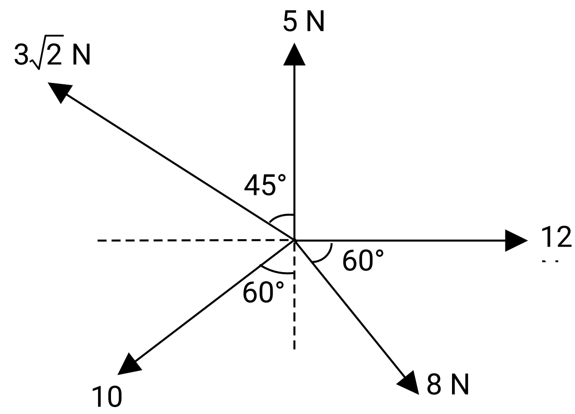
$$\Rightarrow \mathbf{F} = -\,i \text{ N}$$

From **F** = m**a**

$$-i = 2\mathbf{a} \Rightarrow \mathbf{a} = -\frac{1}{2}i \text{ m s}^{-2}$$

Example 3

The figure below shows a system of forces acting on a particle of mass 5 kg.



Find the resultant force on the particle and hence its acceleration.

Solution:

$$\mathbf{F} = \begin{pmatrix} 12 + 0 - 3\sqrt{2}\sin 45 - 10\sin 60 + 8\cos 60 \\ 0 + 5 + 3\sqrt{2}\cos 45 - 10\cos 60 - 8\sin 60 \end{pmatrix}$$

$$\Rightarrow \mathbf{F} = \begin{pmatrix} 4\cdot 3397 \\ -3\cdot 9282 \end{pmatrix}$$

$$|F| = \sqrt{4\cdot 3397^2 + (-3\cdot 9282)^2}$$

$$\Rightarrow |F| = 5\cdot 8535\text{N}$$

Alternatively, the forces can be resolved separately;

Resolving horizontally:

$(\rightarrow)$: $X = 12 - 3\sqrt{2}\sin 45 - 10\sin 60 + 8\cos 60$

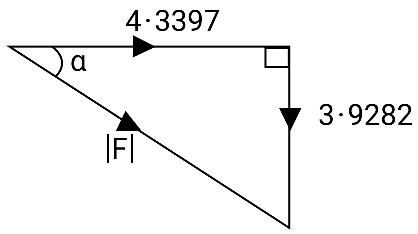
$$\Rightarrow X = 4\cdot 3397 \text{ N}$$

Resolving vertically:

$(\uparrow)$: $Y = 5 + 3\sqrt{2}\cos 45 - 10\cos 60 - 8\sin 60 \Rightarrow Y = -3\cdot 9282$ N

The magnitude and direction of the

resultant is obtained from the following triangle of forces:



$|F| = \sqrt{4{\cdot}3397^2+(-3{\cdot}9282)^2}$

$$\Rightarrow |F| = 5{\cdot}8535 \text{ N}$$

$$\tan\alpha = \frac{3{\cdot}9282}{4{\cdot}3397} \Rightarrow \alpha = 42{\cdot}2°$$

$$|F| = m|a| \Rightarrow 5|a| = 5{\cdot}8535$$

$$\Rightarrow |a| = 1{\cdot}1707 \text{ m s}^{-2}$$

Example 4

Two forces **P** and **Q** act in the directions of the vectors $4\mathbf{i}+3\mathbf{j}$ and $\mathbf{i}-2\mathbf{j}$ respectively and the magnitude of **P** is 25 N. Given that the magnitude of the resultant of **P** and **Q** is also 25 N, find the magnitude of **Q**.

Solution:

$$\mathbf{P} = |P|\hat{P}; \ |P| = 25\text{N}; \ \hat{P} = \frac{4i+3j}{\sqrt{4^2+3^2}} \Rightarrow \hat{P} = \frac{4i+3j}{5}$$

$$\mathbf{P} = 25\times\frac{(4i+3j)}{5} \Rightarrow \mathbf{P} = (20i+15j) \text{ N}$$

$$\text{Similarly; } \mathbf{Q} = |Q|\times\frac{(i-2j)}{\sqrt{1^2+(-2)^2}} \Rightarrow \mathbf{Q} = \frac{|Q|}{\sqrt5}i - \frac{2|Q|}{\sqrt5}j$$

$$\mathbf{P}+\mathbf{Q} = \left(20+\frac{|Q|}{\sqrt5}\right)i + \left(15-\frac{2|Q|}{\sqrt5}\right)j$$

but $|P+Q| = 25$

$$\left(20+\frac{|Q|}{\sqrt5}\right)^2 + \left(15-\frac{2|Q|}{\sqrt5}\right)^2 = 25^2$$

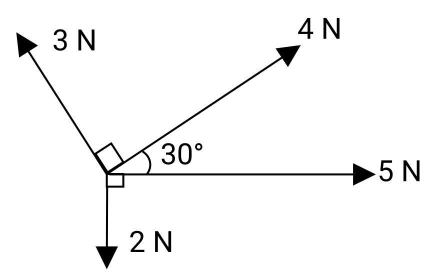
$$|Q|\left(|Q|-4\sqrt5\right) = 0 \text{ Either } |Q| = 0 \text{ or } |Q| = 4\sqrt5$$

$$\text{Hence } |Q| = 4\sqrt5 \text{ N}$$

Exercises

Exercise: 4A

1. Find the magnitudes of the vertical and horizontal components of:
   (a) a force of 20 N acting at 40° to the horizontal.
   (b) a force of 14 N acting at 78° to the horizontal.
   (c) a force of 24 N acting at 36° to the vertical.
2. A set of horizontal forces of magnitude 20 N, 12 N and 30 N act on a particle in the directions due south, due east and N40°E respectively. Find the magnitude and direction of the fourth force which holds the particle in equilibrium.
3. Two forces of magnitude 12 N and 9 N act on a particle producing an acceleration of 3·65 m s^{-2}. The forces act at an angle of 60° to each other. Find the mass of the particle.
4. ABCD is a square. Forces of magnitude 2N, 1N, $\sqrt2$ N and 4N act along AB, BC, AC and DA respectively. The directions of the forces being indicated by the order of the letters. Find the magnitude of the resultant and the angle it makes with AB.
5. Two forces have magnitudes *P*and *Q* and the angle between them is θ. If the resultant of these two forces is R and makes an angle α with P. Show that:
   (a) $R^2 = P^2 + Q^2 + 2PQ\cos\theta$
   (b) $\tan\alpha = \dfrac{Q\sin\theta}{P+Q\cos\theta}$
6. Forces of 3 N and 2 N act along OA and OB respectively, the directions of the forces being indicated by the order of the letters. If $A\hat{O}B = 150°$, find the magnitude of the resultant and the angle it makes with OA.
7. The angle between a force of 6 N and a force of P N is 90°. If the resultant of the two forces has magnitude 8 N, find the value of P.
8. Obtain the magnitude and direction of the resultant of the forces below.



9. Forces 3, 2, 1, 4 N act at a point along lines OA, OB, OC, OD respectively. $A\hat{O}B = 60°$, $B\hat{O}C = 90°$, $C\hat{O}D = 120°$. Find the magnitude

of the resultant and its inclination to OA.

10. Two forces, $(4i\text{-}5j)$ N and $(pi+qj)$ N, act on a particle P of mass m kg. The resultant of the two forces is **R**. Given that **R** acts in a direction which is parallel to the vector $(i\text{-}2j)$:
   (a) find the angle between **R** and the vector **j**.
   (b) show that $2p + q + 3 = 0$.
   Given that $q = 1$ and that P moves with an acceleration of magnitude $8\sqrt{5}$ m s^{-2},
   (c) find the value of m.

Exercise: 4B

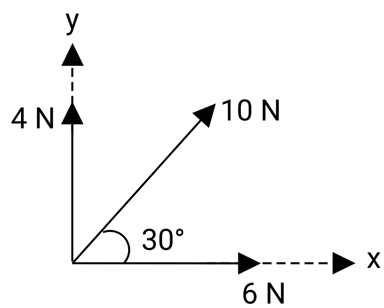
1. Forces 3 N, 5 N, 7 N and 2 N act along sides DA, AB, BC and CD respectively of a square. Calculate the magnitude of their resultant and the angle it makes with AD.

2. Find the angle between the lines of action of two forces of magnitude 7 N and 11 N, given their resultant is of magnitude 8 N.

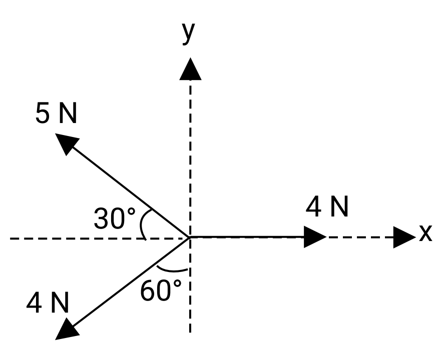
3. *PQRS* is a square. Forces of magnitude 60 N, 40 N, 180 N and 40 N act along the lines PQ, QR, RP and SQ respectively, in each case the direction of the force being given by the order of the letters. Given that SR is horizontal. Determine the:
   (a) magnitude of the resultant force.
   (b) inclination of the resultant to SR.

4. Each of the following diagrams shows a number of forces. Find the magnitude of their resultant and angle it makes with the x-axis.

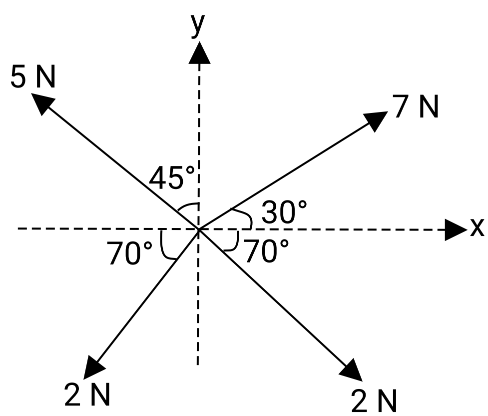
(a)



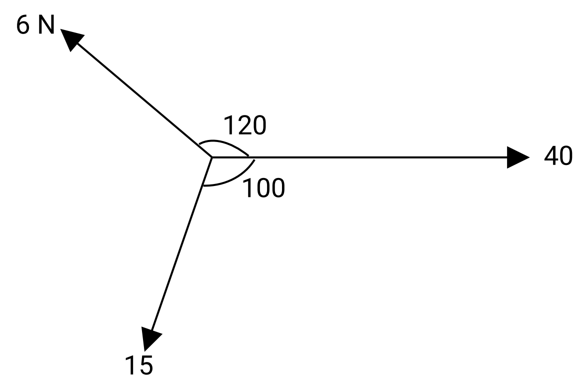
(a)



(c)



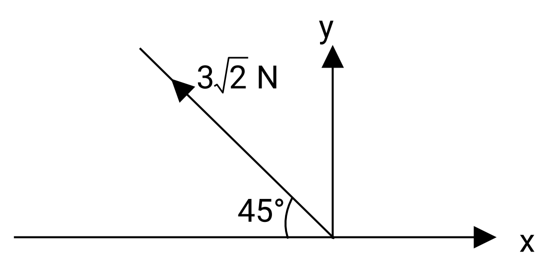
5. Three forces are applied to a point as shown below.



Calculate the resultant of the forces.

6. When two vectors of magnitude P and Q are inclined at an angle θ, the magnitude of their resultant is 2P. When the inclination is changed to (180−θ) the magnitude of their resultant is halved. Find the ratio of P to Q.

7. Express the force shown below in the form ai + bj.



   (b) Forces of magnitudes 4N, 8N, 12 N, 2 N, $2\sqrt{2}$ N and $\sqrt{2}$ N act along the lines AD, DC, CB, BA, AC and BD of a square ABCD in the directions indicated by the order of the letters. Calculate the magnitude and direction of the resultant force.

8. Four forces:
   $F_1 = 8\sqrt{3}$ N due N60°E

$F_2 = 3\sqrt{3}$ N due North west
$F_3 = 2\sqrt{3}$ N due south
$F_4 = 5\sqrt{2}$ N due S72°E
act at a point. Determine the magnitude and direction of the resultant of the forces.

9. Find the resultant of each of the following sets of forces:
   (a) $(2i+3j+2k)$ N, $(2i+4j-8k)$ N
   (b) $(7i-4j+3k)$ N, $(5i-2j+8k)$ N, $(i-k)$ N
   (c) $(2i+3j-7k)$ N, $(2i+5k)$ N, $(3j+4k)$ N

10. Find the magnitude of the resultant of each of the following sets of forces and determine the angle this resultant makes with the direction of **i**.
   (a) $(2i+3j)$ N, $(5i-2j)$ N, $(-3i+3j)$ N
   (b) $(-2i+5j)$ N, $(i+2j)$ N
   (c) $(4i+3j)$ N, $(-i-5j)$ N
   (d) $(2i+4j)$ N, $(-6i-5j)$ N, $(2i+j)$ N

Exercise: 4C

1. A particle of mass 2 kg moves from rest at the origin under the action of two forces **P** and **Q**. **P** has magnitude 22 N and acts in the direction of the vector $2\mathbf{i}-6\mathbf{j}+9\mathbf{k}$ while **Q** has magnitude 30 N and acts in the direction of the vector $4\mathbf{j}-3\mathbf{k}$. Find the:
   (i) resultant force on the particle.
   (ii) acceleration of the particle.

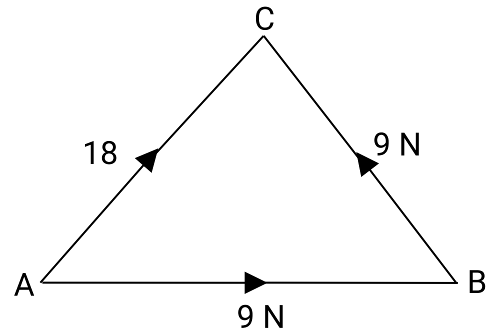
2. Two forces $F_1 = 4\mathbf{j}-5\mathbf{k}$ and $F_2 = 2\mathbf{i}-5\mathbf{j}-\mathbf{k}$ act on a particle. Find the magnitude of the resultant force acting on the particle and the angle it makes with F_1.

3. Four forces $a\mathbf{i}+(a-1)\mathbf{j}$, $3\mathbf{i}+2a\mathbf{j}$, $5\mathbf{i}-6\mathbf{j}$ and $-\mathbf{i}-2\mathbf{j}$ act on a particle. The resultant of the forces makes an angle of 45° with the horizontal. Find the value of a. Hence determine the magnitude of the resultant.

4. Find the magnitude of the resultant of two forces of magnitudes 15 N and $4\sqrt{2}$ N acting in the directions of the vectors $3\mathbf{i}-4\mathbf{j}$ and $\mathbf{i}+\mathbf{j}$ respectively.

5. A force of magnitude 10 N parallel to the vector $4\mathbf{i}+3\mathbf{j}$ is the resultant of two forces parallel respectively to the vectors $2\mathbf{i}+\mathbf{j}$, $\mathbf{i}+\mathbf{j}$. Find the magnitudes of the two forces.

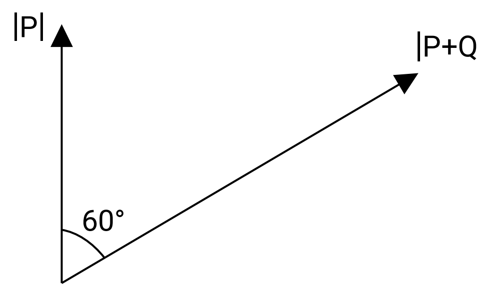
6. The diagram shows forces acting along sides of an equilateral triangle.



Find the magnitude and direction of the resultant force.

7. The resultant of the forces $a\mathbf{i}+2\mathbf{j}$, $3\mathbf{i}+(4-a)\mathbf{j}$ and $4\mathbf{i}-5\mathbf{j}$ acts in the direction $\mathbf{i}+\mathbf{j}$. Find the magnitude of the resultant force.

8.



A particle is freely moved on a horizontal table. It is acted on by constant forces **P** and **Q**. The force **P** has magnitude 5 N and acts due north. The resultant **P** + **Q** has magnitude 9 N and acts in the direction 060°. Calculate the magnitude and direction of **Q**.

9. A force **F** has magnitude 50 N and acts in the direction of the vector $24\mathbf{i}+7\mathbf{j}$. Show that $\mathbf{F} = (48i+14j)$ N. Two forces F_1 and F_2 have magnitudes α N and β N and act in the directions $\mathbf{i}-2\mathbf{j}$ and $4\mathbf{i}+3\mathbf{j}$ respectively. Given that the resultant of F_1 and F_2 is **F**, show that $\alpha = 8\sqrt{5}$ and find β.

10. A force of 7 N and another of 4 N have a resultant of magnitude 9 N when the angle between them is θ. Calculate the:
   (a) value of θ.
   (b) angle between the resultant force and 7N force.

Answers to Exercises

Exercise: 4A

1. (a) $12 \cdot 856$ N ; $15 \cdot 321$ N (b) $13 \cdot 694$ N ; $2 \cdot 911$ N (c) $19 \cdot 416$ N ; $14 \cdot 107$ N
2. $31 \cdot 425$ N ; S$84 \cdot 6°$W 3. $5 \cdot 0$ kg 4. $\sqrt{13}$ N ; $33 \cdot 7°$ below AB 5. (a) (b)
6. $1 \cdot 6148$ N ; $38 \cdot 3°$ 7. $5 \cdot 2915$ N 8. $7 \cdot 433$ N at $20 \cdot 5°$ above the 5 N force
9. $3 \cdot 598$ N ; $29 \cdot 4°$ 10. (a) $153 \cdot 4°$ (b) (c) $\frac{1}{4}$ kg

Exercise: 4B

1. 5 N ; $36 \cdot 9°$ 2. $46 \cdot 5°$ 3. (a) $121 \cdot 965$ N (b) $71 \cdot 4°$
4. (a) $17 \cdot 2$ N ; $31 \cdot 6°$ (b) $3 \cdot 8$ N ; $172 \cdot 5°$ (c) $4 \cdot 1$ N ; $52 \cdot 4°$
5. $35 \cdot 7035$ N at $15 \cdot 6°$ below horizontal 6. $P{:}Q = \sqrt{6}{:}3$ 7. (a) $(-3i+3j)$ N (b) $\sqrt{74}$ N at $35 \cdot 5°$ below AB 8. $15 \cdot 845$ N ; N$71 \cdot 8°$E
9. (a) $(4i+7j-6k)$ N (b) $(13i-6j+10k)$ N (c) $(4i+6j+2k)$ N
10. (a) $4\sqrt{2}$ N ; $45°$ (b) $5\sqrt{2}$ N ; $98 \cdot 1°$ (c) $\sqrt{13}$ N ; $33 \cdot 7°$ (d) 2 N ; $180°$

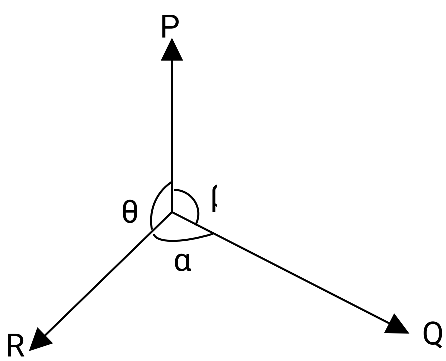
Exercise: 4C

1. (i) $(4i+12j)$ N (ii) $(2i+6j)$ m s^{-2}
2. $\sqrt{41}$ N ; $50 \cdot 6°$ 3. $a = 8$; $15\sqrt{2}$ N
8. $15 \cdot 264$ N **5.** $2\sqrt{5}$ N ; $4\sqrt{2}$ N **6.** 27 N ; $60°$ to AB **7.** $4\sqrt{2}$ N **8.** $\sqrt{61}$ N at $093 \cdot 7°$
9. $\beta = 50$ 10. (a) $73 \cdot 4°$ (b) $25 \cdot 2°$

5. EQUILIBRIUM & ACCELERATION UNDER CONCURRENT FORCES

5.1 Three force problems

A body is said to be in equilibrium when two or more forces act upon it (particle or rigid body) and motion does not take place. When a body is in equilibrium under the action of three forces, **Lami's theorem** can be applied. Consider forces P, Q and R holding a particle or rigid body in equilibrium as shown below.



Lami's theorem can be used to solve the problem and is stated as : $\frac{P}{\sin\alpha} = \frac{Q}{\sin\theta} = \frac{R}{\sin\beta}$

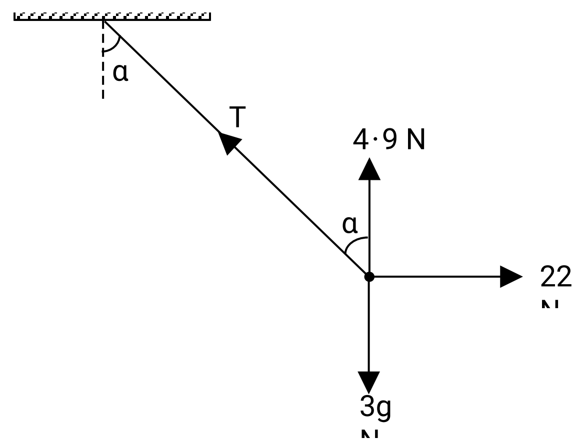
Example 1

(a) A particle of mass 3 kg is attached to the lower end B of an inextensible string. The upper end A of the string is fixed to a point on the ceiling of a roof.
A horizontal force of 22 N and an upward vertical force of 4·9 N act upon the particle making it to be in equilibrium, with the string making an angle α with the vertical. Find the value of α and the tension in the string.

(b) A non-uniform rod of mass 9 kg rests horizontally in equilibrium supported by two light inextensible strings tied to the ends of the rod. The strings make angles of 50° and 60° with the rod. Calculate the tensions in the strings.

Solution

(a)



Resolving horizontally:

$$T\sin\alpha = 22 \text{ ....(i)}$$

Resolving vertically:

$$T\cos\alpha + 4{\cdot}9 = 3g$$
$$T\cos\alpha = 3\times9{\cdot}8-4{\cdot}9$$
$$T\cos\alpha = 24{\cdot}5 \text{ ....(ii)}$$

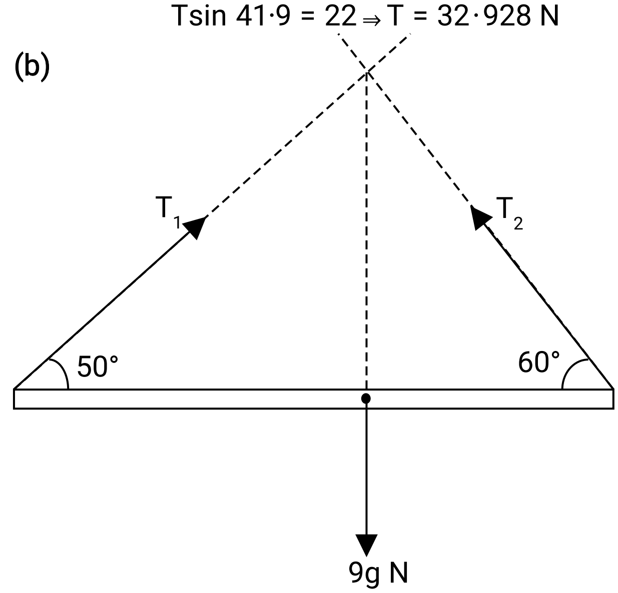
Dividing equation (i) by equation (ii)

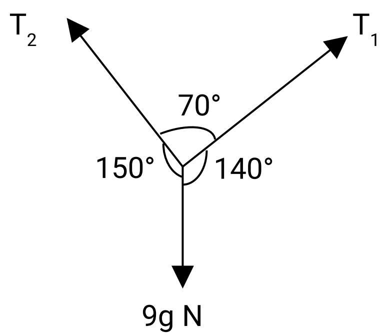
$$\tan\alpha = \frac{22}{24{\cdot}5} \Rightarrow \alpha = 41{\cdot}9°$$

From equation (i);

$$T\sin 41{\cdot}9 = 22 \Rightarrow T = 32{\cdot}928\text{ N}$$

(b)





Using Lami's theorm;

$$\frac{T_1}{\sin 150} = \frac{9g}{\sin 70} = \frac{T_2}{\sin 140}$$

$$T_1 = \frac{9g\sin 150}{\sin 70} = \frac{9\times9{\cdot}8\times\sin 150}{\sin 70}$$

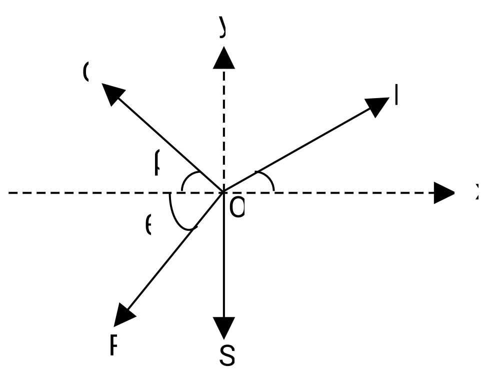
$$T_1 = 46{\cdot}9302 \text{ N}$$

$$T_2 = \frac{9g\sin 140}{\sin 70} = \frac{9\times9{\cdot}8\times\sin 150}{\sin 70}$$

$$T_2 = 60{\cdot}3324 \text{ N}$$

5.2 Equilibrium under any number of forces

When several forces act on a particle, they are said to be concurrent. When a particle is acted upon by any given number of forces and is kept in equilibrium, we can apply the general method below to find the solution to the problem.
Consider forces P, Q, R, S acting at a point O as shown below.



Resolving horizontally:

$$P\cos\alpha - Q\cos\beta - R\cos\theta = 0$$

$$P\cos\alpha = Q\cos\beta + R\cos\theta \text{ ……………………. (i)}$$

Resolving vertically:

$$P\sin\alpha + Q\sin\beta - R\sin\theta - S = 0$$

$$P\sin\alpha + Q\sin\beta = R\sin\theta + S \text{ ………………. (ii)}$$

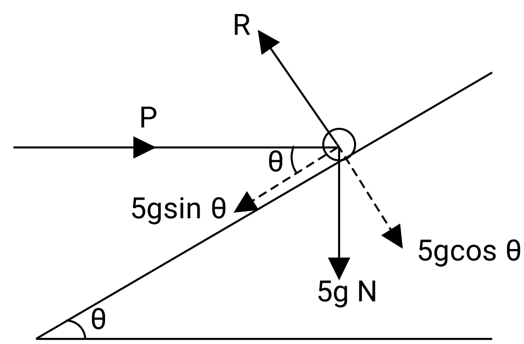
From the above two equations two unknowns can be found.

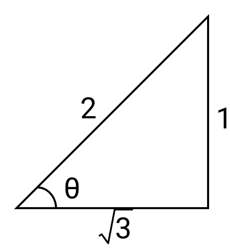
The same procedure can be applied to each particle when dealing with systems involving equilibrium of more than one particle or connected particles.

Example 2

A particle of mass 5 kg is placed on a smooth plane inclined at $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ to the horizontal. Find the magnitude of the force acting horizontally required to keep the particle in equilibrium and the normal reaction to the plane.

Solution





Resolving along plane:

$$P\cos\theta = 5g\sin\theta \Rightarrow P = 5g\tan\theta$$

$$P = 5\times9{\cdot}8\times\frac{1}{\sqrt{3}} \Rightarrow P = \frac{49\sqrt{3}}{3}\text{N}$$

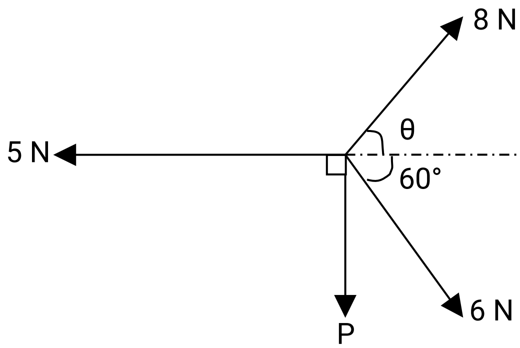
Resolving normal to plane:

$$R = P\sin\theta + 5g\cos\theta$$

$$R = \frac{49\sqrt{3}}{3}\times\frac{1}{2} + 5\times9{\cdot}8\times\frac{\sqrt{3}}{2} \Rightarrow R = \frac{98\sqrt{3}}{3}\text{N}$$

Example 3

Find P and θ if the forces below are in equilibrium.



Solution:
Resolving horizontally:

$$8\cos\theta + 6\cos 60 = 5 \Rightarrow \cos\theta = \frac{1}{4}$$

$$\theta = 75{\cdot}5°$$

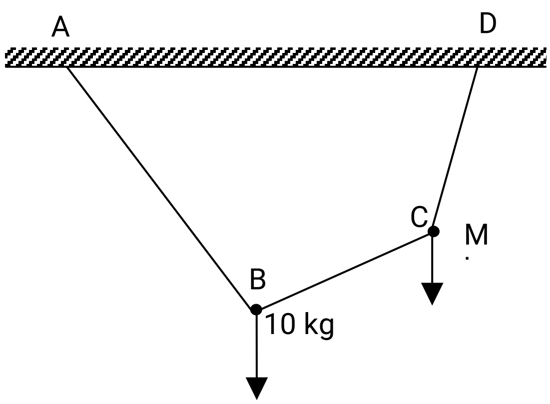
Resolving vertically:

$$P + 6\sin 60 = 8\sin\theta \Rightarrow P = 8\sin 75{\cdot}5 - 6\sin 60$$

$$P = 2{\cdot}5498 \text{ N}$$

Example 4

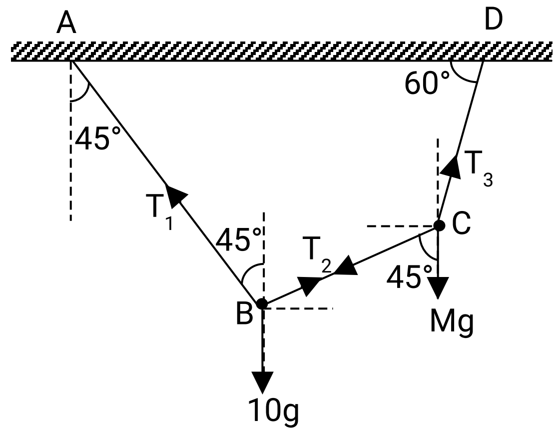
A thin copper wire ABCD is supported at A and D which are on the same horizontal level. At B hangs a mass of 10 kg and at C hangs a mass of M kg. The portion of the string AB is at 45° to the vertical and BC slopes upwards at 45° to the horizontal while CD is at 30° to the vertical as shown below.



(a) Determine the tension in the portion AB, BC and CD in terms of g.

(b) Show that the value of M is $5(\sqrt{3}-1)$ kg.

Solution



For 10 kg mass;
Resolving horizontally:

$$T_1 \sin 45 = T_2 \cos 45$$

$$T_1 = T_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Resolving vertically:

$$T_1 \cos 45 + T_2 \sin 45 = 10g;\ \text{but } T_1 = T_2$$

$$\frac{T_1}{\sqrt{2}} + \frac{T_1}{\sqrt{2}} = 10g$$

$$T_1 = 5\sqrt{2}g \text{ N and } T_2 = 5\sqrt{2}g \text{ N}$$

For M:
Resolving horizontally:

$$T_2 \cos 45 = T_3 \sin 30$$

$$T_3 = \frac{2T_2}{\sqrt{2}} \Rightarrow T_3 = 5\sqrt{2}g \times \frac{2}{\sqrt{2}}$$

Hence $T_3 = 10g$ N

(b) Resolving vertically:

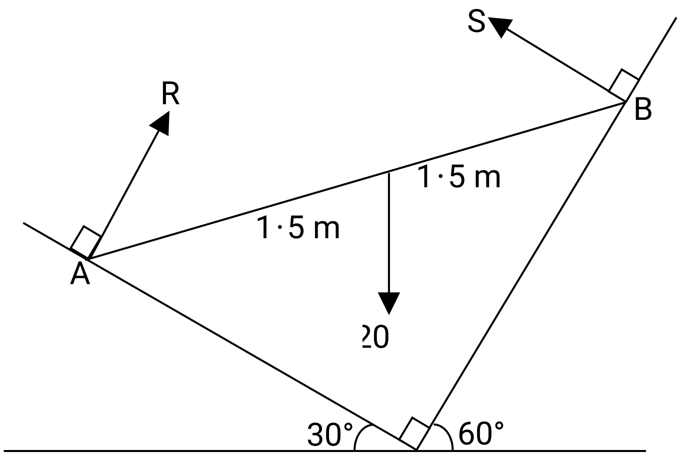
$$Mg + T_2 \sin 45 = T_3 \cos 30$$

$$Mg + 5\sqrt{2}g \times \frac{1}{\sqrt{2}} = 10g \times \frac{\sqrt{3}}{2}$$

$$M + 5 = 5\sqrt{3} \Rightarrow M = 5(\sqrt{3}-1) \text{ kg}$$

Example 5

The diagram below shows a uniform rod, 3 m long of mass 20 kg placed on two smooth planes inclined at 30° and 60° to the horizontal. The rod is resting in equilibrium.



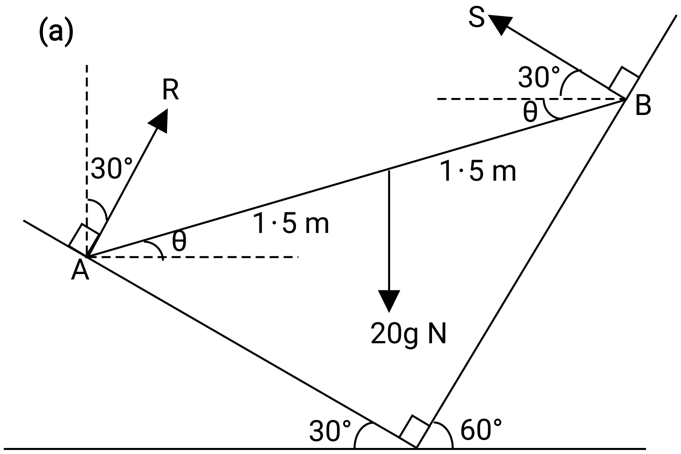
(a) (i) Show that $R = S\sqrt{3}$.

(ii) Calculate the reactions S and R.

(b) By taking moments about A, find the angle of inclination of the rod to the horizontal.

Solution

(a)



(i) Resolving horizontally:

$R\sin 30 = S\cos 30$

$R\tan 30 = S$

$R\times\frac{1}{\sqrt{3}} = S \Rightarrow R = S\sqrt{3}$

(ii) Resolving vertically:

$R\cos 30 + S\sin 30 = 20g$

$R\times\frac{\sqrt{3}}{2} + \frac{1}{2}S = 20g \Rightarrow R\sqrt{3} + S = 40g$

But $R = S\sqrt{3} \Rightarrow (S\sqrt{3})\sqrt{3} + S = 40g$

$4S = 40g \Rightarrow S = 10g = 98$ N

Also $R = 98\sqrt{3}$ N

(b) Taking moments about A:

$S\times 3\sin(\theta+30) = 20g\times 1\cdot 5\cos\theta$

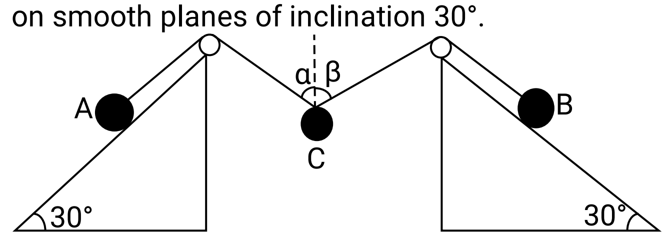
$98\times 3\sin(\theta+30) = 20\times 9\cdot 8\times 1\cdot 5\cos\theta$

$\sin(\theta+30) = \cos\theta \Rightarrow \tan\theta = \frac{1}{\sqrt{3}}$

$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \Rightarrow \theta = 30°$

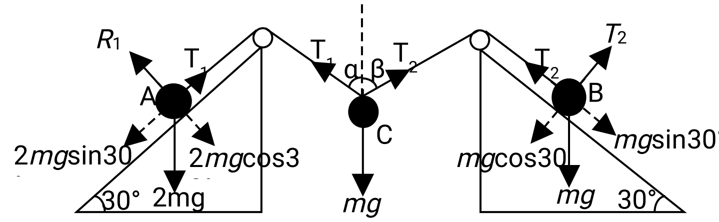
Example 6

The diagram shows masses A and B each lying on smooth planes of inclination 30°.



Light inextensible strings attached to A and B pass along lines of greatest slope, over smooth pulleys and are connected to a third mass C hanging freely. The strings make angles of α and β with the upward vertical as shown. If A, B and C have masses 2m, m and m respectively and the system rests in equilibrium, show that $\sin\beta = 2\sin\alpha$ and $\cos\beta + 2\cos\alpha = 2$. Hence find α and β.

Solution:



For A:

Resolving along plane:

$T_1 = 2mg\sin 30 \Rightarrow T_1 = mg$

For C:

Resolving horizontally:

$T_1\sin\alpha = T_2\sin\beta$……………….(i)

Resolving vertically:

$T_1\cos\alpha + T_2\cos\beta = mg$……………(ii)

For B:

Resolving along plane:

$T_2 = mg\sin 30 \Rightarrow T_2 = \frac{1}{2}mg$

From equation (i): $mg\sin\alpha = \frac{1}{2}mg\sin\beta$

$\sin\beta = 2\sin\alpha$…………………………….(iii)

From equation (ii): $mg\cos\alpha + \frac{1}{2}mg\cos\beta = mg$

$\cos\beta + 2\cos\alpha = 2$ …………………..(iv)

From equation (iv): $\cos\beta = 2(1-\cos\alpha)$ …….(v)

Squaring equation (iii) and equation (v) and adding

$$\sin^2\beta + \cos^2\beta = 4\sin^2\alpha + 4(1-2\cos\alpha + \cos^2\alpha)$$

$$1 = 4(\sin^2\alpha + \cos^2\alpha) + 4-8\cos\alpha \Rightarrow \cos\alpha = \frac{7}{8}$$

$$\alpha = \cos^{-1}\frac{7}{8} \Rightarrow \alpha = 29{\cdot}0°$$

From equation (iv): $\cos\beta + 2\times\frac{7}{8} = 2$

$$\Rightarrow \beta = \cos^{-1}\frac{1}{4} \Rightarrow \beta = 75{\cdot}5°$$

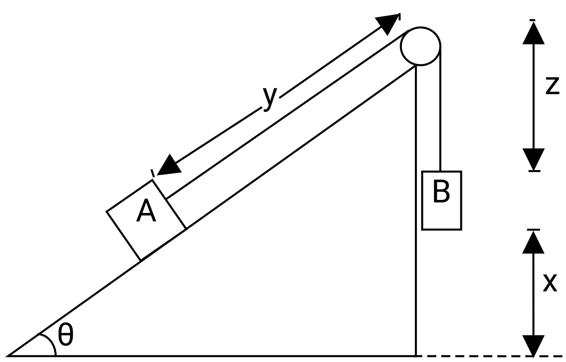
5.3 Acceleration under any number of forces

If a body accelerates under the action of any given number of forces:

- Resolve forces perpendicular to the direction of motion. In this direction the forces are in equilibrium.
- Resolve forces parallel to the direction of motion. Using Newton's second law, obtain the equation for the motion.

Example 7

The body A lies on a smooth slope and body B is freely suspended. The pulley is smooth and the string is light and inextensible.



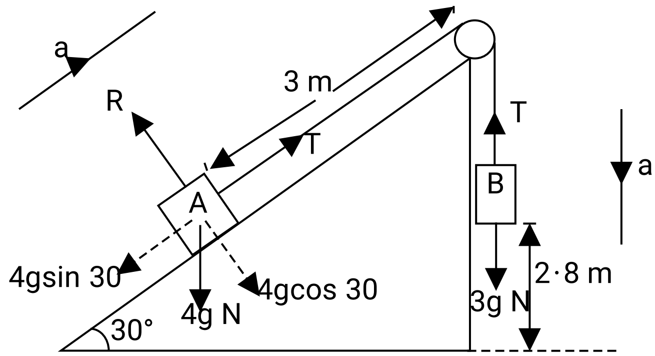
(a) If the mass of A is 4 kg and the mass of B is 3 kg, with $\theta = 30°$, body A will accelerate up the slope. If y = 3 m and x = 2·8 m, find the velocity with which A hits the pulley.

(b) If the mass of A is 2m and the mass of B is m, show that A will accelerate up the slope provided $\sin\theta < \frac{1}{2}$.

With this condition fulfilled and y > x, show that if the system is released from rest, mass B hits the ground with velocity $\sqrt{\frac{2gx(1-2\sin\theta)}{3}}$ and that A reaches the pulley provided $x \geq \frac{3y\sin\theta}{1+\sin\theta}$.

Solution:

(a)



For A:

$$T-4g\sin 30 = 4a$$

$$T-2g = 4a \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

For B:

$$3g-T = 3a \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Adding equation (i) and equation (ii)

$$g = 7a \Rightarrow a = \frac{1}{7}g = \frac{1}{7}\times 9{\cdot}8$$

$$a = 1{\cdot}4 \text{ m s}^{-2}$$

From equation (ii): $T = 3(g-a) = 3(9{\cdot}8-1{\cdot}4)$

$$T = 25{\cdot}2 \text{ N}$$

From $v^2 = u^2 + 2as$

Velocity of A as B hits the ground

$$u_A^2 = 2\times1{\cdot}4\times2{\cdot}8 \Rightarrow u_A = 2{\cdot}8 \text{ m s}^{-1}$$

For A:

$$0-4g\sin 30 = 4a'$$

$$a' = -\frac{1}{2}g = -\frac{1}{2}\times9{\cdot}8 = -4{\cdot}9 \text{ m s}^{-2}$$

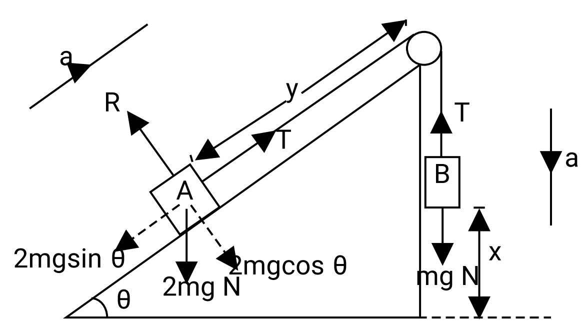
From $v^2 = u^2 + 2as$; $s = 3-2{\cdot}8 = 0{\cdot}2$ m

$$v_A^2 = u_A^2 + 2a's$$

$$\Rightarrow v_A^2 = 2{\cdot}8^2-2\times4{\cdot}9\times0{\cdot}2$$

$$v_A = 2{\cdot}425 \text{ m s}^{-1}$$

(b)



For A to accelerate up and B to move down:

$$2mg\sin\theta < T < mg$$

$$\therefore 2mg\sin\theta < mg \Rightarrow \sin\theta < \frac{1}{2}$$

With this condition fulfilled

For A: $T - 2mg\sin\theta = 2ma$……………. (i)

For B: $mg - T = ma$ …………………. (ii)

Adding equation (i) and equation (ii):

$$mg(1-2\sin\theta) = 3ma$$

$$a = \frac{1}{3}g(1-2\sin\theta)$$

From $v^2 = u^2 + 2as$, $u = 0$

$$v_B^2 = 2\times\frac{1}{3}g(1-2\sin\theta)\times x$$

$$v_B = \sqrt{\frac{2}{3}gx(1-2\sin\theta)}$$

After B hits the ground

For A: $0 - 2mg\sin\theta = 2ma' \Rightarrow a' = -g\sin\theta$

From $v^2 = u^2 + 2as \Rightarrow v_A^2 = u_A^2 + 2a's$;

$$v_A^2 = \frac{2}{3}gx(1-2\sin\theta) - 2g\sin\theta\times(y-x)$$

A reaches the pulley if $v_A \geq 0 \Rightarrow v_A^2 \geq 0$

$$\frac{2}{3}g\,x(1-2\sin\theta) - 2g(y-x)\sin\theta \geq 0$$

On simplifying;

$$x(1+\sin\theta) \geq 3y\sin\theta \Rightarrow x \geq \frac{3y\sin\theta}{1+\sin\theta}$$

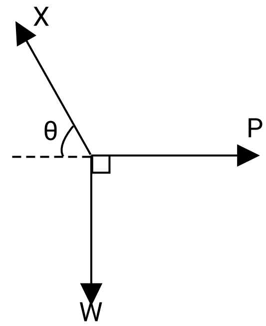
Exercises

Exercise: 5A

1. A particle of weight 8N is attached to a point B of a light inextensible string AB. It hangs in equilibrium with point A fixed and AB at 30° to the downward vertical. A force F at B acting at right angles to AB keeps the particle in equilibrium. Find the magnitude of F and the tension in the string.

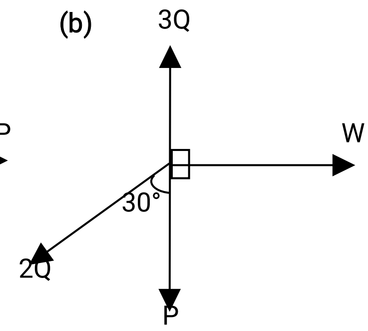
2. Each of the following diagrams shows a particle in equilibrium under the forces shown.

(a)



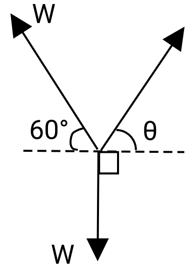
Prove that $P\tan\theta = W$

(b)



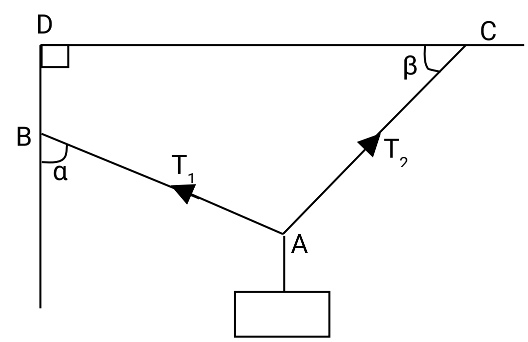
Prove that $P = W(3-\sqrt{3})$

(c)



Prove that $\tan\theta = 2-\sqrt{3}$

3. The figure shows two light inextensible strings AC and AB attached to the ceiling DC and wall DB at C and B respectively. A weight W is suspended from A.



If the strings AB and AC make angles α and β with the wall and ceiling respectively, show that $\frac{T_1}{T_2} = \frac{\cos\beta}{\sin\alpha}$ and find T_1 and T_2 in terms of α, β and W.

4. If the following forces are in equilibrium, find the value of a and b in each case.
   (a) $(ai+3j)$ N, $(2i-5j)$ N, $(-7i+bj)$ N
   (b) $(5i+aj+k)$ N, $(bi-6j-k)$ N

5. One end of a light inextensible string of length 75 cm is fixed to a point on a vertical pole. A particle of weight 12 N is attached to the end of the string. The particle is held 21 cm away from the pole by a horizontal force. Find the

magnitude of this force and the tension in the string.

6. A weight W is suspended by two ropes which make 20° and 70° with the horizontal. If the tension in the first rope is 20 N, find the tension in the other and the value of W.

7. A string AB has end A fixed and end B carrying a particle of mass 5 kg. Another string BC connects B to another particle of mass 10 kg at end C. A horizontal force of 40 N acts at C and the system is in equilibrium with C lower than B. Find the:
   (i) angles that AB and BC make with the vertical.
   (ii) tensions in the strings.

8. A small object of weight 4 N is suspended from a fixed point by a string. The object is held in equilibrium with the string at an angle of 25° to the vertical by a horizontal force. Find the magnitude of this force and the tension in the string.

9. A particle P of weight 50 N is hanging in equilibrium supported by two strings inclined at 20° and 40° to the vertical. Find the tensions in the strings.

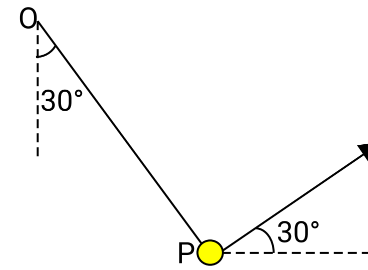
10. A particle of weight 10W is supported by two light inextensible strings attached to fixed points X and Y which lie in the same horizontal plane. Given that PX = 36 cm, PY = 48 cm and XY = 60 cm, find the tension in each string.

Exercise: 5B

1. A particle P of weight 10W is hanging in equilibrium from two strings PA and PB. If $A\hat{P}B = 90°$ and the tensions in the strings are T and 2T respectively, find T in terms of W.

2. A particle P of weight 50 N is supported by two light inextensible strings attached to fixed points A and B. Given that A and B lie 2 m apart in the same horizontal plane and that AP = 2 m, PB = 1 m, find the tension in each string.

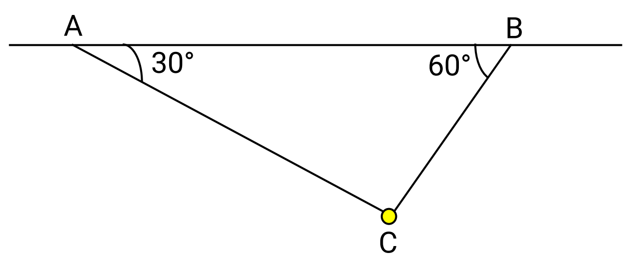
3. A particle P of mass 2 kg is attached to one end of a light string, the other end of which is attached to a fixed point O. The particle is held in equilibrium with OP at 30° to the downward vertical by a force of magnitude F newtons. The force acts in the same vertical plane as the string and acts at an angle of 30° to the horizontal as shown in the figure below.



Find the:
   (i) value of F.
   (ii) tension in the string.

4. A particle of mass 5 kg is placed on a smooth horizontal plane inclined at $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ to the horizontal. Find the magnitude of the force acting horizontally required to keep the particle in equilibrium and the normal reaction to the plane.

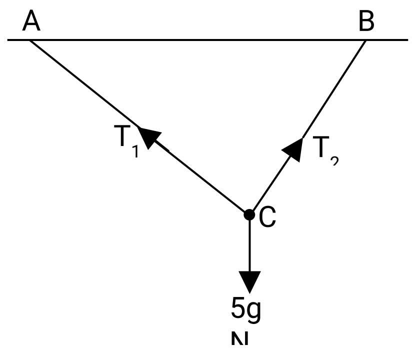
5. A particle of mass m kg is attached at C to two light inextensible strings AC and BC.
The other ends of the strings are attached to fixed points A and B on a ceiling. The particle hangs in equilibrium with AC and BC inclined to the horizontal at 30° and 60° respectively as shown in the figure below.



Given that the tension in AC is 20 N, find the:
   (a) tension in BC.
   (b) value of m. [Use g = 10 m s^{-2}].

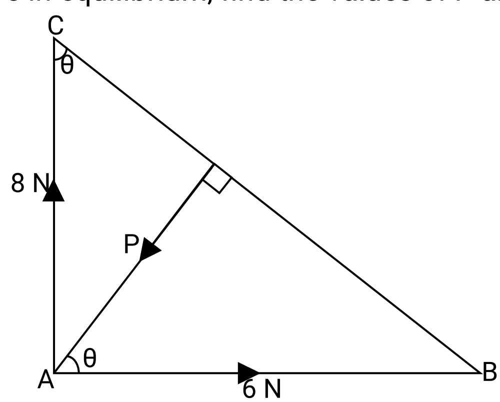
6. The diagram shows a body of mass 5 kg supported by two light inextensible strings, the other ends of which are attached to points A and B on the same horizontal level

and 7 m apart.



The body rests in equilibrium at C, 3 m vertically below AB. If $C\hat{B}A = 45°$, find T_1 and T_2, the tensions in the strings.

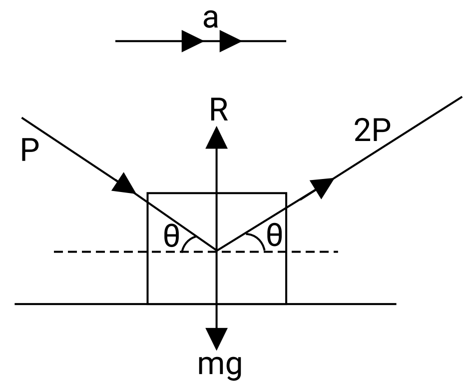
7. ABC is a right angled triangle in which forces of 8 N, 6 N and P N act. Given that the forces are in equilibrium, find the values of P and θ.



8. A body of mass m is released from rest at the top of a rough plane inclined at an angle θ to the horizontal. After time t, the mass has traveled a distance d down the slope. Show that the resistance to motion experienced by the body is given by
$$m\left(g\sin\theta - \frac{2d}{t^2}\right).$$

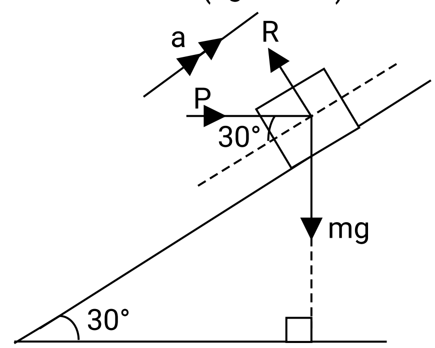
9. Each of the following diagrams shows a mass or masses accelerating in the directions indicated.
(b)

(a)



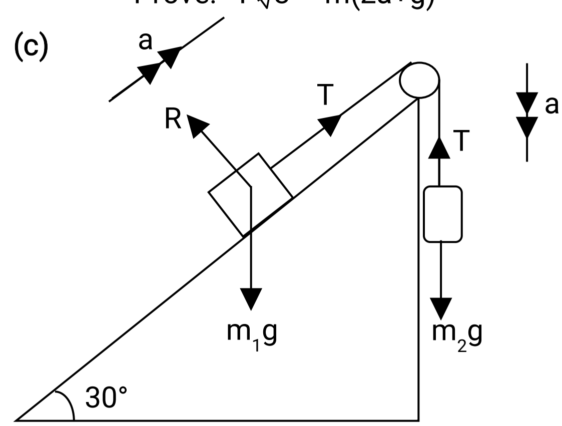
Prove: $3R = m(3g - a\tan\theta)$

(b)



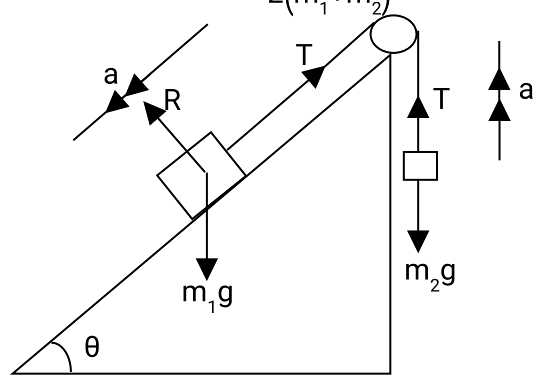
Prove: $P\sqrt{3} = m(2a+g)$

(c)



Prove : $a = \dfrac{(2m_2 - m_1)g}{2(m_1+m_2)}$

(d)



Prove: $T = \dfrac{m_1m_2g(1+\sin\theta)}{m_1+m_2}$

10. A weight W is suspended by two ropes which make 30° and 60° with the horizontal. If the tension in the first rope is 40 N, find the tension in the other rope and the value of W.

Answers to exercises

<u>Exercise: 5A</u>

1. 4 N ; $4\sqrt{3}$ N   2. (a) (b) (c)

3. $T_1 = \dfrac{W\cos\beta}{\cos(\alpha-\beta)}$; $T_2 = \dfrac{W\sin\alpha}{\cos(\alpha-\beta)}$

6. (a) a = 5 ; b = 2 (b) a = 6 ; b = -5   **5.** 3·5 N ; 12·5 N   **6.** 54·95 N ; 58·48

7. (i) 15·2° ; 22·2° (ii) AB : 152·366 N ; BC :105·865 N   8. 1·87 N ; 4·41 N

9. 37·11 N ; 19·75 N   10. XP : 8W ; PY : 6W

<u>Exercise: 5B</u>

1. $2\sqrt{5}$W   2. AP : 12·912 N ; PB : 45·183 N

3. (i) 9·8 N (ii) $\dfrac{49\sqrt{3}}{5}$ N

4. 28·29 N ; 56·58 N   5. (a) $20\sqrt{3}$ N (b) 4 kg   6. 35 N ; $28\sqrt{2}$ N   7. 10 N ; 53·1°

8. 9. (a) (b) (c) (d)   10. $40\sqrt{3}$ N ; 80

6. FRICTION

When one body slides or attempts to slide over another, friction force usually exists between the two surfaces in contact. Friction force acts between rough surfaces in contact. Smooth surfaces in contact are frictionless.

6.1 Laws of friction

The experimental laws below describe the behavior of friction force:

(1) Friction force only exists when one body slides or attempts to slide over another.

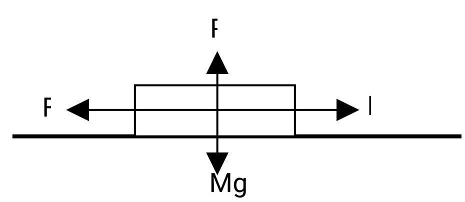
(2) Friction force always opposes the tendency of one body to slide over another.

(3) The magnitude of friction force may vary, always being just sufficient to prevent motion until it reaches a maximum value called limiting value.

(4) The limiting value of friction force is μR, where μ is the coefficient of friction and R is the normal reaction for the surfaces in contact. μ is the measure of roughness of two surfaces in contact and differs for different pairs of surfaces.

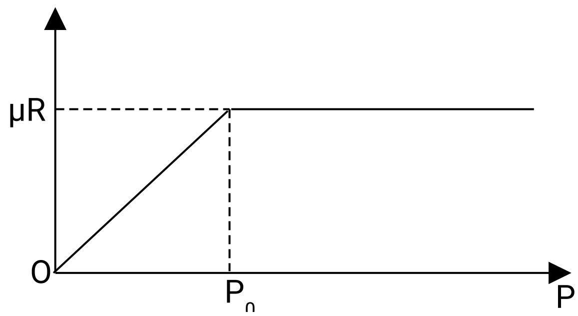
(5) When a body slides over another, the frictional force between them equals the limiting value μR. A consequence of laws (3) and (4) is that the frictional force F obeys the relation F ≤ μR.

Consider a body of mass M, resting on a rough horizontal surface and being pulled by a horizontal force P.



As the magnitude of P is increased gradually from zero, the magnitude of frictional force F will also increase from zero in an attempt to prevent motion.

When motion begins F has reached its maximum, called limiting value μR, and cannot increase any more to prevent motion. So frictional force remains constant, that is, F = μR, whatever the increase in P.



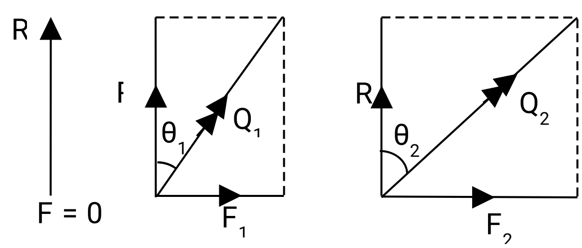
For: $0 \le P < P_0$ the body remains at rest and $F < \mu R$, in addition $F = P$.

$P = P_0$ the body is in limiting equilibrium and *F*= *R*

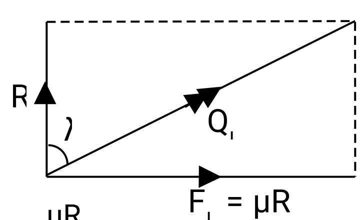
$P > P_0$ the body accelerates and $F = \mu R$.

6.2 Angle of friction

The resultant Q of friction F and normal reaction R is called the total or resultant reaction. It makes an angle θ with the normal reaction, where $\tan\theta = \frac{F}{R}$. The normal reaction R is constant, but the frictional force F may vary.



As the frictional force increases from zero to its maximum value F_L, the limiting value μR, the angle θ increases from zero to a maximum value λ, called the **angle of friction**.



$\tan\lambda = \frac{\mu R}{R}$

$\tan\lambda = \mu \Rightarrow \lambda = \tan^{-1}\mu$

When the frictional force has reached its limiting value F_L, the direction of the resultant reaction Q_L is at an angle λ to the normal reaction R. The magnitude of the resultant reaction Q_L is

$$Q_L = \sqrt{R^2+(\mu R)^2}$$
$$= R\sqrt{1+\mu^2}$$
$$= R\sec\lambda$$

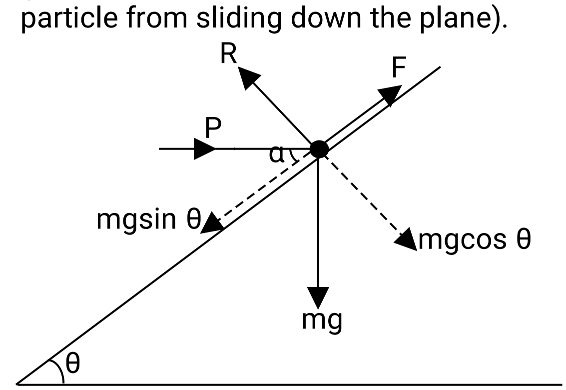
Problem Solving:

The following points are important when solving problems involving a friction force F.

1. Draw a clear diagram. Show the friction force F, do not use μR. Remember F tends to oppose motion.
2. In general $F \le \mu R$, if F has reached its limiting value, then $F = \mu R$ may be used in the solution.
3. Limiting equilibrium indicates that the body is at rest but on the point of moving and then $F = \mu R$.
4. If the body is in equilibrium, then the equations for equilibrium and $F \le \mu R$ are used.
5. If λ is given not μ, then it is often easier to solve the problem by considering the resultant reaction, rather than F and R separately. This case is common in three force problems.

6.3 Least force problems

(a) Least force to keep a particle on a plane (least force required to just prevent the particle from sliding down the plane).



Resolving normal to plane:

$$R = P\sin\alpha + mg\cos\theta$$

Resolving along the plane:

$$F = mg\sin\theta - P\cos\alpha$$

But $F = \mu R$

$$\therefore\ mg\sin\theta - P\cos\alpha = \mu(P\sin\alpha + mg\cos\theta)$$

$$mg\sin\theta - P\cos\alpha = \frac{\sin\lambda}{\cos\lambda}(P\sin\alpha + mg\cos\theta)$$

$P(\sin\alpha\sin\lambda + \cos\alpha\cos\lambda)$

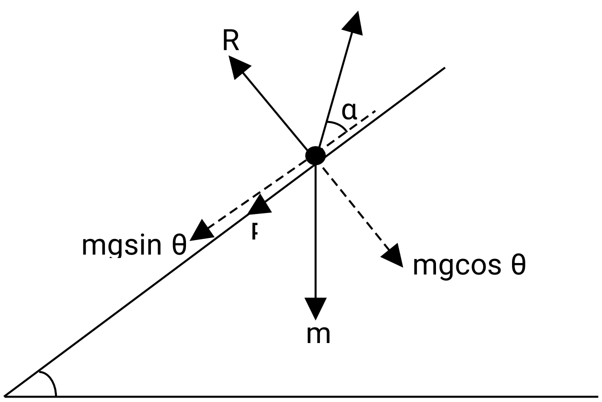
$$= mg(\sin\theta\cos\lambda - \cos\theta\sin\lambda)$$

$$P\cos(\alpha-\lambda) = mg\sin(\theta-\lambda)$$

$$\Rightarrow P = \frac{mg\sin(\theta-\lambda)}{\cos(\alpha-\lambda)}$$

$P_{min} = mg\sin(\theta-\lambda)$, when $\cos(\alpha-\lambda) = 1 \Rightarrow \alpha - \lambda = 0 \Rightarrow \alpha = \lambda$.

(b) Least force to move the particle up the plane. (Least force required to make a particle be at the point of sliding up the plane.)



Resolving normal to the plane:

$$R = mg\cos\theta - P\sin\alpha$$

Resolving along the plane:

$$F = P\cos\alpha - mg\sin\theta$$

But $F = \mu R$

$$P\cos\alpha - mg\sin\theta = \mu(mg\cos\theta - P\sin\alpha)$$

$$P\cos\alpha - mg\sin\theta = \frac{\sin\lambda}{\cos\lambda}(mg\cos\theta - P\sin\alpha)$$

$$P(\cos\alpha\cos\lambda + \sin\alpha\sin\lambda) = mg(\cos\theta\sin\lambda + \sin\theta\cos\lambda)$$

$$P\cos(\alpha-\lambda) = mg\sin(\theta+\lambda) \Rightarrow P = \frac{mg\sin(\theta+\lambda)}{\cos(\alpha-\lambda)}$$

$P_{min} = mg\sin(\theta+\lambda)$, when $\cos(\alpha-\lambda) = 1$.

$$\Rightarrow \alpha - \lambda = 0 \Rightarrow \alpha = \lambda$$

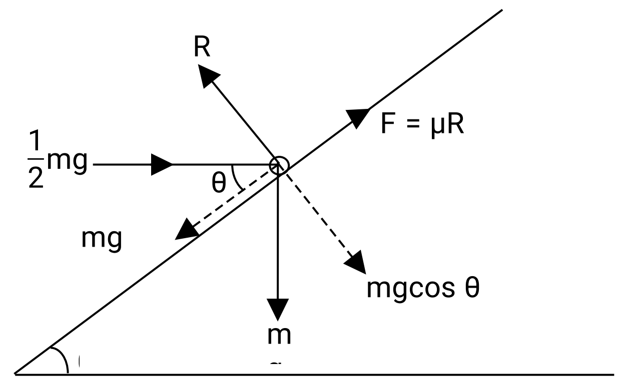
Example 1

A body of mass m kg is just prevented from sliding down a rough inclined plane by a horizontal force of $\frac{1}{2}mg$ N. The coefficient of friction between the body and the plane is μ. Prove that the angle of inclination of the plane to the horizontal is $\tan^{-1}\left[\frac{1+2\mu}{2-\mu}\right]$.

(a) Given that $\mu = \frac{1}{2}$, show that the magnitude of the least force parallel to the plane that will just move the body up the plane is $\frac{11}{10}mg$ newtons.

Solution:

(a)



Normal to plane: $R = \frac{1}{2}mg\sin\theta + mg\cos\theta$

Along plane:

$$mg\sin\theta = \frac{1}{2}mg\cos\theta + \mu R$$

$$mg\sin\theta = \frac{1}{2}mg\cos\theta + \mu(\frac{1}{2}mg\sin\theta + mg\cos\theta)$$

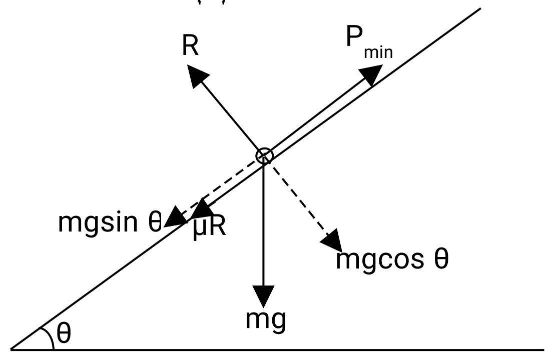
$$2\tan\theta = 1 + \mu\tan\theta + 2\mu X$$

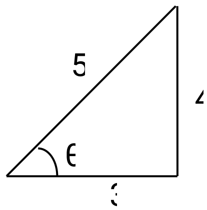
$$\Rightarrow \alpha - \lambda = 0 \Rightarrow \alpha = \lambda$$

(a) If $\mu = \frac{1}{2}$:

$$\theta = \tan^{-1}\left(\frac{1+2\times\frac{1}{2}}{2-\frac{1}{2}}\right)$$

$$\theta = \tan^{-1}\left(\frac{4}{3}\right)$$





Normal plane: R = mgcos θ

Along plane: P_{min} = mgsin θ + μR

$= mg\sin\theta + \mu mg\cos\theta$

$= mg\times\frac{4}{5} + \frac{1}{2}\times mg\times\frac{3}{5}$

$= \frac{11}{10}mg$ N

Example 2

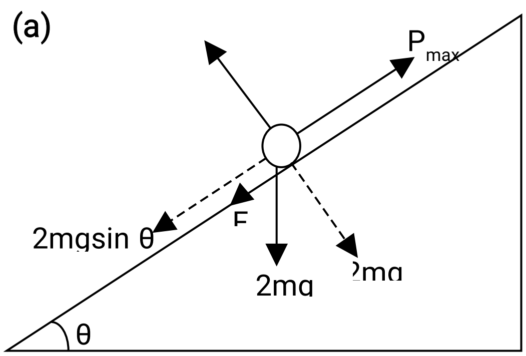
A particle of mass 2m rests on a rough plane inclined to the horizontal at an angle of $\tan^{-1}(3\mu)$, where μ is the coefficient of friction between the particle and the plane. The particle is acted upon by a force of P newtons.

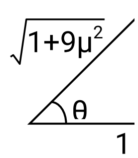
(a) Given that the force acts along a line of greatest slope and the particle is on the point of sliding up, show that the maximum force possible to maintain the particle in equilibrium is $P_{max} = \frac{8\mu mg}{\sqrt{1+9\mu^2}}$.

(b) Given that the force acts horizontally in a vertical plane through a line of greatest slope and the particle is on the point of sliding down the plane, show that the minimum force required to maintain the particle in equilibrium is $P_{min} = \frac{4\mu mg}{1+3\mu^2}$.

Solution:

(a)





Resolving normal to plane: R = 2mgcos θ

Resolving along plane:

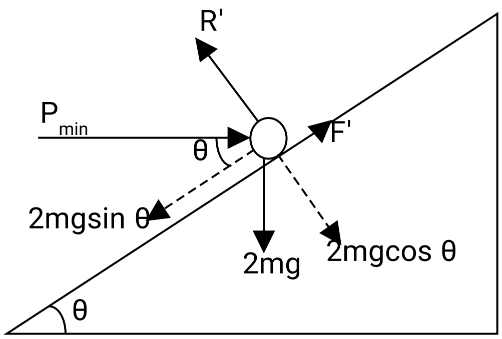
P_{max} = 2mgsin θ + F, F = μR

P_{max} = 2mgsin θ + 2μmgcos θ

$= 2mg\left(\frac{3\mu}{\sqrt{1+9\mu^2}}+\mu\times\frac{1}{\sqrt{1+9\mu^2}}\right)$

$$P_{max} = \frac{8\mu mg}{\sqrt{1+9\mu^2}}$$

(b)



Resolving normal to plane:

$R' = 2mg\cos\theta + P_{min}\sin\theta$ ………………… (i)

Resolving along the plane:

$P_{min}\cos\theta + F' = 2mg\sin\theta$; $F' = \mu R'$

$P_{min}\cos\theta + \mu(2mg\cos\theta + P_{min}\sin\theta) = 2mg\sin\theta$

$P_{min}(1 + \mu\tan\theta) = 2mg(\tan\theta - \mu)$

$P_{min}(1 + \mu\times 3\mu) = 2mg(3\mu - \mu)$

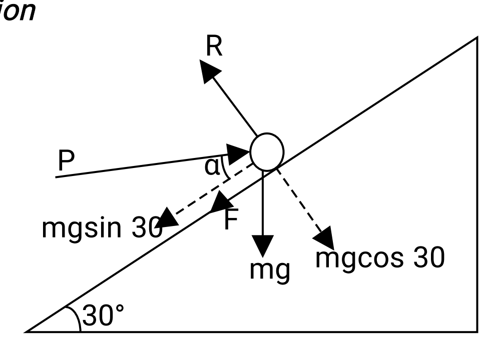
$\Rightarrow P_{min}(1+3\mu^2) = 4\mu mg$

$$P_{min} = \frac{4\mu mg}{1+3\mu^2}$$

Example 3

A particle of mass m is placed on a rough plane inclined at 30° to the horizontal, if the angle of friction between the mass and the plane is λ, show that the minimum force required to move the particle up the plane is given by $\frac{1}{2}mg(\cos\lambda+\sqrt{3}\sin\lambda)$.

Solution



Resolving normal to plane:

R = Psin α + mgcos 30………………… (i)

Resolving parallel to plane:

Pcos α = F + mgsin 30…………. (ii)

But F = μR

From equation (i) and equation (ii):

Pcos α = μ(Psin α + mgcos 30) + mgsin 30

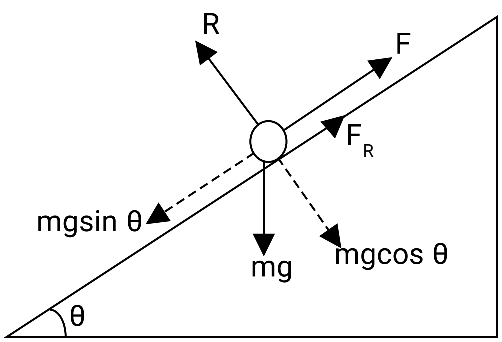
$$P\cos\alpha = \frac{\sin\lambda}{\cos\lambda}\left(P\sin\alpha + mg\cos 30\right) + mg\sin 30$$

$$P(\cos\alpha\cos\lambda - \sin\alpha\sin\lambda)$$
$$= mg(\sin 30\cos\lambda + \cos 30\cos\lambda)$$
$$P\cos(\alpha+\lambda) = mg\sin(30+\lambda)$$
$$\Rightarrow P = \frac{mg\sin(30+\lambda)}{\cos(\alpha+\lambda)}$$
$$P_{min} = mg\sin(30+\lambda), \text{ when } \cos(\alpha+\lambda) = 1$$
$$= mg(\sin 30\cos\lambda + \cos 30\sin\lambda)$$
$$P_{min} = \frac{1}{2}mg(\cos\lambda + \sqrt{3}\sin\lambda)$$

Example 4

A force F acting parallel to and up a rough inclined plane of inclination θ is just sufficient to prevent a body of mass m from sliding down the plane. A force 4F acting parallel to and up the same rough plane causes the mass to be at a point of moving up the plane. If μ is the coefficient of friction between the body and the plane, show that 5μ = 3tan θ.

Solution:

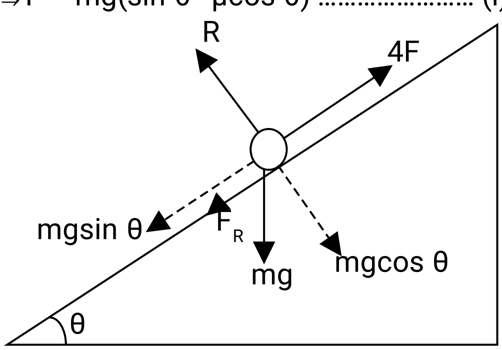


Resolving normal to plane:

$$R = mg\cos\theta$$

Resolving parallel to plane:

$$F = mg\sin\theta - F_R,\quad F_R = \mu R$$
$$F = mg\sin\theta - \mu mg\cos\theta$$
$$\Rightarrow F = mg(\sin\theta - \mu\cos\theta) \text{ .... (i)}$$



Resolving normal to plane:

$$R = mg\cos\theta$$

Resolving parallel to plane:

$$4F = mg\sin\theta + F_R,\ F_R = \mu R$$
$$4F = mg\sin\theta + \mu mg\cos\theta$$
$$\Rightarrow 4F = mg(\sin\theta + \mu\cos\theta) \text{ .... (ii)}$$

From equation (i) and equation (ii):

$$4mg(\sin\theta - \mu\cos\theta) = mg(\sin\theta + \mu\cos\theta)$$
$$\Rightarrow 4(\tan\theta - \mu) = (\tan\theta + \mu)$$
$$\therefore\ 5\mu = 3\tan\theta$$

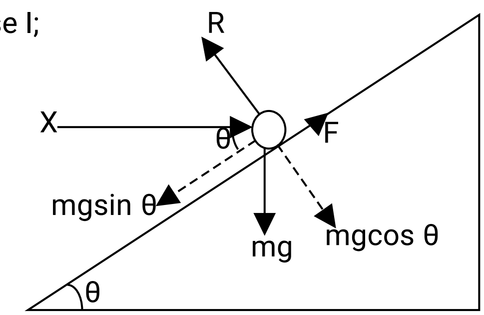
Example 5

A horizontal force X is just sufficient to prevent a body of mass m from sliding down a rough plane of inclination θ. A horizontal force 4X applied to same mass on the same rough plane, causes the mass to be on the point moving up the plane. If μ is the coefficient of friction between the mass and the

$5\mu\tan^2\theta - 3(\mu^2 + 1)\tan\theta + 5\mu = 0.$

Solution

Case I;



Resolving normal to plane:

$$R = X\sin\theta + mg\cos\theta$$

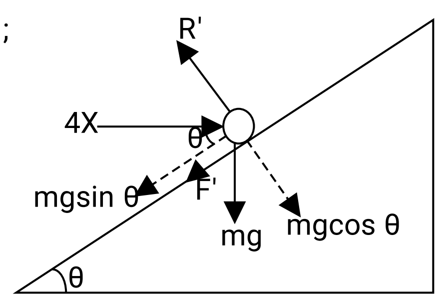
Resolving parallel to plane:

$$X\cos\theta + F = mg\sin\theta,\ F = \mu R$$

Thus

$$X\cos\theta + \mu(X\sin\theta + mg\cos\theta) = mg\tan\theta$$
$$X = \frac{mg(\tan\theta - \mu)}{(1 + \mu\tan\theta)} \text{ ....(i)}$$

Case II;



Resolving normal to plane:

$$R' = 4X\sin\theta + mg\cos\theta$$

Resolving parallel to plane:

$$4X\cos\theta = F' + mg\sin\theta,\quad F' = \mu R'$$
$$4X\cos\theta =$$
$$\mu(4X\sin\theta + mg\cos\theta) + mg\sin\theta$$
$$X = \frac{mg(\tan\theta + \mu)}{4(1 - \mu\tan\theta)} \text{ ....(ii)}$$

From equation (i) and equation (ii):

$$\frac{mg(\tan\theta+\mu)}{4(1-\mu\tan\theta)} = \frac{mg(\tan\theta-\mu)}{(1+\mu\tan\theta)}$$

$(\tan\theta + \mu)(1 + \mu\tan\theta) = 4[(\tan\theta - \mu)(1 - \mu\tan\theta)$

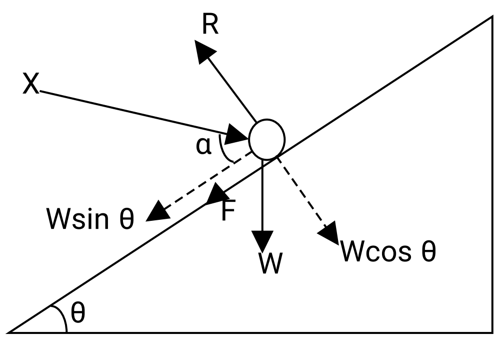
$$5\mu\tan^2\theta - 3(\mu^2 + 1)\tan\theta + 5\mu = 0$$

Example 6

The least force which will move a mass up an inclined plane is P. Show that the least force acting parallel to the plane which will move the mass upwards is $P\sqrt{1+\mu^2}$, where μ is the co-efficient of friction.

Solution

Case I;



Resolving normal to the plane:

$$R = X\sin\alpha + W\cos\theta$$

Resolving parallel to plane:

$$X\cos\alpha = F + W\sin\theta,\ F = \mu R$$

$$X\cos\alpha = \mu(X\sin\alpha + W\cos\theta) + W\sin\theta$$

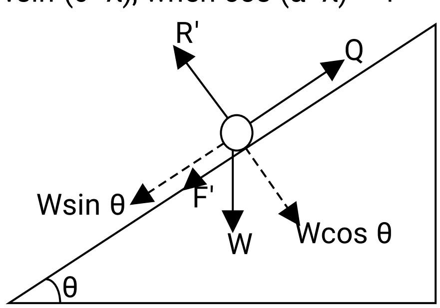
$$X\cos\alpha = \frac{\sin\lambda}{\cos\lambda}(X\sin\alpha + W\cos\theta) + W\sin\theta$$

$$X\cos(\alpha+\lambda) = W\sin(\theta+\lambda)$$

$$\Rightarrow X = \frac{W\sin(\theta+\lambda)}{\cos(\alpha+\lambda)}$$

$X_{min} = P = W\sin(\theta+\lambda)$, when $\cos(\alpha+\lambda) = 1$

Case II;

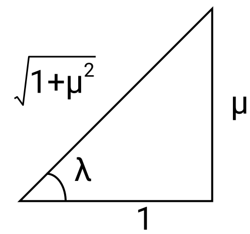


Resolving normal to plane: $R' = W\cos\theta$

Resolving parallel to plane:

$$Q = F + W\sin\theta;\ \text{but } F' = \mu R'$$

$$\therefore\ Q = \mu W\cos\theta + W\sin\theta$$



$$Q = \frac{\sin\lambda}{\cos\lambda}\times W\cos\theta + W\sin\theta$$

$$= \frac{W}{\cos\lambda}[\sin\lambda\cos\theta + \sin\theta\cos\lambda]$$

$$= \frac{W\sin(\theta+\lambda)}{\cos\lambda}$$

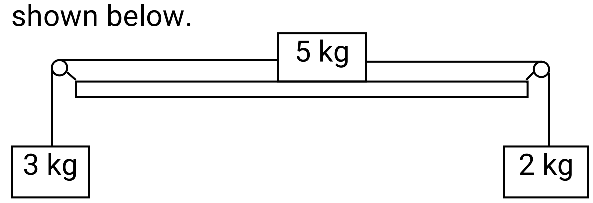
$$Q = \frac{P}{\cos\lambda}$$

$$Q = P \div \frac{1}{\sqrt{1+\mu^2}}$$

$$Q = P\sqrt{1+\mu^2}$$

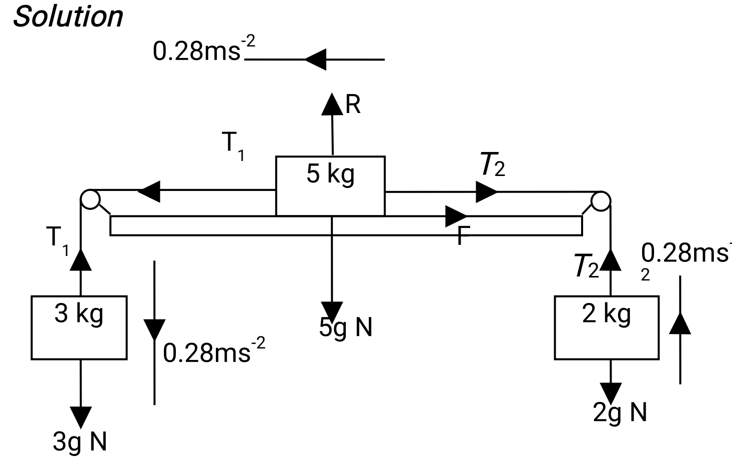
Example 7

Masses of 3 kg, 5 kg and 2 kg are connected as shown below.



The table is rough and when the system is released from rest, it moves with an acceleration of $0{\cdot}28\ \text{m s}^{-2}$. Find the co-efficient of friction between the 5 kg mass and the table.

Solution



For 3 kg mass:

$$3g - T_1 = 3\times0{\cdot}28$$

$$T_1 = 3(9{\cdot}8 - 0{\cdot}28)$$

$$T_1 = 28{\cdot}56\ \text{N}$$

For 2 kg mass:

$$T_2 - 2g = 2\times0{\cdot}28$$

$$T_2 = 2(9{\cdot}8 + 0{\cdot}28)$$

$$T_2 = 20\cdot16 \text{ N}$$

For 5 kg mass:

Resolving horizontally:

$$T_1 - (T_2 + F) = 5\times0\cdot28$$
$$28\cdot56 - 20\cdot16 - F = 1\cdot4$$
$$F = 7 \text{ N}$$

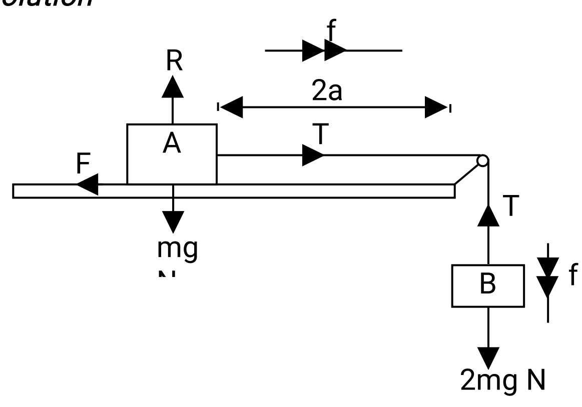
Resolving vertically:

$$R = 5g = 5\times9\cdot8 = 49 \text{ N}$$
$$F = \mu R$$
$$7 = 49\mu \Rightarrow \mu = \frac{1}{7}$$

Example 8

Two particles A and B of mass m and 2m respectively are connected by a light inextensible string which passes over a smooth pulley attached to the edge of a rough horizontal table. The particle A is held at rest on the table at a distance 2a from the pulley with the string taut and B hangs vertically below the pulley. The system is released from rest. The coefficient of friction between A and the table is $\frac{3}{5}$. When A is released, find the magnitude of the acceleration of each particle and the magnitude of the tension in the string. After B has fallen a distance a, it hits a horizontal floor and does not rebound. Show that A comes to rest at a distance $\frac{2}{9}a$ from the pulley.

Solution



For B:

$$2mg - T = 2mf \text{ ……………………… (i)}$$

For A:

$$T - F = mf, \text{ but } F = \mu R$$
$$T - \frac{3}{5}mg = mf \text{ …………………(ii)}$$

Adding equation (i) and equation (ii):

$$\frac{7}{5}mg = 3mf \Rightarrow f = \frac{7}{15}g \text{ m s}^{-2}$$

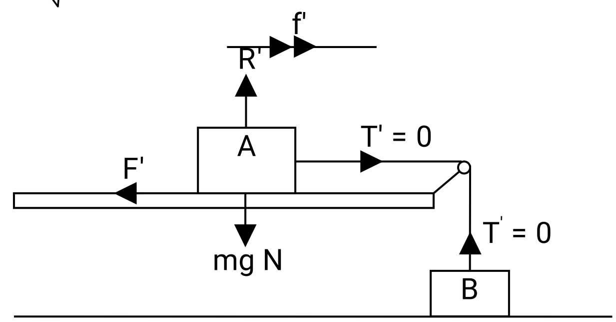
From equation (i): $T = 2m(g - f) = 2m\left(g - \frac{7}{15}g\right)$

$$T = \frac{16}{15}mg \text{ N}$$

From $v^2 = u^2 + 2fs$, f- acceleration

$$v_B^2 = 2\times\frac{7}{5}g\times a$$
$$v_B = \sqrt{\frac{14}{15}ag}$$

Note that for the next part of the motion: $u_A = \sqrt{\frac{14}{15}ag}$



For A:

$$0 - F = mf', \ F = \mu R$$
$$-\frac{3}{5}mg = mf' \Rightarrow f' = -\frac{3}{5}g \text{ m s}^{-2}$$

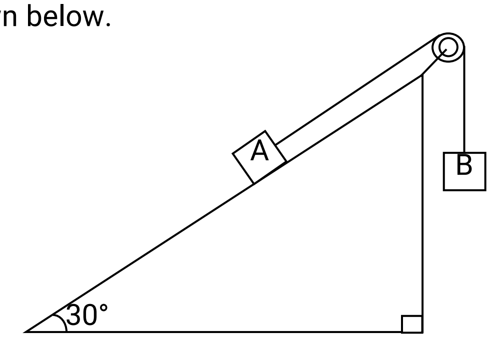
From $v^2 = u^2 + 2f's$, f'− acceleration

$$0 = \left(\sqrt{\frac{14}{15}ag}\right)^2 - 2\times\frac{3}{5}g\times s$$
$$s = \frac{7}{9}a$$

Distance of A from pulley $= a - \frac{7}{9}a = \frac{2}{9}a$

Example 9

Blocks A and B of masses 2 kg and 3 kg respectively are connected by a light inextensible string passing over a smooth fixed pulley as shown below.

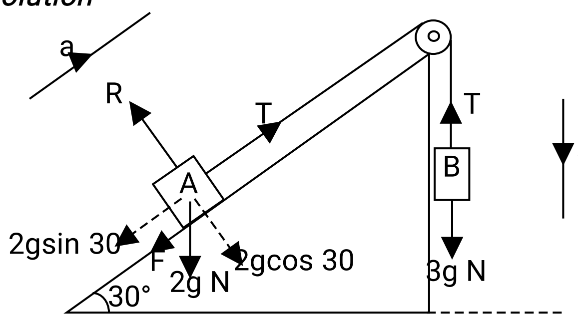


Block A is resting on a rough plane inclined at 30° to the horizontal while block B hangs freely. When the system is released from rest, block B travels a distance of 0.75m before it attains a speed of $2\cdot25 \text{ m s}^{-1}$. Calculate the:

(i) acceleration of the blocks.

(ii) coefficient of friction between block A and the plane.
(iii) reaction of the pulley on the string.

Solution



(i) $u = 0$, $v = 2{\cdot}25$ m s^{-1}, $s = 0{\cdot}75$ m

From $v^2 = u^2 + 2as$

$2{\cdot}25^2 = 2\times0{\cdot}75a$

$a = 3{\cdot}375$ m s^{-2}

(ii) For B:

$3g - T = 3\times3{\cdot}375$

$\Rightarrow T = 3(9{\cdot}8 - 3{\cdot}375)$

$T = 19{\cdot}275$ N

For A:

Along plane:

$T - (2g\sin 30 + F) = 2\times3{\cdot}375$

$19{\cdot}275 - 2\times9{\cdot}8\times\frac{1}{2} - F = 6{\cdot}75$

$F = 2{\cdot}725$ N

Normal to the plane:

$R = 2g\cos 30$

$R = 2\times9{\cdot}8\times\cos 30$

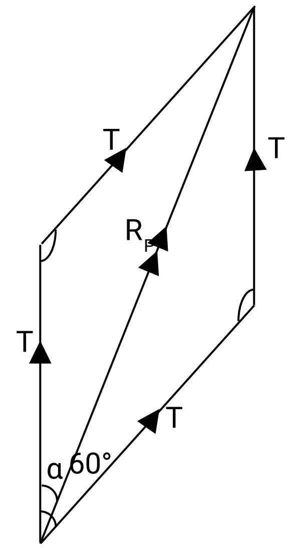
$\Rightarrow R = 16{\cdot}974$ N

From $F = \mu R$

$2{\cdot}725 = \mu\times16{\cdot}974$

$\mu = 0{\cdot}1605$

(iii) **Note**: The reaction of the pulley on the string is equal but opposite the resultant tension on the pulley.



$R_P^2 = T^2 + T^2 - 2T\times T\cos 120 \Rightarrow R_P^2 = 3T^2$

$R_P = T\sqrt{3} \Rightarrow R = 19{\cdot}275\times\sqrt{3}$

$\Rightarrow R = 33{\cdot}385$ N

$$\frac{T}{\sin\alpha} = \frac{R_P}{\sin 120}$$

$$\alpha = \sin^{-1}\left(\frac{T\sin 120}{T\sqrt{3}}\right) = \sin^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{\sqrt{3}}\right) = \sin^{-1}\frac{1}{2}$$

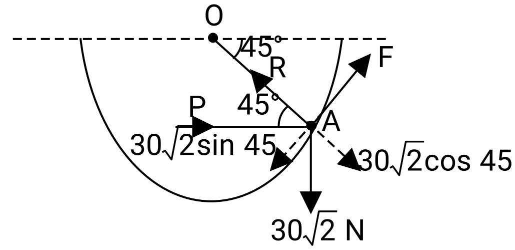
$\alpha = 30°$

Example 10

A fixed hollow hemisphere has centre O and is fixed so that the plane of the rim is horizontal. A particle A of weight $30\sqrt{2}$ N can move on the inside surface of the hemisphere. The particle is acted upon by a horizontal force P, whose line of action is in a vertical plane through O and A. OA makes an angle of 45° with the horizontal. If the coefficient of friction between the particle and the hemisphere is 0·5 and the particle is about to slip downwards, find the:

(a) normal reaction.
(b) value of P.

Solution



Resolving along AO:

$R = P\cos 45 + 30\sqrt{2}\cos 45$

$\Rightarrow R = \frac{P}{\sqrt{2}} + 30$…………(i)

(a) Resolving normal to AO:

$F + P\sin 45 = 30\sqrt{2}\sin 45 \Rightarrow F + \dfrac{P}{\sqrt{2}} = 30$

From equation (i) and $F = 0{\cdot}5R$,

$0{\cdot}5\left(\dfrac{P}{\sqrt{2}}+30\right) + \dfrac{P}{\sqrt{2}} = 30 \Rightarrow P = 10\sqrt{2}$ N

(b) From equation (i):

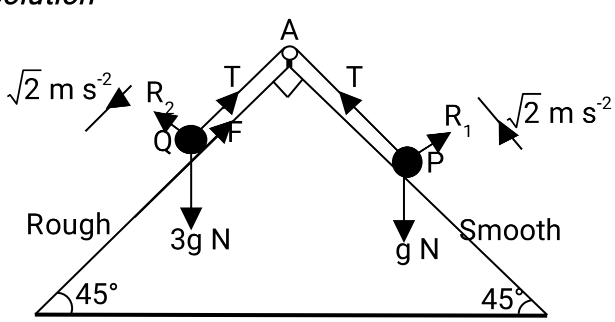
$R = \dfrac{10\sqrt{2}}{\sqrt{2}} + 30 \Rightarrow R = 40$ N

Example 11

A smooth plane and a rough plane, both inclined at 45° to the horizontal, intersect at a fixed horizontal ridge. A particle P of mass 1 kg is held on a smooth plane by a light inextensible string which passes over a smooth frictionless pulley A and is attached to a particle Q of mass 3 kg which rests on a rough plane. The plane containing P, Q and A is perpendicular to the ridge. The system is released from rest with the string taut. Given that the acceleration of each particle is of magnitude $\sqrt{2}$ m s^{-2}, find the:

(a) tension in the string.
(b) coefficient of friction between Q and the rough plane.
(c) magnitude and direction of the force exerted by the string on the pulley. [Take g = 10 m s^{-2}]

Solution



(a) Let T be the tension in the string and F the friction force on Q.
Let R_1 and R_2 be the normal reactions on P and Q respectively.
For P:
Equation of motion:

$T - g\sin 45 = 1\times\sqrt{2}$

$T - 10\times\dfrac{\sqrt{2}}{2} = \sqrt{2}$

$T = 6\sqrt{2}$ N

(b) For Q:
Resolving perpendicular to slope

$R_2 = 3g\cos 45$

$= 3\times10\times\dfrac{\sqrt{2}}{2}$

$R_2 = 15\sqrt{2}$ N

Equation of motion:

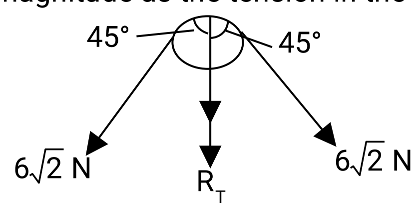
$3g\sin 45 - T - F = 3\times\sqrt{2}$

$F = 30\times\dfrac{\sqrt{2}}{2} - 3\sqrt{2} - 6\sqrt{2}$

$F = 6\sqrt{2}$ N

From $F = \mu R_2 \Rightarrow 6\sqrt{2} = \mu\times15\sqrt{2} \Rightarrow \mu = \dfrac{2}{5}$

(c) Forces on the pulley have the same magnitude as the tension in the strings.



The resultant force on the pulley is

$R_T = 6\sqrt{2}\cos 45 + 6\sqrt{2}\cos 45$

$R_T = 12$ N vertically downwards

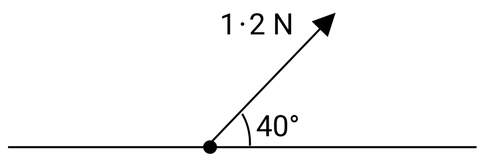
Exercises

Exercise: 6A

1. A box of mass 4·9 kg rests on a rough plane inclined at 60° to the horizontal. If the coefficient of friction between the box and the plane is 0·35, determine the force acting parallel to the plane which will move the box up the plane.
2. A box of mass 2 kg is at rest on a plane inclined at 25° to the horizontal. The coefficient of fiction between the box and the plane is 0·4. What minimum force applied parallel to the plane would move the box up the plane?
3. A body of mass 8 kg rests on a rough plane inclined at an angle θ to the horizontal. If the coefficient of friction is μ, find the least horizontal force in terms of θ, μ and g which will hold the body in equilibrium on the plane.
4. A mass of 5 kg lies on a horizontal rough

surface. If the coefficient of friction between the mass and the surface is 0·4, find the force required to move the mass horizontally if it is applied at 45° above the horizontal.

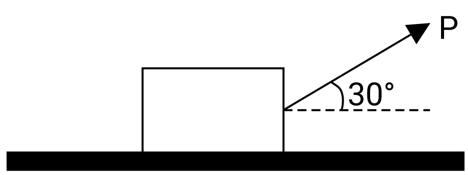
5. A small ring of mass 0·25 kg is threaded on a fixed rough horizontal rod. The ring is pulled upwards by a light string which makes an angle of 40° with the horizontal as shown in the diagram.



If the tension in the string is 1·2 N and the coefficient of friction between the ring and the rod is μ. Find the:

(a) normal reaction between the ring and the rod.
(b) value of μ given that the ring is in limiting equilibrium.

6. The figure shows a body of mass 1 kg on a rough horizontal surface, coefficient of friction 0·5.



A force P acting at an angle of 30° to the horizontal is used to move the body at constant speed.

(i) Indicate all forces acting on the body.
(ii) Find the value of P.

7. A carton of mass 3 kg rests on a rough plane inclined at 30° to the horizontal. The coefficient of friction between the carton and the plane is $\frac{1}{3}$. Find a horizontal force that should be applied to make the carton just about to move up the plane.

8. A particle of mass m, is placed on a rough plane inclined at 30° to the horizontal. Given that the angle of friction is λ, show that the minimum force required to move the particle up the plane is $\frac{1}{2}mg\left(\cos\lambda+\sqrt{3}\sin\lambda\right)$.

If the force is three times the least force that would keep the particle on the plane, show that $\lambda=\tan^{-1}\frac{1}{2\sqrt{3}}$.

9. A particle of mass m rests on a rough plane inclined at an angle θ above the horizontal. A force applied parallel and up the line of greatest slope will just prevent motion down the slope. If this force is doubled, motion is about to take place up the slope, show that if μ is the coefficient of friction, then 3μ = tan θ.

10. A particle of mass m is at the point of being pulled along a horizontal rough floor by a string inclined at an angle θ to the floor. The coefficient of friction between the particle and the floor is μ. Prove that if tan θ = μ, the least tension in the string is $\frac{\mu mg}{\sqrt{1+\mu^2}}$.

11. A particle is at rest on a rough plane inclined at an angle α to the horizontal. The coefficient of friction is 0·25. Given that the particle is on the point of sliding down the plane, find α.

12. A particle of mass $\frac{3}{2}$ kg rests on a rough plane inclined at 45° to the horizontal. It is maintained in equilibrium by a horizontal force Q. Given that the coefficient of friction between the particle and the plane is $\frac{1}{4}$, calculate the value of Q when the particle is on the point of moving:

(a) down the plane.
(b) up the plane.

13. A particle of weight W is at rest on an inclined plane under the action of a force P acting parallel to the line of greatest slope in an upward direction. If λ is the angle of friction and the angle of inclination of the plane is 2λ. Show that:

$$P_{max} = W\tan\lambda(4\cos^2\lambda-1) \text{ and } P_{min} = \mu W.$$

14. A particle of mass m kg is kept at rest on an inclined plane of angle 3λ to the horizontal by a force P acting parallel to the line of greatest slope of the plane, where λ is the angle of friction. Prove that $P_{min} = 2mg\sin\lambda$.

15. A box of mass 1 kg rests on a rough plane inclined at 30° to the horizontal. If the coefficient of friction between the box and the surface is 0· 5, find the horizontal force required to move the box up the plane.

Exercise: 6B

1. A particle of mass 0·8 kg is on a rough plane inclined at an angle α to the horizontal, where $\tan\alpha = \frac{5}{12}$. The particle is acted upon by an upward force of 4 N parallel to a line of greatest slope of the plane. Given that the particle is about to move up the plane, calculate the coefficient of friction between the particle and the plane. Given that the force of 4 N is then removed, find the acceleration of the particle down the plane. (Use $g = 10$ m s^{-2})

2. A particle of mass 4 kg resting on a rough plane of inclination 3 in 5 is connected to another of mass 6 kg by a light inextensible string passing over a smooth fixed pulley at the top of the plane. Given that the 6 kg mass particle is hanging freely and descends with an acceleration of 2 m s^{-2} when the system is released.
   (i) Find the coefficient of friction.
   (ii) If the 6 kg mass particle hits the ground with a speed of 4 m s^{-1}, find the further distance moved by the 4 kg mass before coming to instantaneous rest.

3. A block of mass 12 kg is placed on a rough plane, μ = 0·5, inclined at 30° to the horizontal. If the block is kept in equilibrium by a horizontal force of magnitude P newtons. Find the minimum value of P.

4. A particle of mass 4 kg rests on a rough horizontal surface. The particle is acted on by a force P N acting at 30° to the surface and is in limiting equilibrium. Given that the normal reaction between the particle and the surface is 54 N, find the:
   (i) value of P.
   (ii) coefficient of friction between the particle and the surface.

5. A particle of weight W lies on a rough plane inclined at an angle α to the horizontal where $\sin\alpha = \frac{5}{13}$ and the coefficient of friction between the particle and the plane is $\frac{1}{3}$. A horizontal force F acts on the body. Prove that if the particle is to stay on the plane then $\frac{3}{41}W < F < \frac{27}{31}W$.

6. A force P inclined at 56° to the horizontal pushes a body of mass 2 kg resting on a rough horizontal surface. The coefficient of friction between the body and the surface is $\frac{1}{4}$. Given that the force causes the body to accelerate at 2·1 m s^{-2}, find the value of P.

7. A particle is projected from the bottom of a rough inclined plane of angle 30° with a speed of $\sqrt{20g}$ m s^{-1}, to move up a line of greatest slope. The coefficient of friction between the particle and the plane is $\frac{1}{\sqrt{3}}$.
   (a) Find the distance the particle covers up the plane before coming to instantaneous rest.
   (b) What distance will the particle cover if the speed is halved?

8. 8. A particle P, of mass 2·5 kg rests in equilibrium on a rough inclined plane inclined at 20° to the horizontal. A force X N acts up a line of greatest slope. If the coefficient of friction between P and the plane is 0·4 and the particle is on the point of moving up the plane.
   (i) Calculate the value of X.
   (ii) Given that X is now removed, show that P remains at rest on the plane.

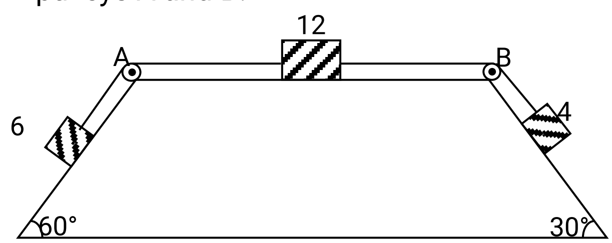
9. Blocks A and B of masses 0·5 kg and 0·8 kg respectively are connected by a light inextensible string passing over a smooth fixed pulley. Block A is resting on a rough horizontal table while block B hangs freely. When the system is released from rest, block B travels a distance of 0·4 m in 0·5 s. Calculate the:
   (a) acceleration of the blocks.
   (b) coefficient of friction between block A and the table.
   (c) reaction of the pulley on the string.

10. A particle of mass 2 kg lies on a rough horizontal table (coefficient of friction $\frac{2}{3}$) and is connected to a light inextensible string which passes over a smooth fixed pulley at the edge of the table, then under a smooth movable pulley of mass 3 kg and over a smooth fixed pulley and a mass of 7 kg hangs freely at its end. If the system is released from rest, find the acceleration of the two masses and the movable pulley. Find

also the force that the string exerts on the pulley at the edge of the table.

11. A particle of mass 50 kg is pulled across a rough horizontal floor by an inextensible rope inclined at 40° to the horizontal. If the tension in the rope is 400 N and the coefficient of friction between the particle and the floor is 0·3. Find the acceleration of the particle.

12. The diagram below shows a 12 kg mass on a horizontal rough plane. The 6 kg and 4 kg masses are on rough planes inclined at angles of 60° and 30° respectively. The masses are connected by light inextensible strings passing over smooth fixed pulleys A and B.

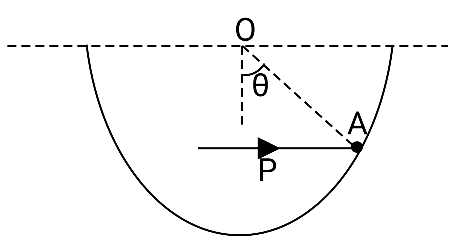


The planes are equally rough with coefficient of friction $\frac{1}{12}$. If the system is released from rest, find:

(a) acceleration of the system.

(b) tensions in the strings.

13. A fixed hollow hemisphere has centre O and the plane of the rim is horizontal. A particle A of weight W can move on the inside surface of the hemisphere. The particle is acted on by a horizontal force of magnitude P, whose line of action is in a vertical plane through O and A. The diagram below shows the situation when A is in equilibrium, the line OA makes an angle θ with the downward vertical.



(a) Given that the inside surface of the hemisphere is smooth, find tan θ in terms of P and W.
(b) Given that the inside surface of the hemisphere is rough, with coefficient of friction μ between the surface and A, and that the particle is about to slip downwards, show that $\tan\theta = \frac{P+\mu W}{W-\mu P}$.

14. A particle of mass 3 kg rests on a rough horizontal table, the coefficient of friction between the particle and the table being $\frac{1}{3}$. The particle is connected by a light inextensible string passing over a smooth fixed pulley at the edge of the table to a second particle of mass 4 kg which hangs freely. If the system is released from rest, find the acceleration of the particles and the tension in the string.

15. A mass of 4 kg rests on a horizontal rough table coefficient of friction $\frac{1}{2}$ and is connected by a light inextensible string passing over a smooth fixed pulley at the edge of the table to a smooth pulley of mass 5 kg hanging freely. Over this hanging pulley passes another light inextensible string to the ends of which are connected particles of mass 2 kg and 3 kg. If the system is released from rest, calculate the:
   (i) accelerations of the masses.
   (ii) tension in each string.

16. A mass of 4 kg rests on a rough horizontal table with coefficient of friction $\frac{1}{2}$. It is connected by a light inextensible string passing over a smooth fixed pulley at the edge of the table to a smooth pulley of mass $\frac{1}{2}$ kg hanging freely, over the hanging pulley passes a light inextensible string to the ends of which are connected masses of 2 kg and 3 kg. If the system is released from rest, find the:
(a) accelerations of the movable pulley, 2 kg and 3 kg masses.
(b) tensions in the strings.

Answers to Exercises

Exercise: 6A

1. 49·99 N 2. 15·3888 N 3. $\frac{8g(\tan\theta-\mu)}{(1+\mu\tan\theta)}$

4. $P > 19 \cdot 8$ N 5. (a) $1 \cdot 68$ N (b) $0 \cdot 548$ 6. (i) (ii) $4 \cdot 39$ N 7. $22 \cdot 453$ N 8. 9. 10. 11. $14 \cdot 0°$

12. (a) $8 \cdot 82$ N (b) $24 \cdot 5$ N 13. 14. 15. $14 \cdot 84$ N

Exercise: 6B

1. $\frac{1}{8}$; $\frac{35}{13}$ m s^{-2} 2. (i) $0 \cdot 4872$ (ii) $0 \cdot 825$ m 3. $7 \cdot 06$ N 4. (i) $29 \cdot 6$ (ii) $0 \cdot 475$

5. 6. $11 \cdot 873$ N 7. (a) 10 m (b) $\frac{5}{2}$ m
8. (i) $17 \cdot 5885$ N (ii)

9. (a) $3{\cdot}2$ m s^{-2} (b) $0 \cdot 751$ (c) $7 \cdot 467$ N at 45° to the horizontal

10. 2 kg mass : $2{\cdot}558$ m s^{-2} ; 7 kg mass : $7{\cdot}2024$ m s^{-2}(downwards) ; Movable pulley : $2{\cdot}3222$ m s^{-2}(upwards) ; $25 \cdot 715$ N at 45° below the horizontal

11. $4{\cdot}731$ m s^{-2} 12. (a) $0 \cdot 738$ m s^{-2} (b) $44 \cdot 04$ N ; $25 \cdot 38$ N

13. (a) $\frac{P}{W}$ (b) 14. $4{\bullet}2$ m s^{-2} ; $22 \bullet 4$ N

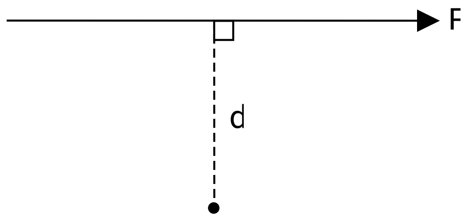
15. (i) 4 kg mass: $5{\bullet}5391$ m s^{-2} ;2 kg mass: $4{\bullet}6869$ m s^{-2} (downwards);3 kg mass: 6.3913ms^{-2} (downwards) (ii) Upper string: 41.7564N; Lower string: $10 \bullet 2262$ N

16. (a) $3{\bullet}4775$ m s^{-2}(downwards) ; 2.213ms^{-2} (downwards) ; $4{\bullet}742$ m s^{-2}(downwards)
(b) Upper string: 33.61N; Lower string: 15.174N

7. MOMENTS

7.1 Moment of a force

Moment of a force about a point is obtained by multiplying the magnitude of the force by the perpendicular distance from the point to the line of action of the force. The moment of a force F about point A is F×d.



If the line of action of the of the force passes through a certain point, the moment of the force about that point is zero.

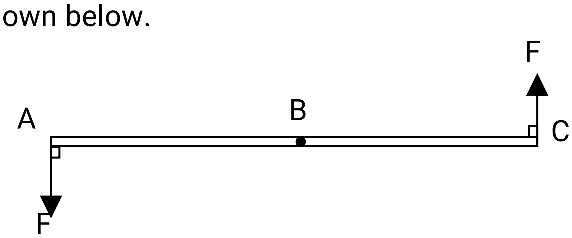
Since moment of a force about a point causes a turning effect which may be clockwise or anticlockwise, by convention clockwise moments are negative and anticlockwise moments are positive.

7.2 Couples

A couple is formed by two parallel unlike forces of equal magnitude whose lines of action do not coincide.

7.2.1 Moment of a Couple

Consider two forces each of magnitude *F* as shown below.



Taking moments about:

A: G = F×AC

B: G = F×AB + F×BC = F×AC

C: G = F×AC

Where G is the sum of moments.

Hence the moment of a couple is the same about all points in the plane of the forces forming the couple. Note that forces forming a couple yield no resultant force but there is a resultant moment.

A given system of forces is said to reduce to a couple if the resultant of the forces is zero yet they produce a resultant moment.

That is: X = 0, Y = 0 but G ≠ 0.

X- algebraic sum of forces in horizontal direction.

Y- algebraic sum of forces in vertical direction.

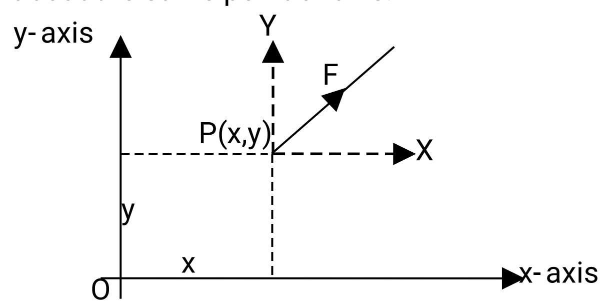
G−sum of moments of all forces about a point or axis.

7.2.2 Forces in Equilibrium

A given system of forces is said to be in equilibrium if the resultant of the forces is zero and the sum of moments of the forces about a given point or axis is also zero.

7.3 Equation of line of action of the Resultant

If a given system of forces is not in equilibrium, the single force which can represent all the forces is called their resultant. The resultant force causes the same translation effect as that which could be caused by the individual forces and has a net turning effect as that which would be caused by all the individual forces. If $\mathbf{F} = \begin{pmatrix} X \\ Y \end{pmatrix}$ is the resultant of any given number of forces and G is the sum of moments of the individual forces about a point (we shall call the origin). The equation of the line of action of the resultant is obtained by using the fact that the moment of the resultant force about a point or axis is equal to the sum of moments of the individual forces about the same point or axis.



Taking moments about O:

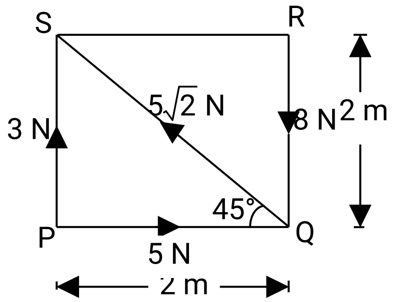
$$G = xY - yX$$

Hence G = xY - yX is the equation of the line of action of the resultant. It can also be obtained from $G = \begin{vmatrix} x & y \\ X & Y \end{vmatrix}$.

Example 1

In a square PQRS of side 2 m, forces of magnitude 5, 8, 3 and $5\sqrt{2}$ newtons act along PQ, RQ, PS and QS respectively. Show that the system is equivalent to a couple and find the moment of the couple. If a fifth force of magnitude $6\sqrt{2}$ newtons acts along PR, find the equation of the line of action of the resultant of the enlarged system.

Solution



$$R = \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 5+0+0-5\sqrt{2}\cos 45 \\ 0-8+3+5\sqrt{2}\sin 45 \end{pmatrix}$$

$$= \begin{pmatrix} 5-5 \\ -8+8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

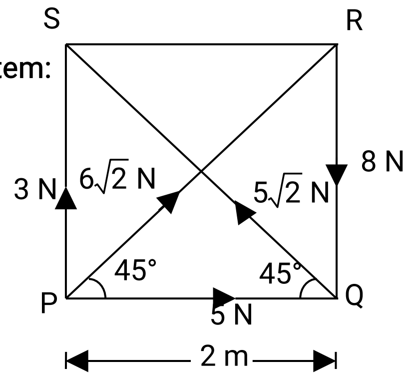
Taking moments about P:

Sum of moments, $G = -(8\times 2) + \left(5\sqrt{2}\times 2\sin 45\right)$

$= -6$ N m

Hence the system reduces to a couple since the resultant force is zero but the sum of moments is not zero.

Enlarged system:



$$R = \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0+6\sqrt{2}\cos 45 \\ 0+6\sqrt{2}\sin 45 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$$

Taking moments about P:

$G = -(8\times 2) + (5\sqrt{2}\times 2\sin 45)$

$G = -6$ N m

$G = \begin{vmatrix} x & y \\ X & Y \end{vmatrix}$

$G = xY - yX$

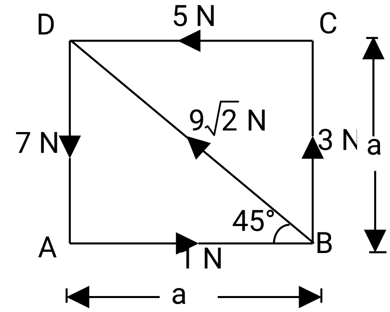
$-6 = 6x - 6y$

$y = x + 1$

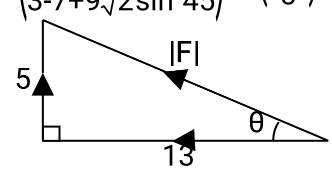
Example 2

Forces of 1 N, 3 N, 5 N, 7 N and $9\sqrt{2}$ N act along the sides AB, BC, CD, DA and diagonal BD of a square of side a, their directions being given by the order of the letters. Taking A as the origin and AB, AD as x and *y*-axes respectively. Find the resultant force and show that its line of action has equation $5x + 13y = 17a$.

Solution:



$$F = \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 1-5-9\sqrt{2}\cos 45 \\ 3-7+9\sqrt{2}\sin 45 \end{pmatrix} = \begin{pmatrix} -13 \\ 5 \end{pmatrix}$$



$|F| = \sqrt{(-13)^2 + (5)^2}$

$= \sqrt{194}$

$= 13\cdot 9284$ N

Taking moments about A:

$G = 3\times a + 5\times a + 9\sqrt{2}\times a\sin 45$

$= 17a$

From $G = \begin{vmatrix} x & y \\ X & Y \end{vmatrix}$

$G = xY - yX \Rightarrow 17a = 5x + 13y$

$\therefore \quad 5x + 13y = 17a$

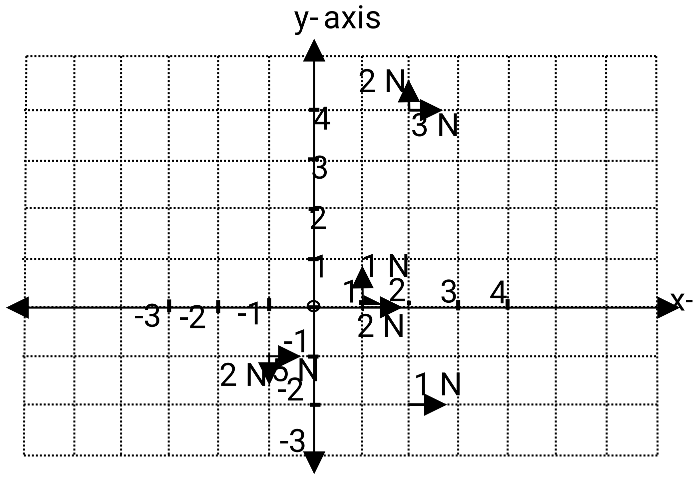
Example 3

The forces $2i + j$, $3i + 2j$, $5i - 2j$ and i N act at points with position vectors i, $2i + 4j$, $-i - j$ and $2i - 2j$ m respectively. Find the equation of the line of action of the resultant force.

Solution

$F = (2i + j) + (3i + 2j) + (5i - 2j) + i$

$= 11i + j$



Taking moments about the origin:

$$G = (2\times2)-(3\times4) + (1\times1) + (2\times0) + (5\times1) + (2\times1) + (1\times2)$$

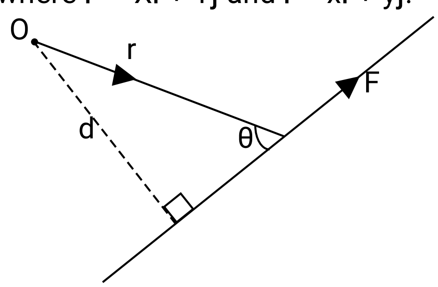
$$G = 2 \text{ N m}$$

From $G = \begin{vmatrix} x & y \\ X & Y \end{vmatrix} = xY - yX$

$$2 = x - 11y \Rightarrow x - 11y = 2$$

7.4 Moment as a Vector

Let a force **F** pass through a point with position vector **r**, where **F** = X**i** + Y**j** and **r** = x**i** + y**j**.



The moment of **F** about O is of magnitude $= |F|d = |F||r|\sin\theta$

From the cross product of vectors:

$$\mathbf{r}\times\mathbf{F} = |r||F|\sin\theta\,\hat{u}$$

$|\mathbf{r}\times\mathbf{F}| = |r||F|\sin\theta$, since $|\hat{u}| = 1$

Hence the moment in vector form, $\vec{G} = \mathbf{r}\times\mathbf{F}$.

This implies that; $\vec{G} = \mathbf{r}\times\mathbf{F} = \begin{vmatrix} i & j & k \\ x & y & 0 \\ X & Y & 0 \end{vmatrix}$

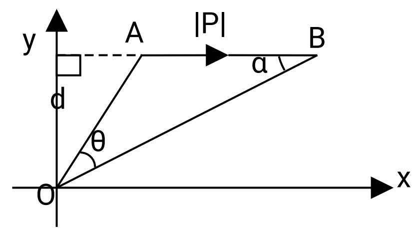
$$\vec{G} = \mathbf{i}\begin{vmatrix} y & 0 \\ Y & 0 \end{vmatrix} + \mathbf{j}\begin{vmatrix} 0 & x \\ 0 & X \end{vmatrix} + \mathbf{k}\begin{vmatrix} x & y \\ X & Y \end{vmatrix}$$

Hence $\vec{G} = (xY - yX)\mathbf{k}$

Example 4

If a line AB represents the force P both in magnitude and direction, show that the moment of P about a point O is represented in magnitude by twice the area of triangle AOB.

Solution



Area of triangle

$$OAB = \frac{1}{2}\times|\vec{OA}|\times|\vec{OB}|\sin\theta \text{ .... (i)}$$

Taking moments about O:

$$|G| = |P|\times d$$

$$= |P||\vec{OB}|\sin\alpha, \text{ but } \frac{|\vec{OA}|}{\sin\alpha} = \frac{|\vec{AB}|}{\sin\theta}$$

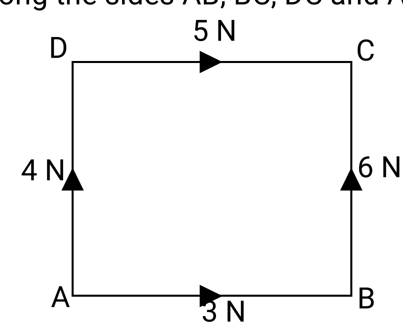
$$|G| = |\vec{AB}||\vec{OB}|\times\frac{|\vec{OA}|}{|\vec{AB}|}\sin\theta = |\vec{OA}||\vec{OB}|\sin\theta \text{ .... (ii)}$$

From equation (i) and equation (ii)

$$|G| = 2\times \text{ area of triangle OAB}$$

Example 5

The diagram shows a square ABCD of side 1 m with forces of magnitude 3 N, 6 N, 5 N and 4 N acting along the sides AB, BC, DC and AD.



Find the:

(i) magnitude of the resultant of the forces.
(ii) total moment of the forces about A.
(iii) perpendicular distance from A to the line of action of the resultant.

Solution:

(i) $\mathbf{F} = \begin{pmatrix} 3+5 \\ 4+6 \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \end{pmatrix}$

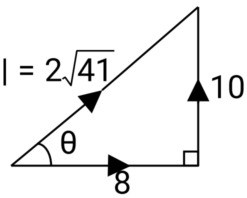
$|F| = \sqrt{8^2+10^2} = 2\sqrt{41}$ N

(ii) Taking moments about A:

$G = (6\times1)-(5\times1) = 1$ N m

$= 1 \text{ N m Anticlockwise}$

(iii)



From $G = \begin{vmatrix} x & y \\ X & Y \end{vmatrix} = xY - yX$

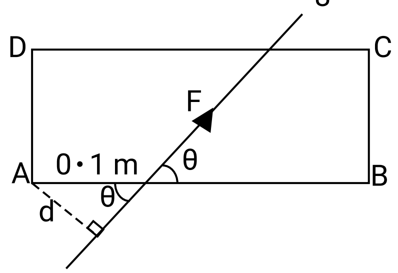
$1 = 10x - 8y$

Along AB, $y = 0$

$1 = 10x \Rightarrow x = 0 \cdot 1 \text{ m}$

Along BC, $x = 1$

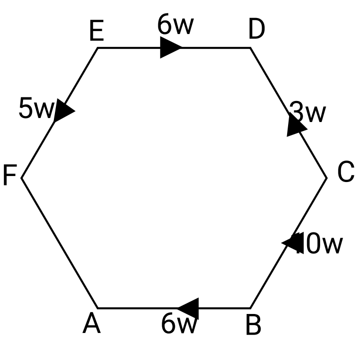
$1 = 10 \times 1 - 8y \Rightarrow y = \frac{9}{8} \text{ m}$



$\sin\theta = \frac{d}{0 \cdot 1} \Rightarrow d = 0 \cdot 1 \times \frac{10}{2\sqrt{41}}$

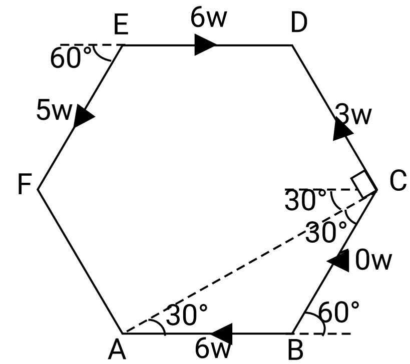
$\Rightarrow d = \frac{1}{2\sqrt{41}} \text{ m}$

Example 6



The diagram shows a regular plane hexagon with sides of length a. Forces having magnitudes 6w, 10w, 3w, 6w and 5w act along five of the sides of the hexagon as shown. Prove that the resultant of this system of forces intersects AB produced at a point P distant $\frac{1}{8}a$ from B. The magnitude of the resultant is R and it makes an angle θ with AB. Find R and θ.

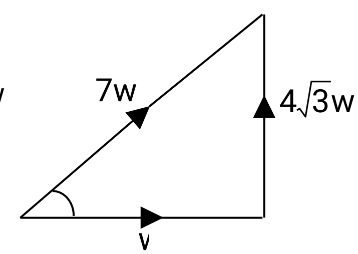
Solution



$$R = \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -6w+10w\cos 60-3w\cos 60+6w-5w\cos 60 \\ 0+10w\sin 60+3w\sin 60+0-5w\sin 60 \end{pmatrix}$$

$$R = \begin{pmatrix} w \\ 4\sqrt{3}w \end{pmatrix}$$

$$R = \sqrt{w^2 + \left(4\sqrt{3}w\right)^2} = 7w$$



$\cos\theta = \frac{w}{7w}$

$\theta = 81 \cdot 8°$

s about A:

$G = 10w \times a\sin 60 + 3w \times a\sqrt{3} - 6w \times a\sqrt{3} + 5w \times a \sin 60$

$G = \frac{9\sqrt{3}}{2}aw$

From $G = \begin{vmatrix} x & y \\ X & Y \end{vmatrix} = xY - yX$

$4\sqrt{3}wx - wy = \frac{9\sqrt{3}}{2}aw \Rightarrow y = 4\sqrt{3}x - \frac{9\sqrt{3}}{2}a$

Along AB, $y = 0$

$\therefore\ 0 = 4\sqrt{3}x - \frac{9\sqrt{3}}{2}a \Rightarrow x = \frac{9}{8}a$

Distance from B $= \frac{9}{8}a - a = \frac{1}{8}a$

Example 7

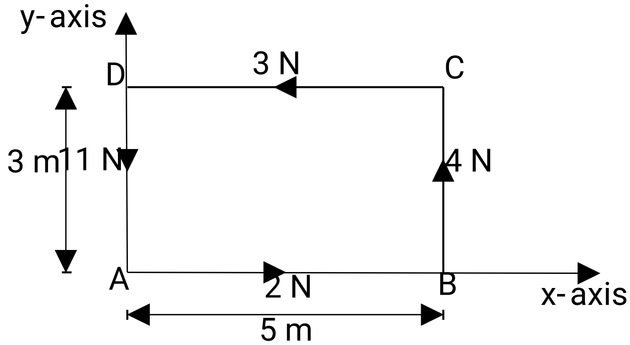
ABCD is a rectangle in which AB = 5 m, BC = 3 m forces of 2 N,4 N,3 N and 11 N act along AB,BC,CD and DA respectively, their directions being given by the order of the letters.

(a) Taking AB as the x-axis and AD as the y-axis, find the resultant force and the equation of its line of action.

(b) If the forces are replaced by a single force at the origin and a couple, determine the moment and direction of the couple.

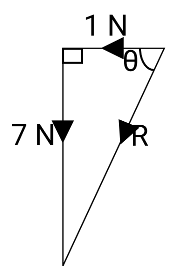
Solution:

(a)



$(\rightarrow)$: $X = 2 - 3 = -1$ N

$(\uparrow)$: $Y = 4 - 11 = -7$ N



$$R = \sqrt{1^2 + 7^2} = 5\sqrt{2}\text{ N}$$

$$\tan\theta = \frac{7}{1} \Rightarrow \theta = \tan^{-1} 7 \Rightarrow \theta = 81{\cdot}9°$$

Taking moments about A:

$$G = 2\times0 + 4\times5 + 3\times3 + 11\times0$$

$$G = 29\text{ N m}$$

Equation of line of action:

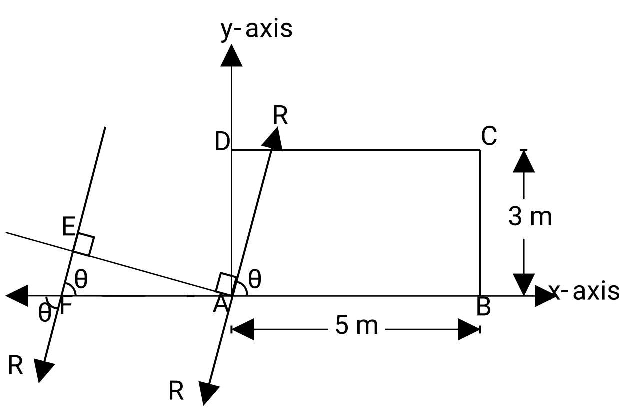
$$G = xY - yX$$

$$29 = -7x + y \Rightarrow y = 7x + 29$$

(b) Along AB, $y = 0$

$$0 = 7x + 29 \Rightarrow x = \frac{-29}{7}$$

$$F\left(\frac{-29}{7}, 0\right)$$



Let us introduce two equal and opposite forces at origin A, parallel to the resultant $R = 5\sqrt{2}$ N. Then one of the forces and the resultant forms a couple and the other is the single force at A.

The moment of the couple about A, $G' = 5\sqrt{2}\times AE$, $AE = AF\sin\theta$

$$G' = 5\sqrt{2}\times AF\sin 81{\cdot}9$$

$$G' = 5\sqrt{2}\times\frac{29}{7}\times\frac{7}{5\sqrt{2}}$$

$$G' = 29\text{ N m}$$

Hence the moment of the couple is 29 N m anticlockwise.

Example 8

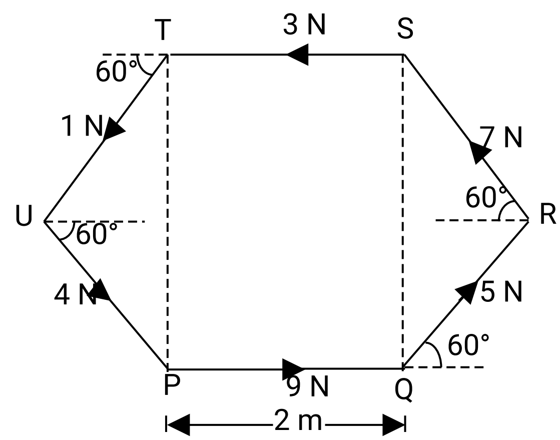
Six forces 9 N, 5 N, 7 N, 3 N, 1 N and 4 N act along the sides PQ, QR, RS, ST, TU and UP of a regular hexagon of side 2 m, their directions being indicated by the order of the letters. Taking PQ as the reference axis, express each of the forces in vector form. Hence find the:

(i) magnitude and direction of the resultant of the forces.

(ii) distance from P where the line of action of the resultant cuts PQ.

Solution:

(i)

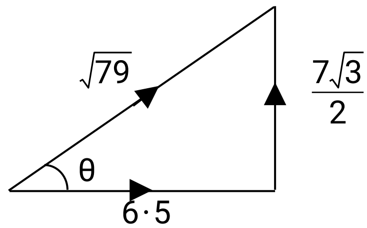


$$R = \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$= \begin{pmatrix} 9+5\cos 60-7\cos 60-3-1\cos 60+4\cos 60 \\ 0+5\sin 60+7\sin 60+0-1\sin 60-4\sin 60 \end{pmatrix}$$

$$R = \begin{pmatrix} 6{\cdot}5 \\ \frac{7\sqrt{3}}{2} \end{pmatrix}$$

$$|R| = \sqrt{6{\cdot}5^2 + \left(\frac{7\sqrt{3}}{2}\right)^2} = \sqrt{79}\text{ N}$$



$$\cos\theta = \frac{6\cdot 5}{\sqrt{79}} \Rightarrow \theta = 43\cdot 0°$$

(ii) Taking moments about P:

$G = 2\times 5\sin 60 + 7\times 2\sqrt{3} + 3\times 2\sqrt{3} + 1\times 2\sin 60$

$G = 26\sqrt{3}$ N m

From $G = \begin{vmatrix} x & y \\ X & Y \end{vmatrix}$

$G = xY - yX$

$26\sqrt{3} = \frac{7\sqrt{3}}{2}x - \frac{13}{2}y$

Along PQ, $y = 0 \Rightarrow 26\sqrt{3} = \frac{7\sqrt{3}}{2}x \Rightarrow x = 7\cdot 4286$ m

Hence the line of action of the resultant cuts PQ-produced at a distance of 7·4286 m from P.

Example 9

The centre of a regular hexagon ABCDEF of side 2a metres is O. Forces of magnitude 4 N, s N, t N, 1 N, 7 N and 3 N act along the sides AB, BC, CD, DE, EF and FA respectively. Their directions are in the order of the letters.

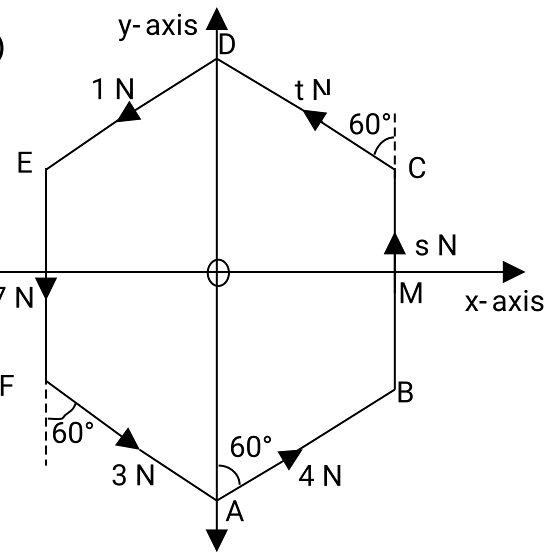
(a) Given that the resultant of the six forces is of magnitude $2\sqrt{3}$ N acting in a direction perpendicular to BC, determine the values of s and t.

(b) (i) Show that that the sum of moments of the forces about O is given by $27a\sqrt{3}$ N m.

(ii) If the midpoint of BC is M, find the equation of the line of action of the resultant, refer to OM as x-axis and OD as y-axis.

Solution

(a)



$$R = \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 4\sin 60 + 0 - t\sin 60 - 1\sin 60 + 0 + 3\sin 60 \\ 4\cos 60 + s + t\cos 60 - 1\cos 60 - 7 - 3\cos 60 \end{pmatrix}$$

$$R = \begin{pmatrix} \frac{\sqrt{3}}{2}(6-t) \\ s + \frac{1}{2}t - 7 \end{pmatrix}$$

Since R is perpendicular to BC, that is, it is horizontal: $s + \frac{1}{2}t - 7 = 0$

$$|R| = \sqrt{\left(\frac{\sqrt{3}(6-t)}{2}\right)^2 + 0^2} = 2\sqrt{3}$$

$$\frac{3}{4}(6-t)^2 = 12 \Rightarrow 6 - t = \pm\sqrt{16}$$

$$6 - t = \pm 4$$

Either t = 2 or t = 10

When t = 2, s = 6 and when t = 10, s = 2

(b) (i) Taking moments about O: (When t = 2 and s = 6)

$G_1 = a\sqrt{3}(4+6+2+1+7+3) = 23a\sqrt{3}$ N m

(When t = 10 and s = 2)

$G_2 = a\sqrt{3}(4+2+10+1+7+3) = 27a\sqrt{3}$ N m

Hence since $G = 27a\sqrt{3}$ N m $\Rightarrow$ t = 10, s = 2

(ii) From $G = \begin{vmatrix} x & y \\ X & Y \end{vmatrix} = xY - yX$

$$27a\sqrt{3} = 0\times x - y\times\frac{\sqrt{3}}{2}(6-10) \Rightarrow y = \frac{27}{2}a$$

Example 10

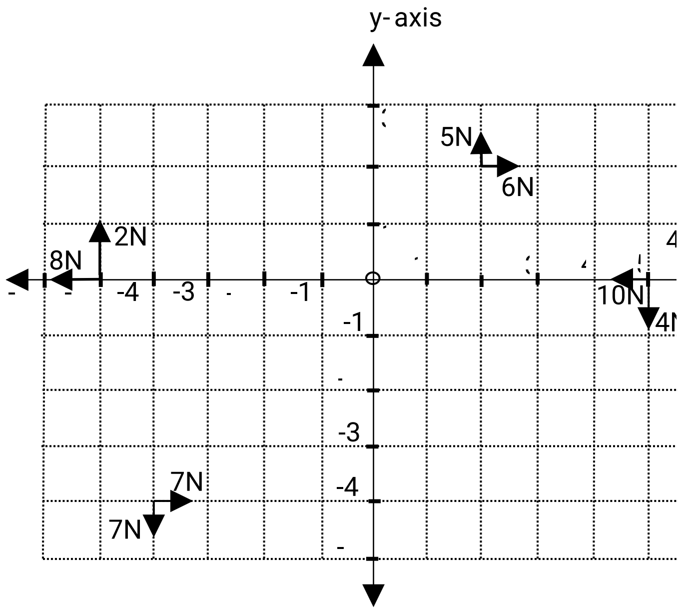
The forces $F_1 = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$, $F_2 = \begin{pmatrix} -10 \\ -4 \end{pmatrix}$, $F_3 = \begin{pmatrix} 7 \\ -7 \end{pmatrix}$, $F_4 = \begin{pmatrix} -8 \\ 2 \end{pmatrix}$ and $F_5 = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ N act at the points (2,2), (5,0), (-4,-4), (0,-5) and (6,0) respectively.

(a) Show that the system reduces to a couple and state the moment of the couple.

(b) The force F_3 is removed and a couple added to the new system. Given that the line of action of the resultant of the forces cuts the y-axis at +10 units, find the couple.

<u>Solution:</u>

(a) $\mathbf{F} = \begin{pmatrix}6\\5\end{pmatrix} + \begin{pmatrix}-10\\-4\end{pmatrix} + \begin{pmatrix}7\\-7\end{pmatrix} + \begin{pmatrix}-8\\2\end{pmatrix} + \begin{pmatrix}5\\4\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$



Taking moments about the origin:

$G = 5\times2 - 6\times2 - 4\times5 + 7\times4 + 7\times4$
$\quad - 2\times5 + 4\times6$
$\quad = 48$ N m

Since $\mathbf{F} = \mathbf{0}$ and $G \neq 0$ the system reduces to a couple of moment 48 N m anticlockwise.

(b) When F_3 is removed the resultant of the remaining forces is:

$$R = \begin{pmatrix}6\\5\end{pmatrix} + \begin{pmatrix}-10\\-4\end{pmatrix} + \begin{pmatrix}-8\\2\end{pmatrix} + \begin{pmatrix}5\\4\end{pmatrix} = \begin{pmatrix}-7\\7\end{pmatrix}$$

$$|R| = \sqrt{(-7)^2 + 7^2} \Rightarrow |R| = 7\sqrt{2} \text{ N}$$

$$\tan\alpha = \frac{7}{7} \Rightarrow \alpha = 45°$$

Taking moments about the origin:

$G' = 5\times2 - 6\times2 - 4\times5 + 2\times5 + 4\times6 \Rightarrow G' = -8$ N m

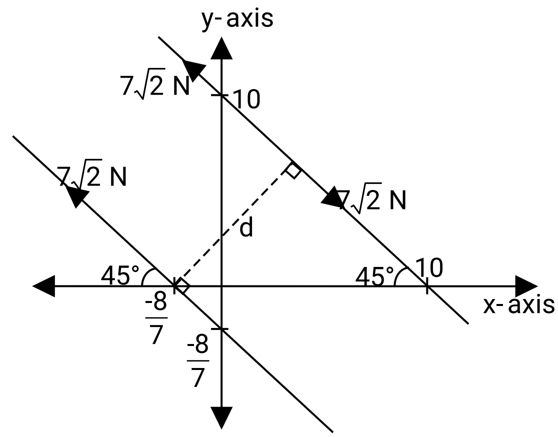
Equation of the line of action of this resultant:

$$G' = \begin{vmatrix}x & y\\-7 & 7\end{vmatrix} \Rightarrow -8 = \begin{vmatrix}x & y\\-7 & 7\end{vmatrix}$$

$7x + 7y = -8$

Along the x-axis, $y = 0 \Rightarrow x = \frac{-8}{7}$ and along the y-axis $x = 0 \Rightarrow y = \frac{-8}{7}$

Now two equal and opposite forces (couple) are introduced parallel to the resultant to shift the line of action of the resultant force.



$$\sin 45 = \frac{d}{\left(10+\frac{8}{7}\right)} \Rightarrow d = \frac{78}{7}\times\frac{1}{\sqrt{2}} \Rightarrow d = \frac{78}{7\sqrt{2}}$$

Moment of the couple; $G'' = -\left(7\sqrt{2}\times d\right)$

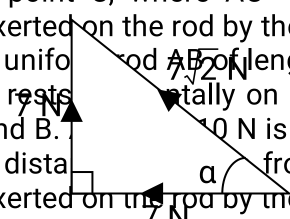
$$\Rightarrow G'' = -\left(7\sqrt{2}\times\frac{78}{7\sqrt{2}}\right) = -78 \text{ N m}$$

Hence the moment of the couple is 78 N m clockwise

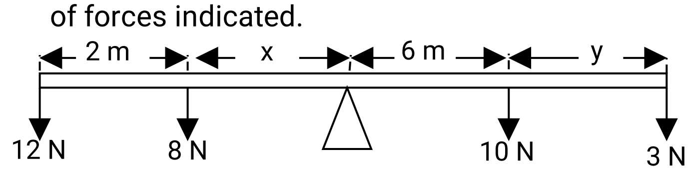
Exercises

Exercise: 7A

1. A uniform rod AB of length 3 m and weight 50 N rests horizontally on smooth supports at A and B. A load of 24 N is attached to the rod at a point C, where AC = 1 m, find the forces exerted on the rod by the supports.
2. A unifo rod AB of length 2 m and weight 20 N rests ntally on smooth supports at A and B. 0 N is attached to the rod at a dista from A. Find the forces exerted on the rod by the supports at A and B.



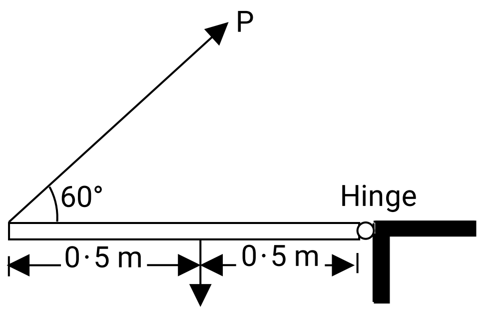
3. A light beam AB of length 13 m balances horizontally about the pivot under the action of forces indicated.



Find the values of x and y.

4. A uniform beam AB of mass m and length 2l has its lower end A resting on a rough horizontal ground and is kept in equilibrium, at an angle of 45° to the horizontal by a rope attached to end B. If the rope makes an angle of 60° with BA and T is the tension in the rope, show that $T = \frac{mg}{\sqrt{6}}$.

5. (a) A uniform beam AB of length 8 m has a mass of 30 kg, masses of 24 kg and 50 kg are suspended from its ends, at what point must the beam be supported so that it rests horizontally?
   (b) A hinged trap door of mass 15 kg and length 1 m is to be opened by applying a force P at an angle of 60° as shown below. Calculate the value of P.

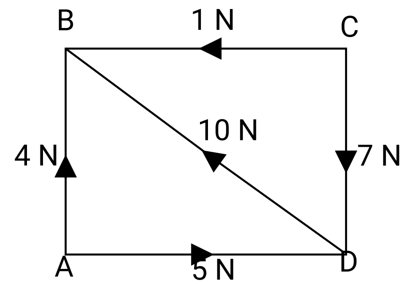


6. A uniform bar of length 2·8 m and weight 80 N has loads of 20 N and 40 N attached to its ends A and B respectively. If the bar balances in a horizontal position when smoothly supported at P, find the distance AP.

7. A uniform rod APQB of length 2 m rests horizontally on smooth supports at P and Q, where PQ = 1·2 m. If he reaction at P is twice the reaction at Q find the distance AP. Given that when a weight of 5 N is attached to the rod at B the reactions at P and Q are equal, find the weight of the rod.

8. Forces 2i−3j, 7i + 9j, -6i−4j, -3j−2j act along a lamina at points (1,-1), (1,1), (-1,-1), (-1,1) respectively. Determine the:
   (i) resultant of the forces.
   (ii) sum of their moments about (0,0). What effect do the forces have on the body?

9. Forces of magnitude 1 N, 7 N, 10 N, 4 N and 5 N act along the lines CB, CD, DB, AB and AD respectively of rectangle ABCD of side AB = 3 m and BC = 4 m as shown in the diagram.



Find the equation of the line of action the resultant force.

10. Find the moments of the force (-8i+12j) N acting at (2,3) about:
    (i) (0,0)
    (ii) (-2,-1)
    (iii) (-4,2)

Exercise: 7B

1. Three forces of -3i + bj, 2bi + 3j and 3i−4j act on a plane at the points A(-3,3), B(4,1) and C(-3,-3) respectively. The sum of the moments of the forces about the origin is 32 N m clockwise. Find the value of b. If instead the sum of moments of the forces about the origin is 32 N m anticlockwise, find the value of b.

2. Two forces having magnitudes of 4 N and 3 N act along the sides AB and AD respectively of a square ABCD. If the square is of side 2 m, calculate the perpendicular distance of their resultant from the midpoint of DC.

3. ABC is a triangle such that AB = 6 m, BC = CA = 4 m. Forces of 9 N, 6 N and 6 N act along AB, BC and CA respectively. Show that the system is equivalent to a couple and find its moment.

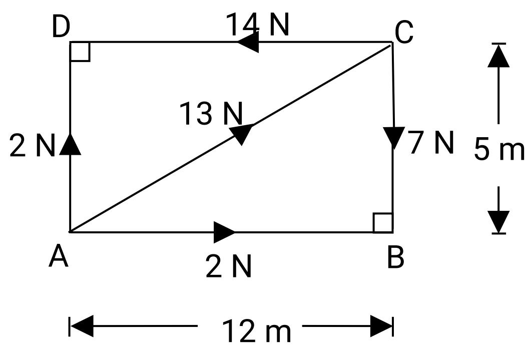
4. A force of (3i−5j) N acts at a point with position vector (6i+j) m and a force of (-3i+5j) N acts at a point with position vector (4i+j) m. Show that these forces reduce to a couple and find the moment of the couple.

5. Forces of (ai+bj) N and (6i−4j) N act at points with position vectors (-2i−2j) m and (3i−j) m respectively. If the forces reduce to a couple, find a and b and the moment of the couple.

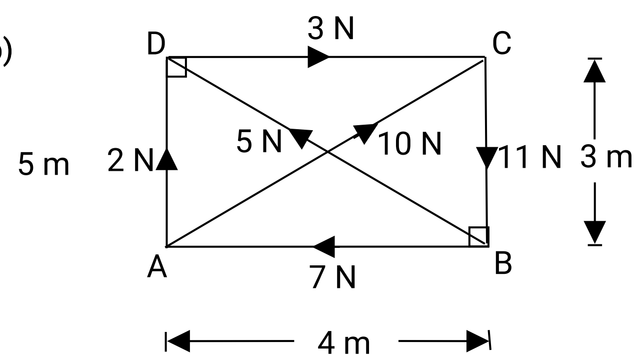
6. Forces of 6j N and -6j N act at the origin and at position vector 2i m respectively. Show that the moment of the forces about any point P(x,y) is independent of x and y.

7. Forces of magnitude 3 N, 4 N, 4 N, 3 N and 5 N act along the lines AB, BC, CD, DA and AC respectively of a square ABCD whose side has a length of a units. The directions of the forces are indicated by the order of the letters.
   (a) Find the magnitude and direction of the resultant force.
   (b) If the line of action of the resultant force cuts AB produced at E, find the length AE.
8. Show that each of the following systems of forces is equivalent to a couple and in each case find the moment of the couple.
   (a) (b)

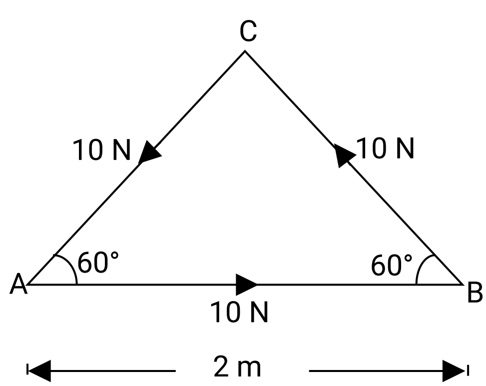
(a)



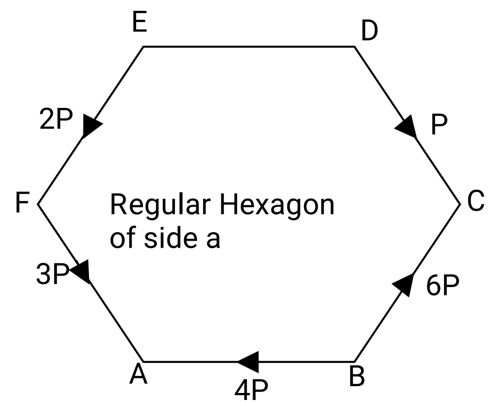
(b)



(c)



(d)



9. ABCD is a rectangle in which AB = 4 m and BC = 3 m. Forces of 16 N, 12 N, 4 N, 9 N and 10 N act along AB, BC, CD, AD and DB respectively, their directions given by the order of the letters.
   (a)(i) Find the magnitude and direction of the resultant force.
   (ii) Determine the equation of the line of action of the resultant force.
   (b) A couple is now added to the system in the sense ABCD. If the moment of the couple is 24 N m, show that the resultant of the enlarged system will now act through B.
10. A force acting in the x-y plane has moments 4, $\frac{5}{2}$ and 12 N m about (0,0),(1,0) and (0,2) respectively in its plane.
   (a) Calculate the magnitude of the resultant force.
   (b) Show that the equation of its line of action is $3x-8y = 8$.

Exercise: 7C

1. ABCD is a rectangle in which AB = 2a and BC = a. Forces of magnitude 2P, 5P, 4P and 3P act along AB, BC, CD and DA respectively, the direction of the forces being given by the order of the letters.
   (i) Find the magnitude and direction of the resultant force.
   (ii) Determine the equation of the line of action of the resultant force and find the distance from A where the line of action of the resultant cuts AB.
   (iii) If the forces are to be replaced by a single force and a couple at the centre of the rectangle, determine the magnitude

of the single force, magnitude and direction of the couple.

(iv) If instead a couple was added to the system and the line of action of the enlarged system passes through B, find the moment and sense of the couple.

2. ABCD is a rectangle in which AB = 3 m, BC = 4 m. Forces of 3 N, 2 N, 10 N and 5 N act in the directions AB, AD, AC and BD respectively. Find their resultant and equation of its line of action.

3. Forces of 3**i**, 4**j**, 5**i** and -10**j** act at points (0,0), (3,4), (-1,2) and (2,5) respectively. Find the equation of the line of action of their resultant.

4. Forces of magnitude 6 N, 3 N, 4 N and 5 N act along the sides PQ, QR, RS and SP of a square of side 2 m. Find the:
   (i) magnitude and direction of the resultant force.
   (ii) equation of the line of action of the resultant force.

   If it is required to replace the given system by a single force passing through the midpoint of PQ and a couple, find the single force and the moment of the couple.

5. Forces of magnitude 8w, 7w, 5w and 3w act along sides AB,BC, CD and DA respectively of a square ABCD of side 2 m. Find the:
   (a) magnitude of the resultant.
   (b) equation of the line of action of the resultant and where it cuts AB.

6. Forces w, 4w,2w, 6w act along the sides AB, BC, CD, DA of a square of side a. Find the magnitude of their resultant and prove that the equation of its line of action referred to AB and *AD* as *x* and *y*-axes respectively is y = 2x + 6a.

7. ABCDEF is a regular hexagon of side 1 m. Forces of magnitude 4, 6, 10, 8, P and 2 newtons act along AB, BC, CD, ED, FE and AF respectively, directions being indicated by order of the letters.
   (a) If the resultant passes through the centre of the hexagon, find the value of P and the magnitude and direction of the resultant.
   (b) The force along FE is replaced by another so that the new system reduces to a couple. Find the magnitude of the force and the moment of the couple indicating its sense.

8. ABCD is a square of side 2 m. Forces of magnitude 2 N, 1 N, 3 N, 4 N and $2\sqrt{2}$ N act along AB, BC, CD, DA and BD respectively. In order to maintain equilibrium a force F, whose line of action cuts AD produced at E has to be applied. Find the:
   (a) magnitude of F,
   (b) angle F makes with AD, and
   (c) length AE.

9. (a) Forces of $\begin{pmatrix}1\\1\end{pmatrix}$, $\begin{pmatrix}-4\\1\end{pmatrix}$ and $\begin{pmatrix}3\\-2\end{pmatrix}$ N act at points (2,2), (-1,4) and (4,-2) respectively. Show that the forces reduce to a couple and find the moment of the couple.
   (b) Four forces of 12, 7, 16 and 20 N act along AB, DA, DC and BD respectively. ABCD is a rectangle with AB = 4 m, BC = 3 m, find:
   (i) the magnitude and direction of the resultant of the system of forces.
   (ii) where the line of action of the resultant cuts the side AB.

10. ABCDEF is a regular hexagon of side a meters and centre O. Forces of magnitude P, 2P, 3P, 4P, kP and hP act along AB, BC, CD, DE, EF and FA respectively. Given that the resultant of the forces has magnitude 3P in a direction parallel to EF:
   (i) determine the values of k and h.
   (ii) show that the sum of the moments of the forces about O is $9\sqrt{3}$Pa.
   (iii) find the equation of the line of action of the resultant, refer to OM as x-axis and OA as y-axis, where M is the midpoint of EF.

11. OABC is a square of side 2 m, O,A,B and C are the points (0,0),(2,0)(2,2) and (0,2) respectively. Forces of magnitude 5,8,3 and $5\sqrt{2}$ newtons act along OA,BA,OC and AC respectively. Show that this system of forces is equivalent to a couple and find its moment. If a fifth force of magnitude $6\sqrt{2}$ N acts along OB, find the equation of the line of action of the resultant of the enlarged system.

Answers to Exercises

<u>Exercise: 7A</u>

1. R_A = 41 N ; R_B = 33 N 2. 18 N ; 12 N 3.

3 ; 2   4.   5. (a) 3 m from B

(b) $49\sqrt{3}$ N 6. 1·6 m 7. 0·6 m ; 20 N 8. (i) $0\mathbf{i} + 0\mathbf{j}$ (ii) 4 N m anticlockwise; Will cause rotation of the lamina about the origin.

9. $3x + 4y = -1$

10. (i) 48 N m anticlockwise (ii) 80 N m anticlockwise
(iii) 80 N m anticlockwise

<u>Exercise: 7B</u>

1. 14·8 ; 2   2. 1 m   3. $\mathbf{F} = \mathbf{0}$ and $G \neq 0$; $9\sqrt{7}$ N m anticlockwise

4. $\mathbf{F} = \mathbf{0}$ and $G \neq 0$; 10 N m clockwise **5.** $a = -6$; $b = 4$; 26 N m clockwise

6. 12 N m clockwise, independent of x and y   7. (a) 5·196 N ; 60·8° to AB
(b) 1·764a   8. (a) 14 N m in sense ADCB
(b) 41 N m in sense ADCB
(c) $10\sqrt{3}$ N m in sense ABC   (d) $3\sqrt{3}$aP in sense ABCDEF

9. (a) (i) 25 N ; 36·9° to AB (ii) $15x - 20y = 36$ (b)

10. (a) $\frac{\sqrt{73}}{2}$ N (b)

<u>Exercise: 7C</u>

1. (i) $2\sqrt{2}$P at 45° to BA (ii) $x + y = 7a$; cuts AB- produced at 7a from A
(iii) $2\sqrt{2}$P ; 11aP in sense ABCD (iv) 10aP in sense ADCB

2. 15·232 N at 66·8° to AB ; $7x - 3y = 6$

3. $3x + 4y = 9$ 4. (i) $2\sqrt{2}$ N ; 45° below PQ (ii) $x + y = -7$; $2\sqrt{2}$ N ;16 N m in sense PQRS

5. (a) 5W (b) $4x - 3y = 24$; cuts AB-produced at 6 m from A

6. $\sqrt{5}$W   7. (a) 10 ; 28 N along AD (b) 18 N; $14\sqrt{3}$ N m in sense ABCDEF

8. (a) $\sqrt{10}$ N (b) 71·6° (c) 4 m

9. (a) $\mathbf{F} = \mathbf{0}$ and $G \neq 0$; 13 N m anticlockwise
(b) (i) 13 N at 22·6° to AB
(ii) Passes through A   10. (i) 2 ; 6
(ii) (iii) $x = 3\sqrt{3}a$

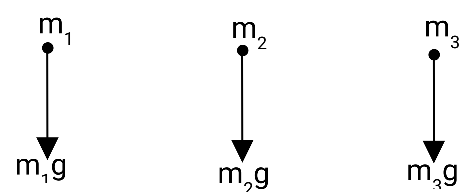
11. 6 N m clockwise ; $y = x + 1$

8. CENTRE OF GRAVITY

The force of attraction of an object towards the earth is called the weight of the object. This force acts through a point in the object called its **centre of gravity**. The **centre of mass** of a body is the point at which the mass of the body is considered to act. The **centroid** of a body is its geometric centre. The centre of gravity and centre of mass coincide in a uniform gravitational field. The centre of mass and the centroid of a body coincide for a uniform body. A body is said to be uniform if it has uniform density. C.O.G will be used as a short form for centre of gravity.

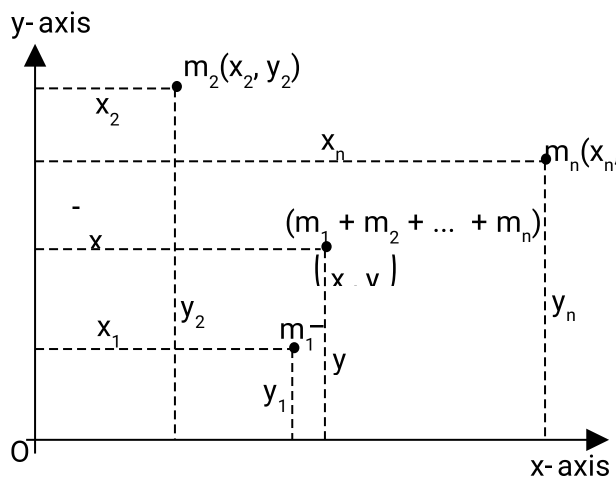
8.1 Centre of gravity of a system of particles

For a system of particles, the location of each particle is taken as the location of its centre of gravity, since particles are very small. Below are particles of masses m_1,m_2 and m_3.



Note that the weight always acts vertically downwards for each particle. Since the location of each particle is also the location of its centre of gravity, we can apply moments to find the location of the centre of gravity of the system of particles.

Consider particles of masses m_1, m_2,..., m_n located at points (x_1,y_1), (x_2,y_2),..., (x_n,y_n). Their centre of gravity $\left(\bar{x}, \bar{y}\right)$ can be obtained by taking moments about the coordinate axes.



Taking moments about the y-axis:

$$(m_1 + m_2 + \ldots + m_n)g\times \bar{x} = m_1g\times x_1 + m_2g\times x_2 + \ldots + m_ng\times x_n$$

$$(\Sigma w)\,\bar{x} = \Sigma wx$$

$$\bar{x} = \frac{\Sigma Wx}{\Sigma W}$$

Taking moments about the x-axis:

$$(m_1 + m_2 + \ldots + m_n)g\times \bar{y} = m_1g\times y_1 + m_2g\times y_2 + \ldots + m_ng\times y_n$$

$$(\Sigma w)\,\bar{y} = \Sigma wy$$

$$\bar{y} = \frac{\Sigma Wy}{\Sigma W}$$

Note: If locations are given in form of position vectors, then:

$$r_1 = x_1\mathbf{i} + y_1\mathbf{j},\ r_2 = x_2\mathbf{i} + y_2\mathbf{j},\ \ldots,r_n = x_n\mathbf{i} + y_n\mathbf{j}$$

The position vector $\bar{r}$ of the center of gravity is given by:

$$\bar{r} = \bar{x}\,\mathbf{i} + \bar{y}\,\mathbf{j}$$

$$\bar{r} = \left(\frac{\Sigma Wx}{\Sigma W}\right)\mathbf{i} + \left(\frac{\Sigma Wy}{\Sigma W}\right)\mathbf{j}$$

$$= \frac{1}{\Sigma W}\{(w_1x_1 + w_2x_2 + \ldots + w_nx_n)\mathbf{i} + (w_1y_1 + w_2y_2 + \ldots + w_ny_n)\mathbf{j}\}$$

$$= \frac{1}{\Sigma W}\{w_1(x_1\mathbf{i} + y_1\mathbf{j}) + w_2(x_2\mathbf{i} + y_2\mathbf{j}) + \ldots + w_n(x_n\mathbf{i} + y_n\mathbf{j})\}$$

$$= \frac{1}{\Sigma W}(\Sigma w\mathbf{r})$$

$$\bar{r} = \frac{\Sigma Wr}{\Sigma W}$$

Example 1

Particles of weight 12 N, 8 N and 4 N act at points (1,-3),(0,2) and (1,0) respectively. Find the centre of gravity of the particles.

Solution

$$\text{C.O.G} = (\bar{x}, \bar{y})$$

$$\bar{x} = \frac{\Sigma Wx}{\Sigma W} \text{ and } \bar{y} = \frac{\Sigma Wy}{\Sigma W}$$

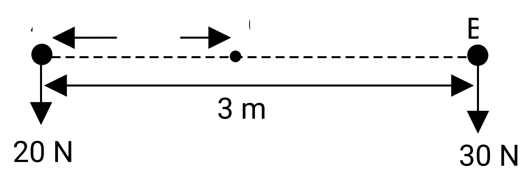
$$\bar{x} = \frac{12\times1+8\times0+4\times1}{12+8+4} = \frac{2}{3}$$

$$\bar{y} = \frac{12\times-3+8\times2+4\times0}{12+8+4} = -\frac{5}{6}$$

$$\text{C.O.G} = \left(\frac{2}{3}, -\frac{5}{6}\right)$$

Example 2

Find the centre of gravity of two particles of weight 20 N and 30 N at points A and B respectively, given AB = 3 m.



$$\bar{x} = \frac{\Sigma Wx}{\Sigma W} = \frac{20\times0+30\times3}{20+30}$$

$$\bar{x} = 1\cdot8 \text{ m (from A)}$$

Example 3

Find the coordinates of the centre of gravity of particles of weight 5 N, 7 N, 1 N and 3 N at points (1,0), (3,1), (6,3) and (0,2) respectively.

Solution

$$\text{C.O.G} = (\bar{x}, \bar{y})$$

$$\bar{x} = \frac{\sum Wx}{\sum W}, \ \bar{y} = \frac{\sum Wy}{\sum W}$$

$$\bar{x} = \frac{5\times1+7\times3+1\times6+3\times0}{5+7+1+3} = 2$$

$$\bar{y} = \frac{5\times0+7\times1+1\times3+3\times2}{5+7+1+3} = 1$$

G(2,1)

Example 4

Find the position vector of the centre of gravity of three particles of weight 7 N, 9 N and 4 N with position vectors 4**i** + **j**, 3**j** and 3**i** + 4**j**.

Solution

$$\bar{r} = \frac{\sum wr}{\sum w}$$

$$\bar{r} = \frac{7(4i+j)+9\times3j+4(3i+4j)}{7+9+4}$$

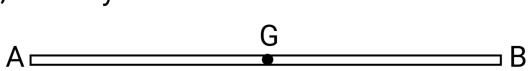
$$\bar{r} = 2\mathbf{i} + 2\cdot5\mathbf{j}$$

8.2 Centre of gravity of a rigid body

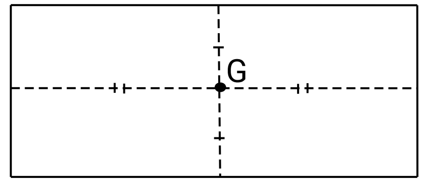
Rigid bodies are categorised as linear, lamina and solid objects. A rigid body is considered to be uniform if the distribution of weight is proportional to the length, area or volume for linear, lamina and solid objects respectively. The centre of gravity of a uniform body lies along every line or plane of symmetry of the body. If a uniform body has two or more lines or planes of symmetry, then their point of intersection gives the location of its centre of gravity. A lamina object is a flat object having negligible thickness compared to its length and width.

Some locations of centre of gravity:

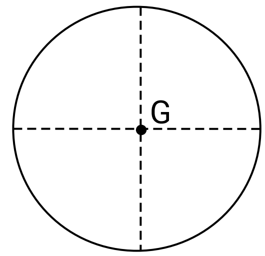
1. **Uniform rod:**
   The centre of gravity is located at its midpoint G, midway between ends A and B.



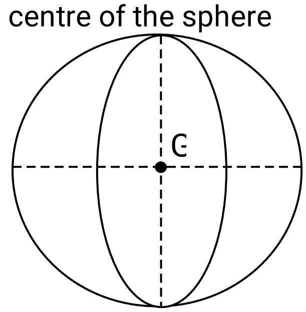
2. **Uniform rectangular lamina:**
   G is at the point of intersection of its lines of symmetry.



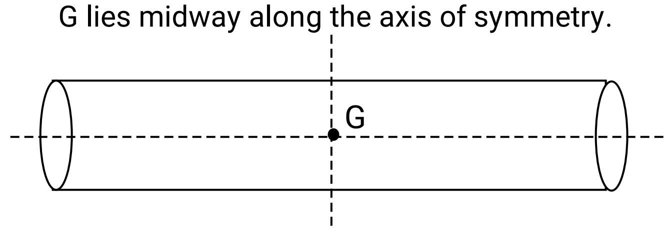
3. **Uniform circular lamina:**
   G is at the centre of the lamina.



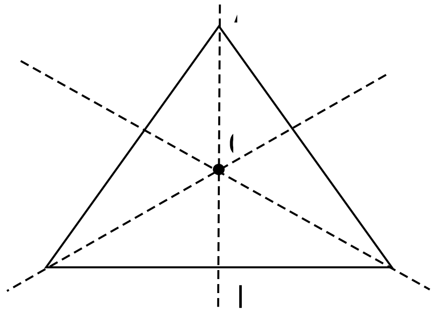
4. **Uniform sphere:**
   G is at the centre of the sphere



5. **Uniform right circular cylinder:**
   G lies midway along the axis of symmetry.



6. **Uniform triangular lamina:**
   G lies along the medians, at their point of intersection, leaving two thirds of the distance from the vertices to the opposite sides of the triangle. AG = $\frac{2}{3}$AB or BG = $\frac{1}{3}$AB.

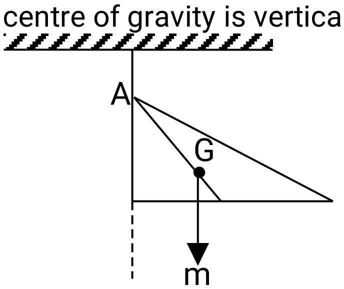


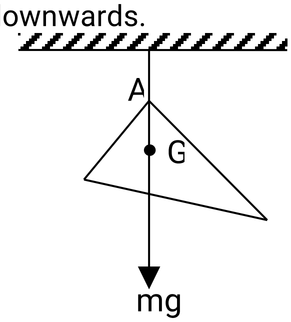
Standard results: A list of standard locations of centre of gravity is provided at the back of the book in the **formulae for reference.** However, proof of the standard results is given later on in this topic.

8.3 Equilibrium, toppling and sliding

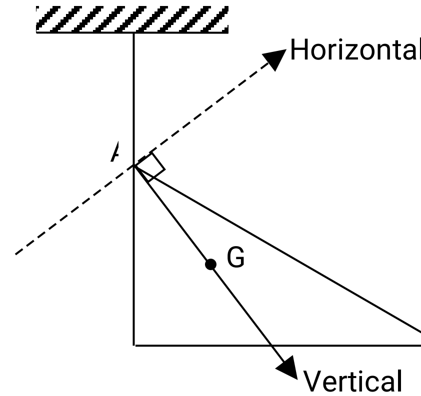
8.3.1 Equilibrium of a suspended lamina

When a lamina is freely suspended, it rests when the line from the point of suspension through the centre of gravity is vertically downwards.





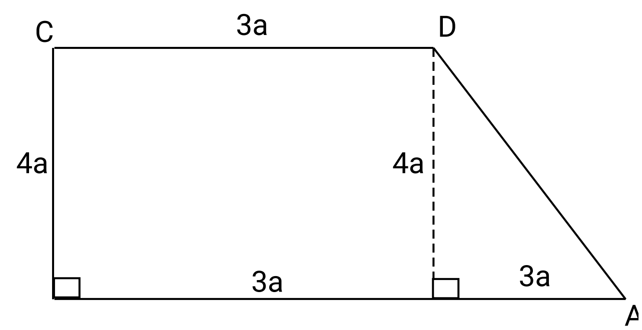
This condition can be used in finding the angle a given side of the lamina makes with the vertical or horizontal when it rests in equilibrium when suspended from a point. The line from the point of suspension through the centre of gravity is the vertical and any line perpendicular to it is the horizontal.



Example 5

A uniform wire is bent to form a trapezium ABCD in which, $A\hat{B}C = B\hat{C}D = 90°$, AB = 6a, BC = 4a and CD = 3a. Find the distances of the centre of gravity from AB and BC. If the trapezium is freely suspended from A, find the angle which AB makes with the vertical.

Solution



$$DA = \sqrt{(4a)^2+(3a)^2} = 5a$$

Let w be the weight per unit area

Portion	Weight	Distance of C.O.G from:	
		BC	BA
AB	6aw	3a	0
BC	4aw	0	2a
CD	3aw	$\frac{3}{2}a$	4a
DA	5aw	$\frac{9}{2}a$	2a
Whole	18aw	$\bar{x}$	$\bar{y}$

Taking moments about BC:

$$18aw\times\bar{x} = 6aw\times3a + 4aw\times0 + 3aw\times\frac{3}{2}a + 5aw\times\frac{9}{2}a$$

$$\bar{x} = \frac{5}{2}a$$

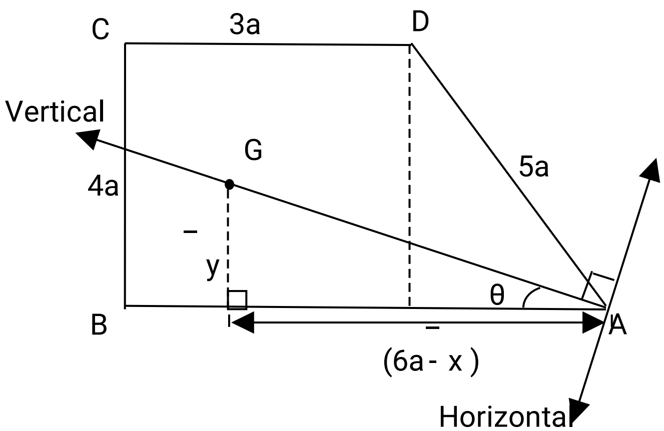
Taking moments about BA:

$$18aw\times\bar{y} = 6aw\times0 + 4aw\times2a + 3aw\times4a + 5aw\times2a$$

$$\bar{y} = \frac{5}{3}a$$

The centre of gravity is $\frac{5}{3}a$ from AB and $\frac{5}{2}a$ from BC.

When the trapezium is freely suspended from A:

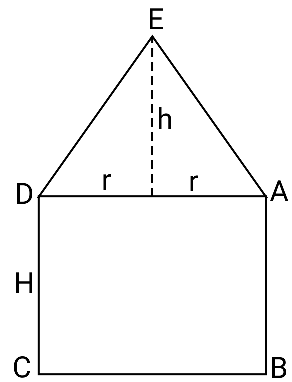


- $$\tan\theta = \frac{\bar{y}}{6a - \bar{x}}$$

$$\tan\theta = \frac{\frac{5}{3}a}{6a - \frac{5}{2}a} = \frac{10}{21} \Rightarrow \theta = 25{\cdot}5°$$

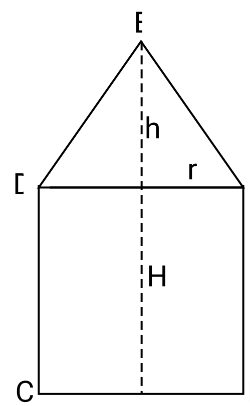
Example 6

The figure ABCDE below shows a solid cone of radius r, height h, joined to a solid cylinder of the same material with the same radius and height H.



If the centre of mass of the whole solid lies in the plane of the base of the cone where the two solids are joined, find H. If instead H = h and $r = \frac{1}{2}h$, find the angle AB makes with the horizontal if the body is hang from A.

Solution



Let w be the weight per unit volume of the solid

Body	Weight	Distance of C.O.G from CB
cylinder	πr^2Hw	$\frac{1}{2}H$
Cone	$\frac{1}{3}\pi r^2hw$	$\frac{1}{4}(4H + h)$
Whole	$\frac{1}{3}\pi r^2w(3H + h)$	$\bar{y}$

Taking moments about CB:

$$\frac{1}{3}\pi r^2w(3H + h)\times\bar{y} = \pi r^2Hw\times\frac{1}{2}H + \frac{1}{3}\pi r^2hw\times\frac{1}{4}(4H + h)$$

$$\bar{y} = \frac{6H^2+4Hh+h^2}{4(3H+h)}$$

If the centre of gravity of the solid lies in the plane where the two solids were joined then $\bar{y}$ = H and hence:

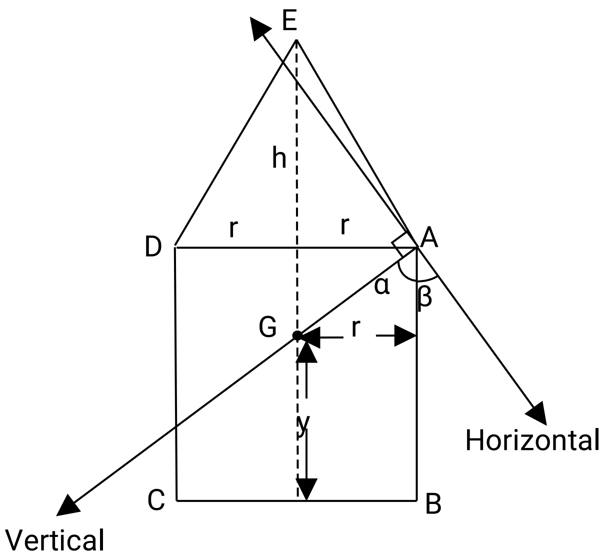
$$H = \frac{6H^2+4Hh+h^2}{4(3H+h)} \Rightarrow 12H^2 + 4Hh = 6H^2 + 4Hh + h^2$$

$$H = \frac{h\sqrt{6}}{6}$$

If H = h and $r = \frac{1}{2}h$

Then $\bar{y} = \frac{6H^2+4Hh+h^2}{4(3H+h)}$

$$\bar{y} = \frac{11h}{16}$$



$$\tan \alpha = \frac{r}{H-\bar{y}}$$

$$= \frac{\frac{1}{2}h}{h-\frac{11}{16}h}$$

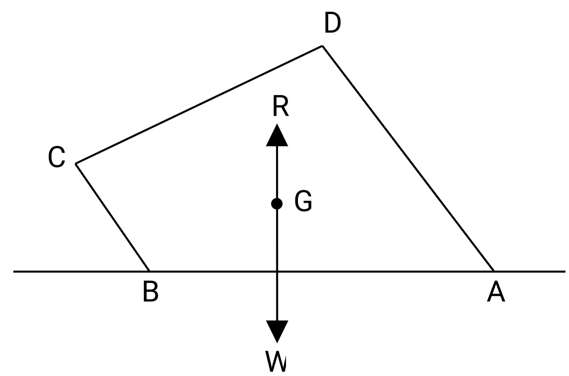
$\alpha = 58{\cdot}0°$
$\alpha + \beta = 90$
$58{\cdot}0 + \beta = 90 \Rightarrow \beta = 32{\cdot}0°$

Hence AB makes an angle of 32·0° with the horizontal.

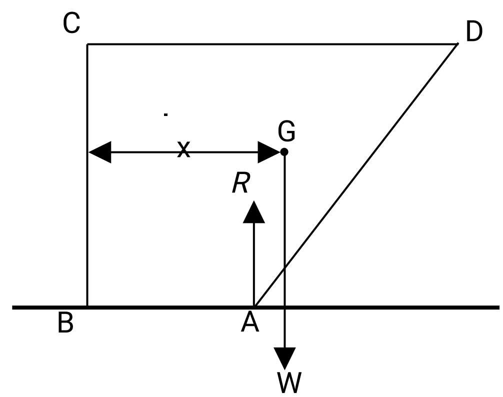
8.3.2 Equilibrium of a lamina on a surface

When a body rests in equilibrium on a horizontal plane, the weight of the body and the force exerted on the body must act along the same line; which is the vertical line through the centre of gravity.

In the case of a lamina with edge AB resting on a smooth horizontal plane, the force R exerted on the lamina is the resultant of normal reactions at the points of contact along AB. Therefore R must act through some point on AB. It follows that, if the vertical line through the centre of gravity G passes through a point on the edge AB, then R can act to maintain equilibrium. Otherwise equilibrium is not possible and the lamina will topple.



Lamina in equilibrium



The lamina will topple

Note: The lamina topples if $\bar{x}$ > AB. The lamina will remain in equilibrium if $\bar{x} \le$ AB and in particular, it will be at a point of toppling if $\bar{x}$ = AB. The lamina topples about the point of contact nearest to the line of action of the weight.

Example 7

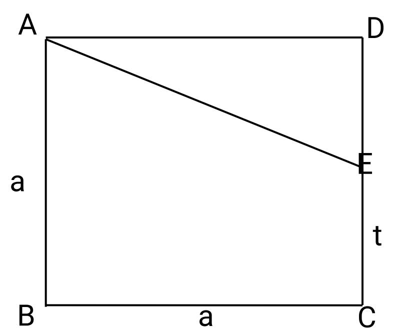
ABCD is a square lamina of side a from which a triangle ADE is removed, E being a point on CD a distance t from C.

(i) Show that the centre of mass of the remaining lamina is at a distance $\frac{a^2+at+t^2}{3(a+t)}$ from BC.

(ii) Hence show that if this lamina is placed in a vertical plane with CE resting on a horizontal table, equilibrium is possible if $t < \frac{a(\sqrt{3}-1)}{2}$

Solution

(i)



Let w be the weight per unit area

Body	Weight	Distance of C.O.G from BC
ABCD	a^2w	$\frac{1}{2}a$
ADE	$\frac{1}{2}a(a-t)w$	$\frac{1}{3}(2a+t)$
ABCE	$\frac{1}{2}a(a+t)w$	$\bar{y}$

Taking moments about BC:

$$\frac{1}{2}a(a+t)w\times\bar{y}$$

$$= a^2w\times\frac{1}{2}a - \frac{1}{2}a(a-t)w\times\frac{1}{3}(2a+t)$$

$$\bar{y} = \frac{a^2+at+t^2}{3(a+t)}$$

(ii) Equilibrium is possible if $\bar{y} < CE$

Thus $\frac{a^2+at+t^2}{3(a+t)} < t$

$2t^2 + 2at - a^2 > 0$

$$t^2 + at - \frac{a^2}{2} > 0 \Rightarrow \left(t+\frac{1}{2}a\right)^2 - \frac{a^2}{4} - \frac{a^2}{2} > 0$$

$$\left(t+\frac{1}{2}a\right)^2 > \frac{3a^2}{4}$$

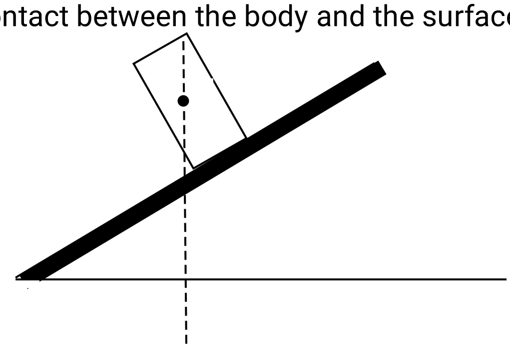
$$t+\frac{1}{2}a > \frac{-a\sqrt{3}}{2} \Rightarrow t > \frac{-1}{2}a\left(1+\sqrt{3}\right) \text{ and}$$

$$t+\frac{1}{2}a < \frac{a\sqrt{3}}{2} \Rightarrow t < \frac{1}{2}a\left(\sqrt{3}-1\right)$$

$$\text{since } t \geq 0 \Rightarrow t < \frac{1}{2}a\left(\sqrt{3}-1\right)$$

8.3.3 Equilibrium of a body on an inclined plane:

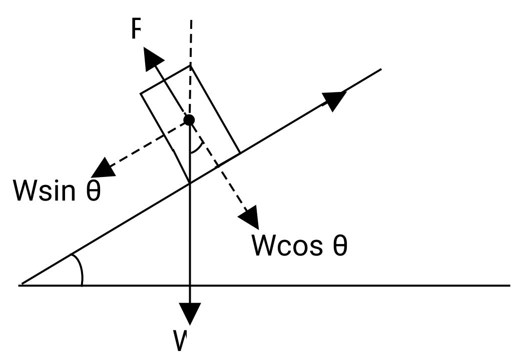
For a body to rest in equilibrium without toppling on a surface, the line of action of its weight should be contained between the extreme points of contact between the body and the surface.



For the block above equilibrium is not possible since the vertical line through the centre of gravity G falls outside the base of the block, hence it will topple. On an inclined plane, the equilibrium of a body may be broken by sliding rather than toppling.

8.3.4 Toppling and sliding

Consider a block of weight W on a rough plane coefficient of friction μ whose inclination is gradually increased.



The body will not slide as along as $W\sin\theta < F_{max}$

Where θ is the inclination of the plane.

$\Rightarrow W\sin\theta < \mu W\cos\theta \Rightarrow \tan\theta < \mu$

When the angle of inclination is θ as shown

above, the block is at a point of toppling.
The block slides before it topples if:

$$W\sin\theta > F_{max} \Rightarrow W\sin\theta > \mu W\cos\theta \Rightarrow \tan\theta > \mu$$

The body topples before it slides if:

$$W\sin\theta < F_{max} \Rightarrow W\sin\theta < \mu W\cos\theta \Rightarrow \tan\theta < \mu$$

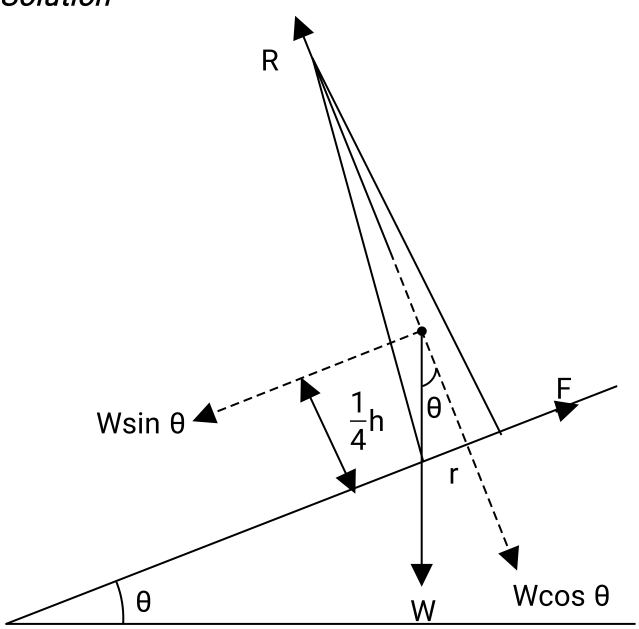
The body simultaneously topples and slides if:

$$W\sin\theta = F_{max} \Rightarrow W\sin\theta = \mu W\cos\theta \Rightarrow \tan\theta = \mu$$

Example 8

A uniform solid cone of height h and base radius r, is placed with its base on a rough plane. The coefficient of friction between the cone and the plane is μ. The inclination of the plane is gradually increased and the cone begins to slide and topple simultaneously, show that $\mu = \frac{4r}{h}$.

Solution



Resolving normal to plane:

$R = W\cos\theta$

Resolving parallel to plane:

$F = W\sin\theta$

But $F = \mu R$

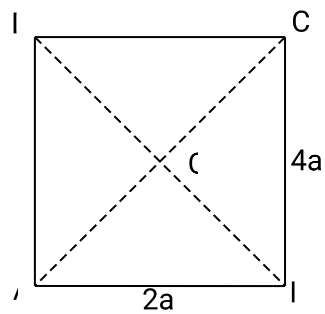
$\mu W\cos\theta = W\sin\theta \Rightarrow \tan\theta = \mu$ but $\tan\theta = \frac{r}{\left(\frac{h}{4}\right)}$

Hence $\mu = \frac{4r}{h}$

Example 9

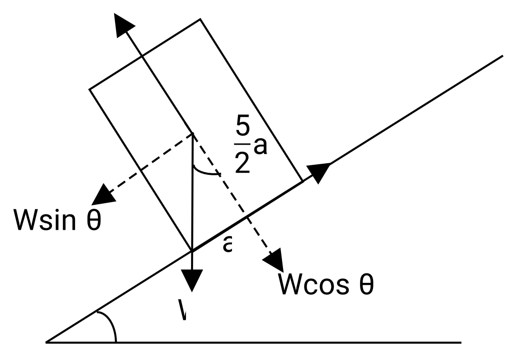
(a) A uniform solid cylinder of height 5a and radius a stands on a rough horizontal plane, coefficient of friction $\frac{1}{2}$. Show that if the plane is tilted slowly, the cylinder will topple before it slides.

(b) A uniform rectangular lamina ABCD has AB = 2a and BC = 4a. The triangular lamina BOC is removed, where O is the point of intersection of the diagonals. If the remainder is freely suspended from A, find the angle AD makes with the vertical.



Solution

(a)



When the cylinder is at a point of toppling:

$$\tan\theta = \frac{a}{\left(\frac{5}{2}a\right)} = \frac{2}{5}$$

$$\theta = 21\cdot8°$$

Resolving normal to the plane:

$R = W\cos\theta$

From: $F_{max} = \mu R$

$\Rightarrow F_{max} = \frac{1}{2}W\cos\theta = \frac{1}{2}W\cos 21\cdot8$

$= 0\cdot4642W$

Resolving along plane:

$F = W\sin\theta \Rightarrow F = W\sin 21.8$

$= 0\cdot3714W$

Since F < F_{max}, when the plane is further tilted, the cylinder will topple before it slides.

Note: Toppling occurs when the line of action of the weight does not pass between the extreme points of contact of the cylinder and the plane.

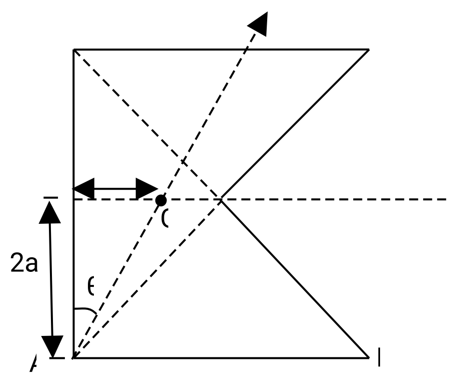
(b) Let *w* be the weight per unit area

Body	Weight	Distance of C.O.G from AD
ABCD	$8a^2w$	a
BOC	$2a^2w$	$\frac{5}{3}a$
Remainder	$6a^2w$	$\bar{x}$

Taking moments about AD:

$$6a^2w\times\bar{x} = 8a^2w\times a - 2a^2w\times\frac{5}{3}a$$

$$\bar{x} = \frac{7}{9}a$$



$$\tan\theta = \frac{\bar{x}}{2a}$$

$$\tan\theta = \frac{\left(\frac{7}{9}a\right)}{2a}$$

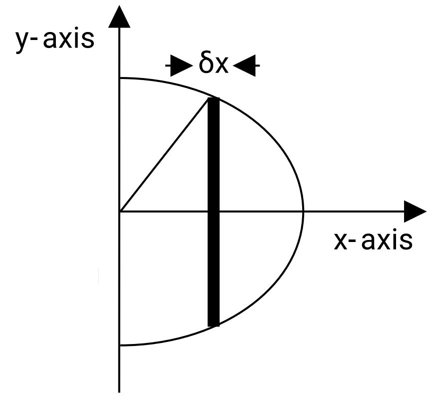
$$\theta = 21\cdot3°$$

8.4 Obtaining standard results by integration

In order to obtain the centre of gravity of a body, we use the principle that the sum of moments of components of weights that constitute the body about a given axis is equal to the moment of the weight of the body about the same axis. The following examples give the ways of obtaining the standard results using calculus.

8.4.1 Standard results

1. Prove that the centre of gravity of a uniform semi-circular lamina of radius r, is at a distance $\frac{4r}{3\pi}$ from the centre.



Let w be the weight per unit area

Body	Weight	Distance of centre of gravity from y- axis
Lamina	$\frac{1}{2}\pi r^2 w$	$\bar{x}$
Element	$2y\delta xw$	x

Taking moments about the y- axis:

$$\Sigma(2y\delta x\, w)x \simeq \frac{1}{2}\pi r^2 w\,\bar{x}$$

$$\frac{1}{2}w\pi r^2\,\bar{x} = \int_0^r 2wxydx$$

$$\frac{1}{2}w\pi r^2\,\bar{x} = 2w\int_0^r x(r^2-x^2)^{\frac{1}{2}}dx$$

Let $u = r^2-x^2 \Rightarrow du = -2xdx$

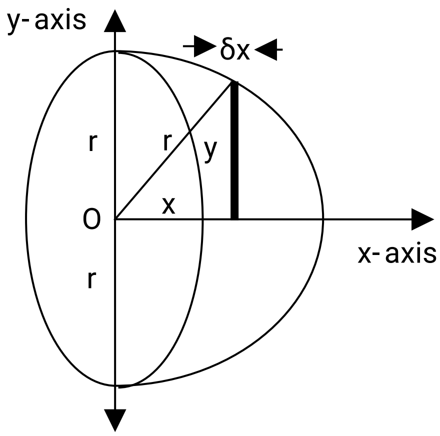
$$\frac{1}{2}\pi r^2\,\bar{x} = -\int_{r^2}^{0} u^{\frac{1}{2}}du \Rightarrow \frac{1}{2}\pi r^2\,\bar{x}$$

$$= -\left[\frac{2}{3}u^{\frac{3}{2}}\right]_{r^2}^{0}$$

$$\bar{x} = \frac{4r}{3\pi}$$

2. Prove that the centre of gravity of a uniform solid hemisphere of radius r, is a distance $\frac{3}{8}r$ from the centre.

Solution



The centre of gravity is located along the axis of symmetry (x- axis)
It is obtained by cutting strips each of thickness δx and length y being rotated about the x- axis.

Let w be the weight per unit volume

Body	Weight	Distance of centre of gravity from y- axis
Solid	$\frac{2}{3}\pi r^3 w$	$\bar{x}$
Element	$\pi y^2 \delta x w$	x

Taking moments about the y- axis:

$$\frac{2}{3}\pi r^3 w\,\bar{x} \simeq \Sigma(\pi y^2 \delta x w)x$$

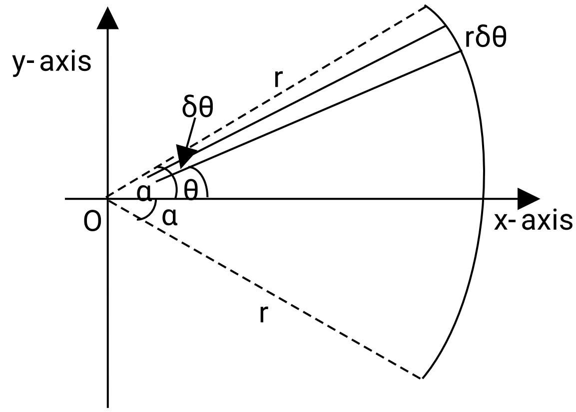
$$\frac{2}{3}r^3\,\bar{x} = \int_0^r xy^2dx = \int_0^r x\left(r^2-x^2\right)dx$$

$$\frac{2}{3}r^3\,\bar{x} = \left[\frac{1}{2}r^2x^2-\frac{1}{4}x^4\right]_0^r$$

$$\frac{2}{3}r^3\,\bar{x} = \frac{1}{4}r^4$$

$$\bar{x} = \frac{3}{8}r$$

3. Find the centre of gravity of a uniform wire in form of an arc of a circle of radius r, subtending an angle 2α at the centre.
Solution:
The centre of gravity lies along the the axis of symmetry.



A typical element arc subtends an angle δθ at the centre O and has length rδθ.
Let w be weight per unit length of the wire:

Body	Weight	Distance of C.O.G from y- axis
Element	wrδθ	rcos θ
Whole arc	wr×2α	$\bar{x}$

Taking moments about the y- axis:

$$2\alpha wr\times\bar{x} \simeq \sum\left(wr\delta\theta\times r\cos\theta\right)$$

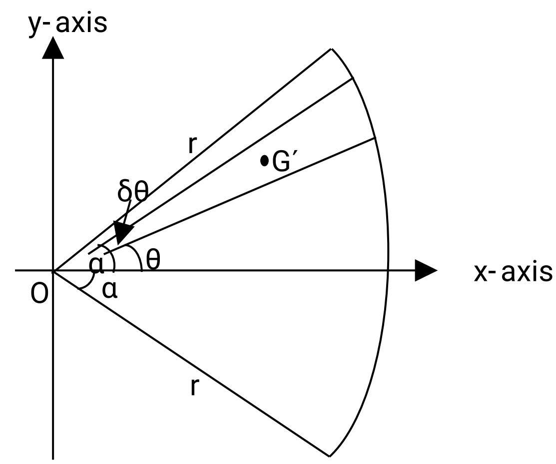
$$\therefore \quad 2\alpha wr\,\bar{x} = \int_{-\alpha}^{\alpha} wr^2\cos\theta d\theta$$

$$= wr^2[\sin\theta]_{-\alpha}^{\alpha} = 2wr^2\sin\alpha$$

$$\therefore \quad \bar{x} = \frac{r\sin\alpha}{\alpha}$$

Hence the centre of gravity of the wire is at a distance $\frac{r\sin\alpha}{\alpha}$ from the centre

4. Find the centre of gravity of a uniform lamina in form of a sector of a circle of radius r, subtending an angle 2α at the centre.
Solution
The centre of gravity lies along the the axis of symmetry.



A typical element is a sector that subtends an angle $\delta\theta$ at the centre O and has area $\frac{1}{2}r^2\delta\theta$. This sector is approximately triangular with its centre of gravity G′ at a distance $\frac{2}{3}r$ from O.

Let w be weight per unit area of the lamina:

Body	Weight	C.O.G from y-axis
Element	$w\times\frac{1}{2}r^2\delta\theta$	$\frac{2}{3}r\cos\theta$
Whole sector	$w\times\frac{1}{2}r^2\times 2\alpha$	$\bar{x}$

Taking moments about the y-axis:

$$w\times\frac{1}{2}r^2\times 2\alpha\times\bar{x} \simeq \sum\left(w\times\frac{1}{2}r^2\delta\theta\times\frac{2}{3}r\cos\theta\right)$$

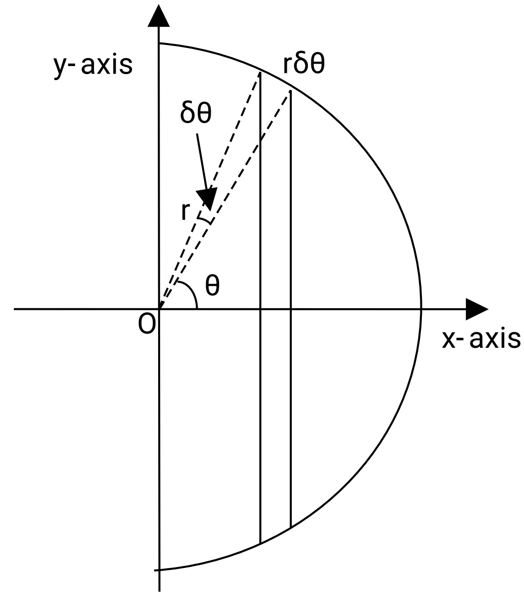
$$\therefore \quad wr^2\alpha\bar{x} = \int_{-\alpha}^{\alpha}\frac{1}{3}wr^3\cos\theta d\theta$$

$$= \frac{1}{3}wr^3[\sin\theta]_{-\alpha}^{\alpha} = \frac{2}{3}wr^3\sin\alpha$$

$$\therefore \quad \bar{x} = \frac{2r\sin\alpha}{3\alpha}$$

Hence the centre of gravity of the wire is at a distance $\frac{2r\sin\alpha}{3\alpha}$ from the centre.

5. Find the centre of gravity of a uniform hemispherical shell of radius r.
Solution
The x-axis is the line of symmetry of the shell.



If the shell is divided into small strips by planes perpendicular to the x-axis, then a typical element strip is approximately cylindrical with radius $r\sin\theta$ and width $r\delta\theta$. Letting w to be the weight per unit area of the shell;

Body	Weight	C.O.G from y-axis
Element	$w\times 2\pi r^2\sin\theta\delta\theta$	$r\cos\theta$
Whole body	$w\times 2\pi r^2$	$\bar{x}$

Taking moments about the y-axis:

$$w\times 2\pi r^2\times\bar{x} \simeq \sum\left(w\times 2\pi r^2\sin\theta\delta\theta\times r\cos\theta\right)$$

$$\therefore \quad 2\pi wr^2\bar{x} = \int_0^{\frac{\pi}{2}}\left(2\pi r^3 w\sin\theta\cos\theta\right)d\theta$$

$$= w\int_0^{\frac{\pi}{2}}\pi r^3\sin 2\theta d\theta$$

$$= \pi wr^3\left[-\frac{1}{2}\cos 2\theta\right]_0^{\frac{\pi}{2}} = \pi wr^3$$

$$\therefore \quad \bar{x} = \frac{1}{2}r$$

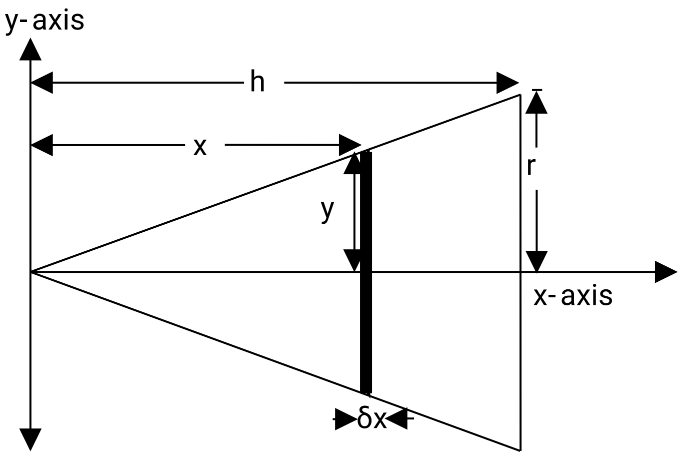
Hence the centre of gravity of the hemispherical shell is at a distance $\frac{1}{2}r$ from its centre.

Example 10

Prove that the centre of mass of a uniform right circular cone of height h and base radius r, is at a distance $\frac{3}{4}h$ from the vertex. Such a cone is joined to a uniform right circular cylinder, of the same material, with base radius r and height H, so that the plane base of the cone coincides with the plane face of the cylinder. Find the centre of mass of the solid thus formed. Show that this solid can rest in equilibrium on a horizontal plane, with the curved surface of the cylinder touching the plane provided $h^2 \le 6H^2$.

Solution

Let the axis of the cone lie along the x-axis, from symmetry the centre of mass lies along the x-axis. If the cone is divided into elements parallel to its base, then each element is approximately a disc, of thickness δx, radius y, having volume $\pi y^2 \delta x$.



Let w be the weight per unit volume of the cone, then we have:

Body	Weight	Distance of C.O.G from y-axis
Element	$(\pi y^2 \delta x)w$	x
Whole cone	$\frac{1}{3}\pi r^2 hw$	$\bar{x}$

Taking moments about the y-axis:

$$\frac{1}{3}\pi r^2 hw \times \bar{x} \simeq \Sigma((\pi y^2 \delta x w)\times x)$$

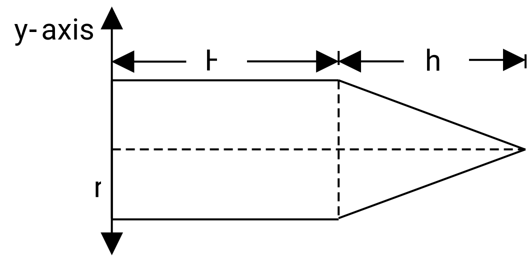
$$\frac{1}{3}\pi r^2 hw \times \bar{x} = \int_0^h \pi w x y^2 dx$$

From similar triangles, $\frac{y}{x} = \frac{r}{h} \Rightarrow y = \frac{r}{h}x$

$$\frac{1}{3}\pi r^2 hw\,\bar{x} = \int_0^h \frac{\pi r^2 w}{h^2} x^3 dx$$

$$\frac{1}{3}\pi r^2 h\,\bar{x} = \frac{\pi r^2}{h^2}\left[\frac{x^4}{4}\right]_0^h \Rightarrow \bar{x} = \frac{3}{4}h$$

Hence the centre of gravity of a cone is on its axis at a distance $\frac{3}{4}h$ from the vertex.



Let w be weight per unit volume of the cone and cylinder

Body	Weight	Distance of C.O.G from y-axis
cylinder	$\pi r^2 Hw$	$\frac{1}{2}H$
Cone	$\frac{1}{3}\pi r^2 hw$	$\frac{1}{4}(4H + h)$
Whole	$\frac{1}{3}\pi r^2 w(3H + h)$	$\bar{x}$

Taking moments about the y-axis:

$$\frac{1}{3}\pi r^2(3H + h)w\,\bar{x}$$

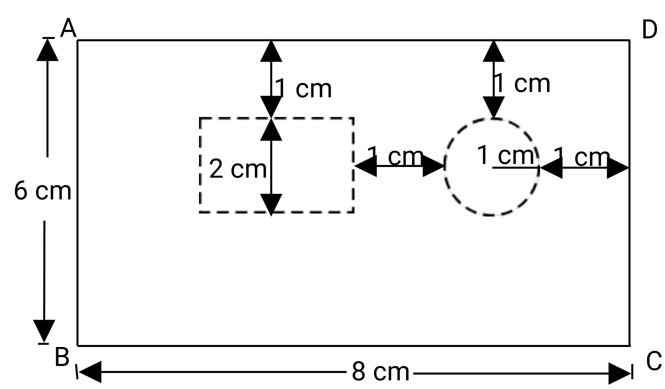
$$= \pi r^2 Hw \times \frac{1}{2}H + \frac{1}{3}\pi r^2 hw \times \frac{1}{4}(4H + h)$$

$$\bar{x} = \frac{6H^2+4Hh+h^2}{4(3H+h)}$$

For equilibrium; $\bar{x} \le H$

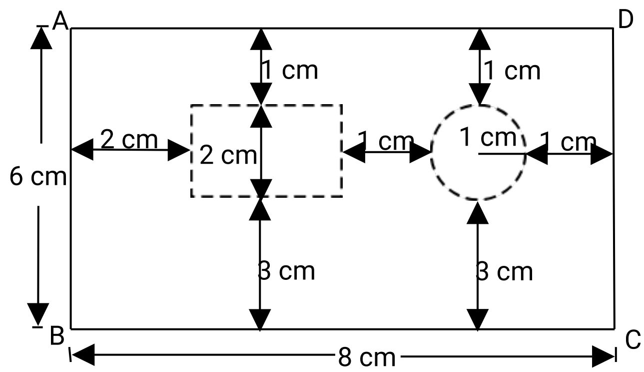
$$\frac{6H^2+4Hh+h^2}{4(3H+h)} \le H \Rightarrow h^2 \le 6H^2$$

Example 11



ABCD is a uniform rectangular sheet of cardboard of length 8 cm and width 6 cm. A square and circular hole are cut off from the cardboard as shown above. Calculate the position of the centre of gravity of the remaining sheet.

Solution



Let w be the weight per cm^2

Body	Weight	Distance of C.O.G from:	
		BA	BC
ABCD	48w	4	3
Square	4w	3	4
Circle	πw	6	4
Remainder	(44−π)w	$\bar{x}$	$\bar{y}$

Taking moments about BA:

$$(44-\pi)\,w\times\bar{x} = 48w\times4-4w\times3-\pi w\times6$$

$$(44-\pi)\,\bar{x} = (180-6\pi)$$

$$\bar{x} = 3\cdot944 \text{ cm}$$

Taking moments about BC:

$$(44-\pi)w\times\bar{y} = 48w\times3-4w\times4-\pi w\times4$$

$$(44-\pi)\,\bar{y} = (128-4\pi)$$

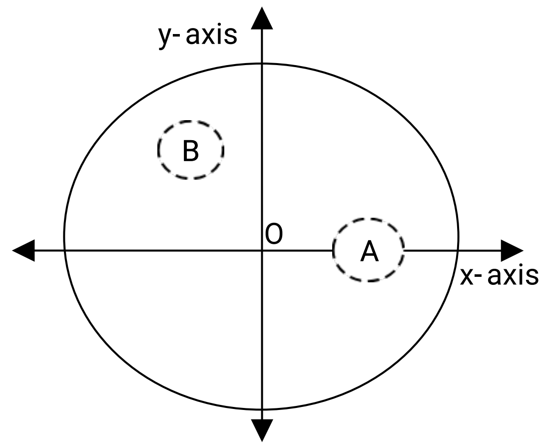
$$\bar{y} = 2\cdot825 \text{ cm}$$

Hence the centre of gravity of the remaining sheet is at a distance 3.944 cm from BA and 2.825 cm from BC.

Example 12

(a) The particles of masses 4, 6 and 2 kg acting at point (2,-1), (2,3) and (-2,5) respectively have their centre of gravity at a point A(a,b). Determine the values of a and b.

(b) A uniform circular lamina of radius 10 cm has two circular holes cut out each of radius 2 cm. The design specification describes these by marking two diameters as co-ordinates axes on the lamina as shown below.



If one hole has its centre at A(6,0) and the other has its centre at B(-2,5). Find the distance of its centre of gravity from the origin (0,0). Give your answer to 4 decimal places.

Solution

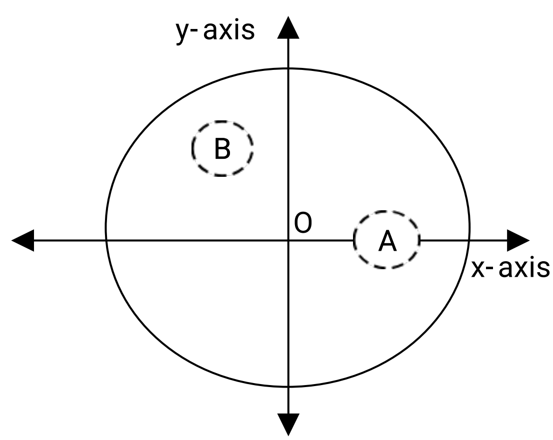
(a) $\bar{x} = \dfrac{\sum wx}{\sum w}$

$$a = \frac{4g\times2+6g\times2+2g\times-2}{4g+6g+2g} \Rightarrow a = \frac{4}{3}$$

$$\bar{y} = \frac{\sum wy}{\sum w}$$

$$b = \frac{4g\times-1+6g\times3+2g\times5}{4g+6g+2g} \Rightarrow b = 2$$

(b)



Let w be the weight per cm^2

Body	Weight	Distance of C.O.G from:	
		y- axis	x- axis
Whole	100πw	0	0
A	4πw	6	0
B	4πw	-2	5
Remainder	92πw	$\bar{x}$	$\bar{y}$

Taking moments about the y- axis:

$$92\pi w\times\bar{x} = 100\pi w\times0-4\pi w\times6-4\pi w\times-2$$

$$\bar{x} = -\frac{4}{23} \text{ cm}$$

Taking moments about the x-axis:

$$92\pi w \times \bar{y} = 100\pi w \times 0 - 4\pi w \times 0 - 4\pi w \times 5$$

$$\bar{y} = -\frac{5}{23} \text{ cm}$$

$$G\left(-\frac{4}{23}, -\frac{5}{23}\right)$$

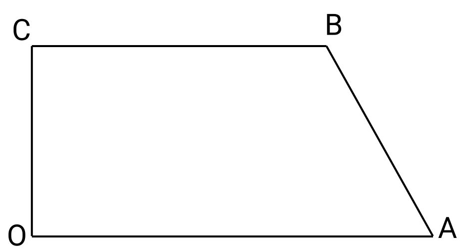
Distance of centre of gravity from origin:

$$= \sqrt{\left(-\frac{4}{23}\right)^2 + \left(-\frac{5}{23}\right)^2}$$

$$= \frac{\sqrt{41}}{23} \text{ cm}$$

$$= 0{\cdot}2784 \text{ cm}$$

Example 13



The figure represents a lamina in which OA = 15 cm, OC = CB = 6 cm.

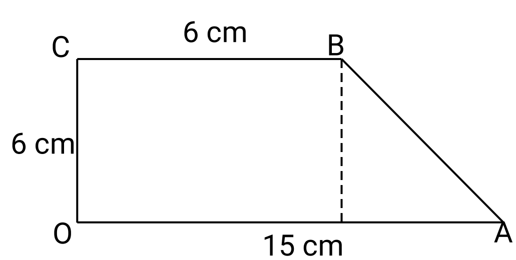
(a) Find the position of the centre of gravity of the lamina with respect to sides OA and OC.

(b) A quadrant of a circle with centre C passing through O and B is cut out of the lamina, find the position of the centre of gravity of the remainder.

(c) The remainder is freely suspended at B. Find the angle side AB makes with the horizontal.

Solution

(a)



Let w be the weight per cm^2

Body	Weight	Distance of C.O.G from:	
		OC	OA
Square	36w	3	3
Triangle	27w	9	2
Whole	63w	$\bar{x}$	$\bar{y}$

Taking moments about OC:

$$63w \times \bar{x} = 36w \times 3 + 27w \times 9$$

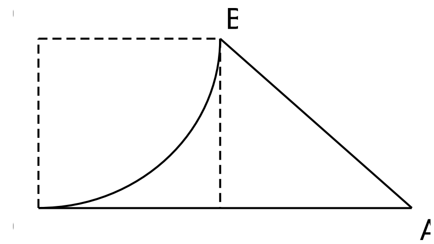
$$\bar{x} = \frac{39}{7} \text{ cm}$$

Taking moments about OA:

$$63w \times \bar{y} = 36w \times 3 + 27w \times 2$$

$$\bar{y} = \frac{18}{7} \text{ cm}$$

(b)



Let w be the weight per cm^2

Body	Weight	Distance of centre of gravity from:	
		OC	OA
Trapezium	63w	$\frac{39}{7}$	$\frac{18}{7}$
Quadrant	9πw	$\frac{8}{\pi}$	$6-\frac{8}{\pi}$
Remainder	(63-9π)w	$\bar{x}$	$\bar{y}$

Taking moments about OC:

$$(63-9\pi)w \times \bar{x} = 63w \times \frac{39}{7} - 9\pi w \times \frac{8}{\pi}$$

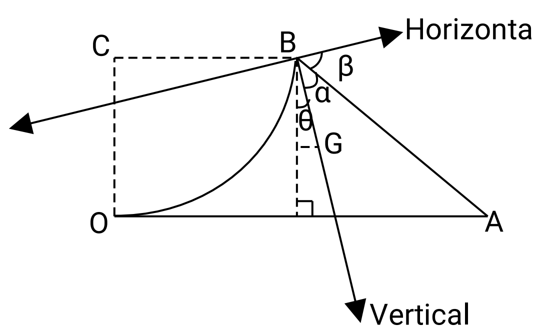
$$\bar{x} = 8{\cdot}0344 \text{ cm}$$

Taking moments about OA:

$$(63-9\pi)w \times \bar{y} = 63w \times \frac{18}{7} - 9\pi w \times \left(6 - \frac{8}{\pi}\right)$$

$$\bar{y} = 1{\cdot}8532 \text{ cm}$$

(c)



$$\tan\theta = \frac{8{\cdot}0344-6}{6-1{\cdot}8532}$$

$$\theta = 26{\cdot}0°$$

$$\tan(\alpha+\theta) = \frac{9}{6}$$

$$\alpha + \theta = 56 \cdot 3°$$

$$\alpha + 26 = 56 \cdot 3 \Rightarrow \alpha = 30.3°$$

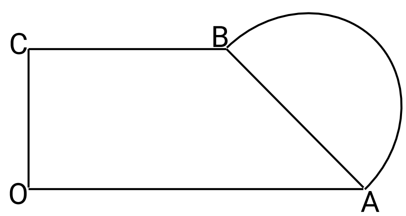
From $\alpha + \beta = 90$

$$\beta = 90 - 30 \cdot 3 \Rightarrow \beta = 59 \cdot 7°$$

Hence AB makes an angle of $59 \cdot 7°$ with the horizontal

Example 14

The figure shows a composite lamina consisting of a trapezium and a semi-circle. Given that OC = CB = 6 cm, OA = 14 cm and $\angle O = \angle C = 90°$.

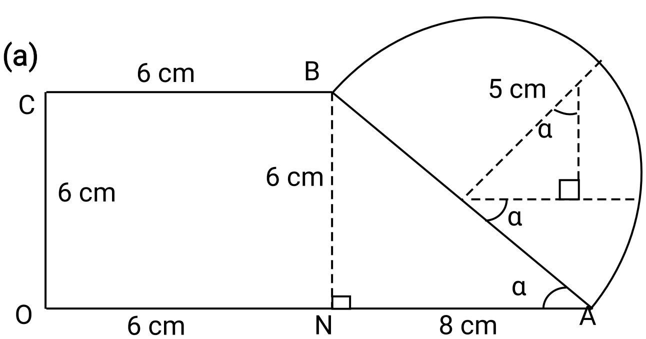


(a) Determine the position of the centre of gravity of the lamina with respect to the sides OA and OC, leaving π in your answer.

(b) The lamina is suspended from C, find the angle OC makes with the vertical at equilibrium.

Solution

(a)



Let w be the weight per cm^2

Body	Weight	Distance of C.O.G from: OC	Distance of C.O.G from: OA
OCBN	36w	3	3
NBA	24w	$\frac{26}{3}$	2
Semi-circle	12.5πw	$\left(10+\frac{4}{\pi}\right)$	$\left(3+\frac{16}{3\pi}\right)$
Whole	(60 + 12.5π)w	$\bar{x}$	$\bar{y}$

Taking moments at OC:

$$(60 + 12 \cdot 5\pi)w \times \bar{x} = 36w \times 3 + 24w \times \frac{26}{3} + 12.5\pi w \times \left(10+\frac{4}{\pi}\right)$$

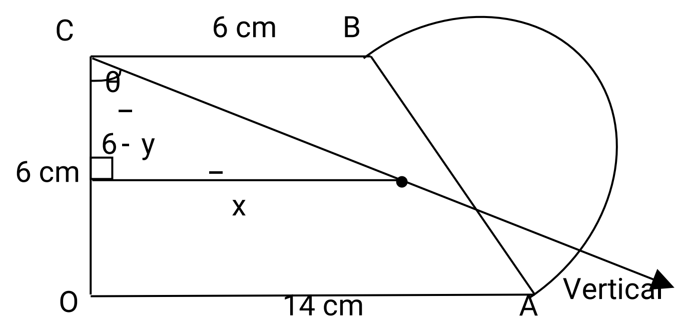
$$\bar{x} = \left(\frac{366+125\pi}{60+12 \cdot 5\pi}\right) \text{cm}$$

Taking moments about OA:

$$(60 + 12 \cdot 5\pi)w \times \bar{y} = 36w \times 3 + 24w \times 2 + 12 \cdot 5\pi w \times \left(3+\frac{16}{3\pi}\right)$$

$$\bar{y} = \left(\frac{\frac{668}{3}+37 \cdot 5\pi}{60+12 \cdot 5\pi}\right) \text{cm}$$

(b)



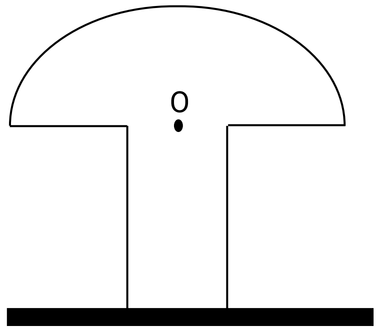
$$\bar{x} = 7 \cdot 6428 \text{ cm}$$

$$\bar{y} = 3 \cdot 4298 \text{ cm}$$

$$\tan\theta = \frac{\bar{x}}{6-\bar{y}}$$

$$\tan\theta = \frac{7 \cdot 6428}{6 - 3 \cdot 4298} \Rightarrow \theta = 71 \cdot 4°$$

Example 15



A uniform wooden "mushroom", used in a game, is made by joining a solid cylinder to a solid hemisphere. They are joined symmetrically, such that the centre O of the plane face of the hemisphere coincides with the centre of one of the ends of the cylinder. The diagram shows the cross-section through a plane of symmetry of the mushroom, as it stands on a horizontal table.
The radius of the cylinder is r, and the radius of the hemisphere is 3r, and the centre of mass of the mushroom is at the point O.

(a) Show that the height of the cylinder is $\frac{9r}{\sqrt{2}}$.

The table top, which is rough to prevent the mushroom from sliding, is slowly tilted until the mushroom is about to topple.

(b) Find to the nearest degree, the angle with the horizontal through which the table has to be tilted.

<u>Solution:</u>

(a) Let w be the weight per unit volume:

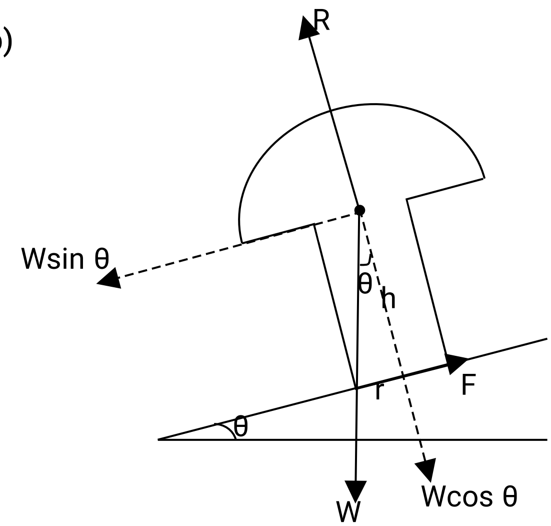
Body	Weight	Distance of C.O.G from table
Cylinder	$\pi r^2 hw$	$\frac{1}{2}h$
Hemisphere	$18\pi r^3 w$	$\left(\frac{9}{8}r+h\right)$
Whole	$\pi r^2(18r + h)w$	h

Taking moments about the table:

$$\pi r^2 w(18r + h)\times h = \pi r^2 hw\times\frac{1}{2}h + 18\pi r^3 w\times\left(\frac{9}{8}r+h\right)$$

$$\frac{1}{2}h^2 = \frac{81}{4}r^2 \Rightarrow h = \frac{9r}{\sqrt{2}}$$

(b)



$$\tan\theta = \frac{r}{h} = \frac{r}{\frac{9r}{\sqrt{2}}}$$

$$\theta = 8\cdot 9°$$

Exercises

Exercise: 8A

1. Find the distance from the origin of the centre of mass of masses $\frac{1}{2}$ kg, $\frac{3}{2}$ kg and 2 kg placed at points with position vector 6i–3j, 2i + 5j and 3i + 2j respectively.

2. Particles of mass 4m, $\frac{1}{2}$m, 3m, $\frac{1}{2}$m and 2m are at points with position vectors **i** + 2j, 3**i** - **j**, -**i** + 5**j**, 4i + **j** and -2**i**–3j respectively. Calculate the centre of gravity of the system of particles.

3. Find the coordinates of the centre of mass of particles of mass 9 kg, 4 kg, 6 kg and 5 kg at the points (4,3), (6,-6), (-3,0) and (6,-3) respectively.

4. (a) Particles of mass 4, 5 and 6 kg are placed at (0, 0), (4,3) and (5,-2) respectively in the x-y plane. Find the coordinates of their centre of mass.
   (b) Find the position of the centre of gravity of three particles of masses 1 kg, 5 kg and 2 kg which lie on the y-axis at points (0,2), (0,4) and (0,5) respectively.

5. Find the position of the centre of gravity of

three particles of masses 5 kg, 2 kg and 3 kg which act at points with position vectors; 3**i**−**j**, 2**i** + 3**j** and -2**i** + 5**j** respectively.

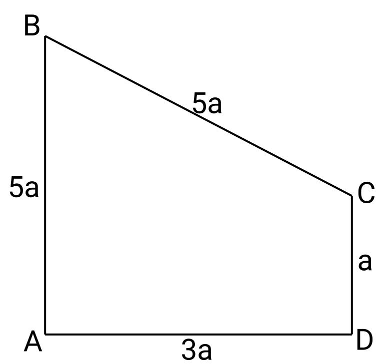
6. Four particles of masses 2, 2, 5 and 1 kg are at the points A, B, C and D whose position vectors are 3**i** + 5**j**, 3**i**−3**j**, -2**i**−3**j** and 2**i**−**j**. Find the position of the centre of gravity in vector form.

7. Particles of 2 kg, 3 kg, 3 kg and 2 kg are at points (-2,5), (4,3), (2,-1) and (-2,2) respectively in the x-y plane. Find the coordinates of their centre of gravity.

8. A uniform rod AB is 4 m long and has a mass of 6 kg, and masses are attached to it as follows; 1 kg at A, 2 kg at 1 m from A, 3 kg at 2 m from A, 4 kg at 3 m from A and 5 kg at B. Find the distance from A of the centre of gravity of the system.

9. A straight piece of uniform wire of length $6(2+\sqrt{2})$ cm is bent so as to form a right angled triangle ABC, with AB = BC = 6 cm and $A\hat{B}C = 90°$. Find the perpendicular distance of the centre of gravity of the triangle from AB and BC.

10. A uniform straight wire of length 14a is bent into the shape shown in the diagram below.



   (a) Find the distance of centre of gravity from AB and AD.
   (b) If the wire is freely suspended from B and hangs in equilibrium, find the angle of inclination of BC to the vertical.

Exercise: 8B

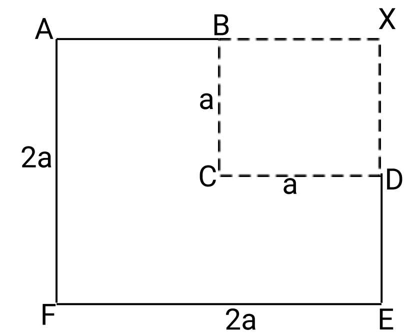
1. A uniform triangular lamina PQR has PQ = PR = 17 cm and QR = 16 cm. Find the distance of its centre of gravity from QR.

2. (a) The table below shows four particles A, B, C and D, their masses and points at which they act.

Particles	Mass(kg)	Point of action
A	4	(0,0)
B	5	(4,3)
C	6	(5,-2)
D	2	(2,-3)

Find the coordinates of their centre of gravity.

(b)



The figure shows a square lamina AFEX from which portion BCDX was cut off. Find the distance of the centre of gravity of the remaining portion from FA and FE.

3. (a) At points (4,1), (-2,-1) and (-1,λ) there are particles of mass 5 kg, a kg and 4 kg respectively. If the centre of gravity of the particles is at the origin, find the values of λ and a.
   (b) A compound lamina consists of a triangle ABC, where A is at (-4,0) B at (4,0) and C at (4,6) and a semi-circle with BC as the diameter.
      (i) Find the centre of gravity of the lamina.
      (ii) The lamina is suspended from A. Find the angle AB makes with the vertical in equilibrium.

4. (a) The table below shows four particles A, B, C and D, their masses and points of action.

Particle	Mass(kg)	Point of action
A	3	(1,6)
B	5	(-1,5)
C	2	(2,-3)
D	4	(-1,-4)

Find the coordinates of their center of gravity.
   (b) PQRS is a uniform square lamina of side 2l. T is a point on PS such that TS = h, if the portion TRS is removed.
      (i) Show that the centre of gravity of

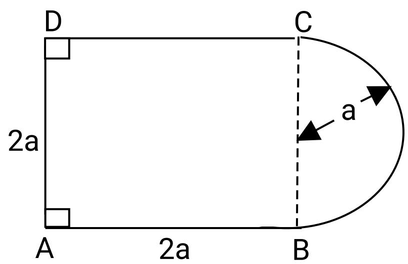
the remainder is a distance $\frac{12l^2-6lh+h^2}{3(4l-h)}$ from side PQ.

(ii) If the remaining lamina is placed in a vertical plane with PT on a rough horizontal surface, show that it will topple if $h > l(3-\sqrt{3})$.

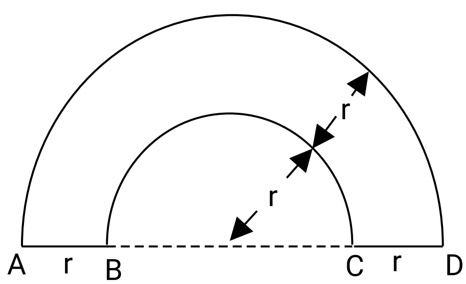
5. A frustrum is cut from a solid right circular cone of base radius r, and height 2h by a plane parallel to the base at a distance h from it. Find the distance of the centre of gravity of the frustrum from the base.

6. In the diagrams below find the distance from AD of the centre of gravity of the given uniform lamina.

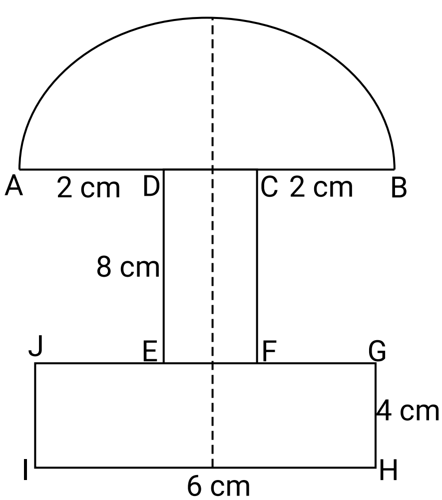
(a)



(b)



7.



The figure ABCDEFGHIJ shows a symmetrical composite lamina made up of a semi-circular lamina of radius 3 cm, a rectangle CDEF 2 cm×8 cm and another rectangle GHIJ 6 cm×4 cm. Find the distance of the centre of gravity of this lamina from IH. If the lamina is freely suspended from H, calculate the angle of inclination of HG to the vertical.

8. A uniform square lamina ABCD of side 2a, from which the isosceles triangle ABE is cut away; in this triangle AE = BE and the distance of E from AB is h. Prove that the centre of gravity of the remaining portion is at a distance $\frac{12a^2-h^2}{3(4a-h)}$ from AB. If $h = \frac{a}{2}$ and AEBCD is suspended by a vertical string from A, find the angle which AD makes with the vertical.

9. A square lamina ABCD of side 2a has a semi-circular lamina of radius a removed from side BC. Find the angle AB makes with the horizontal if the remaining lamina is freely hang from point D.

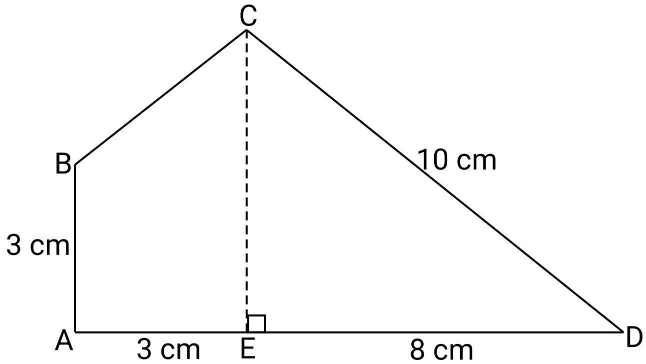
10. ABC is a uniform triangular lamina right angled at B, AB = 2t and BC = 3t. Show that the centre of gravity of ABC is at a distance t from AB. The midpoints P and Q of CB and CA respectively are joined and the portion PQC is cut off. Find the distance from AB and BC of the centre of gravity of the lamina ABPQ. When the lamina is freely suspended from vertex A, AB is at an angle θ to the vertical. Find tan θ.

11. A rectangular metal sheet PQRS has PQ = 4 m and QR = 3 m. T is a point on RS such that RT = 3 m. The sheet is folded about the line QT until R lies on PQ.
   (a) Find the position of the centre of gravity of the folded sheet from PS and PQ.
   (b) The sheet is freely suspended from the point S. Find the angle of inclination of PS to the vertical.

12. The base of a heavy truck is made of a uniform rectangular sheet ABCD such that AB = 5 m, BC = 10 m. Given that masses of 10 kg, 20 kg, 30 kg and 40 kg are fixed at the points A, B, C and D respectively, and the mass of the sheet is 1000 kg.
   (a) Find the centre of gravity of the sheet, taking AB and AD as the x and y-axes respectively.
   (b) If the sheet is suspended from point A,

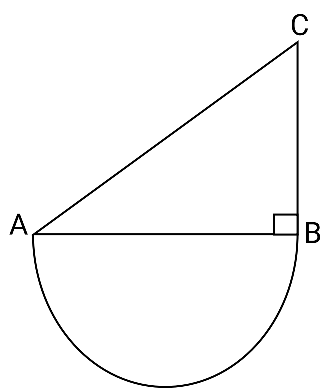
calculate the angle that the diagonal AC makes with the vertical.

13. Find the coordinates of centre of mass of the lamina shown below. Take A as the origin and AD, AB as x and y-axes respectively.



14. A uniform circular lamina of radius 10 cm is perforated with a circular hole of radius 2 cm, whose centre is 6 cm from the centre of the lamina. Determine the centre of gravity of the remaining lamina.

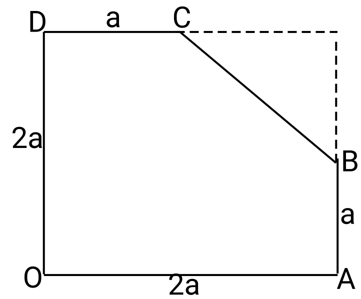
15. The figure below shows a compound lamina consisting of a right angled triangle and a semi-circle. A, B and C are points (-4,0), (4,0) and (4,6) respectively.



(a) Find the coordinates of the centre of gravity of the lamina.

(b) The lamina is suspended from A, find the angle AB makes with vertical when the lamina is in equilibrium.

16. OABCD is a uniform lamina with OA = OD = 2a, AB = DC = a as shown below.

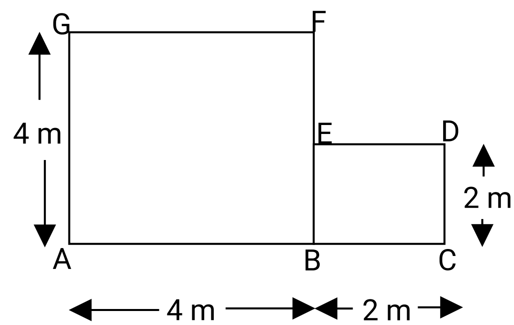


Find the:

(a) Distance of its centre of gravity from O.

(b) Angle a plumbline hang from A makes with OA.

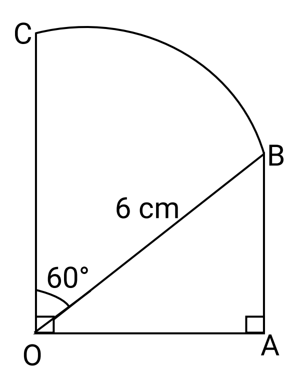
17. The diagram shows two uniform squares, ABFG and BCDE joined together. The mass per unit area of BCDE is twice that of ABFG.



Find the distance of the centre of gravity of the composite body from AB and AG.

18. Find the position vector of the centre of gravity of a uniform lamina in the form of a triangle whose vertices are (2,2), (4,6) and (0,3).

19.



The figure represents a uniform lamina consisting of a sector OBC of a circle centre O and of radius 6 cm and triangle

OAB, $A\hat{O}C = 90°$.

(a) Find the distances of the centre of gravity of the lamina from OC and OA.

(b) The lamina is suspended freely from point C, determine the angle which OC makes with the vertical in equilibrium.

20. (a) particles of mass 1,2,1 and a kg are placed at points (6,4),(-1,λ),(5,-1) and (1,0) respectively. Given that their centre of gravity is located at (2,1·4), find the values of a and λ .

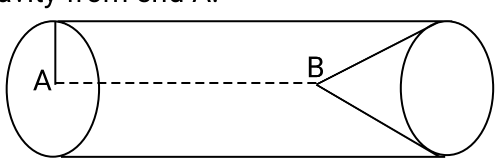
(b) A square lamina ABCD of side 2a is made of a uniform thin material. A semi-circular lamina with AB as diameter is cut away.

(i) Show that the centre of gravity of the remainder is a distance $\frac{20a}{3(8-\pi)}$ from AB.

(ii) The remainder is suspended from C by a light string and hangs in equilibrium. Find the angle CB makes with the vertical.

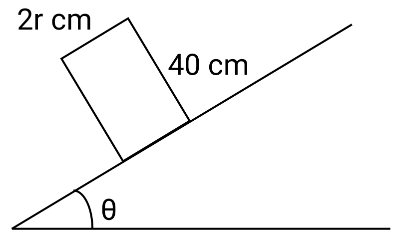
Exercise: 8C

1. The figure below shows a uniform cylindrical solid block of radius r and height 4r. A conical hole of radius r and height r is drilled from the block. Find the distance of the centre of gravity from end A.



2. A uniform right circular cone has its top removed by cutting the cone by a plane parallel to its base, leaving a frustrum of height h, the radii of its ends being r and 4r. Show that the centre of gravity of the frustrum is at a distance $\frac{9}{28}h$ from its broader end.

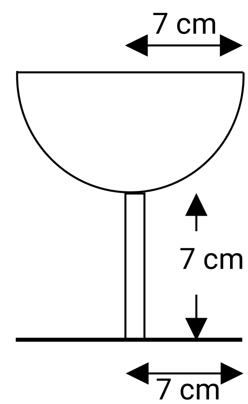
3. A uniform right cylinder has a height of 40 cm and a base radius r cm. It is placed with its axis vertical on a rough horizontal plane. The plane is slowly tilted and the cylinder topples when the angle of inclination θ is 20°. Find r.



What can be said about the coefficient of friction between the cylinder and the plane?

4. (a) Show that the centre of gravity of a uniform thin hemispherical cup of radius r is at a distance $\frac{r}{2}$ from the base.

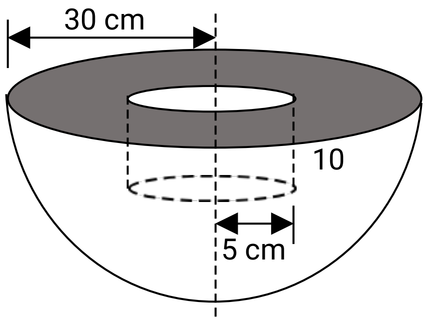
(b) The figure below is made up of a thin hemispherical cup of radius 7 cm. It is welded to a stem of length 7 cm and then to a circular base of the same material and of radius 7 cm. The weight of the stem is one-quarter that of the cup.



Find the distance from the base of the centre of gravity of the figure.

5. A circular cylindrical can without a lid is made of a thin metal sheeting of uniform thickness and with a mass per unit area of 1 g cm^{-2}. The radius of the can is 10 cm and its height is 20 cm. The can is placed with its base on a horizontal plane and is half-filled with a liquid of density 1•5 g cm^{-3}. Calculate the height of the centre of gravity of the can together with the liquid above the base of the can.

6. The figure below shows a uniform hemispherical solid of radius 30 cm with a cylindrical hole of radius 5 cm and height 10 cm centrally drilled in it.



(a) Find the distance of the centre of gravity of the figure from the flat surface.

(b) If the figure is placed on a rough inclined plane with the flat surface in contact with the plane, calculate the angle that the plane should be inclined before toppling occurs, assuming that sliding does not occur.

Answers to Exercises

Exercise: 8A

1. $3\cdot9051$ units 2. $\frac{1}{20}\mathbf{i} + \frac{17}{10}\mathbf{j}$ 3. $\left(3,-\frac{1}{2}\right)$ 4. (a) $\left(\frac{10}{3},\frac{1}{5}\right)$ (b) $(0,4)$

5. $1\cdot3\mathbf{i} + 1\cdot6\mathbf{j}$ 6. $0\cdot4\mathbf{i} - 1\cdot2\mathbf{j}$ 7. $(1,2)$ 8. $\frac{52}{21}$ m 9. $\frac{3\sqrt{2}}{2}$ cm ; $\frac{3\sqrt{2}}{2}$ cm

10. (a) $\frac{15}{14}a$; $2a$ (b) $17\cdot2°$

Exercise: 8B

1. 5 cm 2. (a) $\left(\frac{54}{17},-\frac{3}{17}\right)$ (b) $\frac{5}{6}a$; $\frac{5}{6}a$ 3. (a) $\frac{3}{4}$; 8

(b) (i) $(2\cdot7938, 2\cdot3707)$ (ii) $19\cdot2°$ 4. (a) $\left(-\frac{1}{7},\frac{3}{2}\right)$ (b) (i) (ii)

5. $\frac{11}{28}h$ 6. (a) $\frac{2a(14+3\pi)}{3(8+\pi)}$ (b) $\frac{28r}{9\pi}$ 7. $6\cdot717$ cm ; $24\cdot1°$ 8. $41\cdot8°$

9. $32\cdot1°$ 10. $\frac{2}{3}t$; $\frac{7}{9}t$; $\tan\theta = \frac{6}{11}$ 11. (a) $\frac{13}{8}$ m ; $\frac{9}{8}$ m (b) $40\cdot9°$

12. (a) $\frac{57}{11}$ m ; $\frac{5}{2}$ m (b) $0\cdot8°$ 13. $(4\cdot23, 2\cdot12)$ 14. $\frac{1}{4}$ cm from O along the line of symmetry 15. (a) $\left(\frac{4}{\pi+3},\frac{2}{3(\pi+3)}\right)$ (b) $1\cdot3°$ 16. (a) $1\cdot28a$ units (b) $39\cdot6°$ 17. $\frac{5}{3}$ m ; 3 m 18. $2\mathbf{i} + \frac{11}{3}\mathbf{j}$ 19. (a) $2\cdot3645$ cm ; $2\cdot6328$ cm (b) 35.1° 20. (a) 1; 2 (b) (ii) 57.9°

Exercise: 8C

1. $\frac{81}{44}r$ 3. 7.28 cm ; $\mu \geq \tan 20°$ 4.(b) 6.5 cm

5. $5\cdot75$ cm 6. (a) $11\cdot338$ cm (b) $69\cdot3°$

9. GENERAL EQUILIBRIUM OF A RIGID BODY

For a rigid body forces may act at different points such that even if the resultant force is zero the forces may produce a turning effect about a point. Hence for a rigid body to be in equilibrium there should be no resultant force and there should be no resultant moment.

9.1 Three force problems

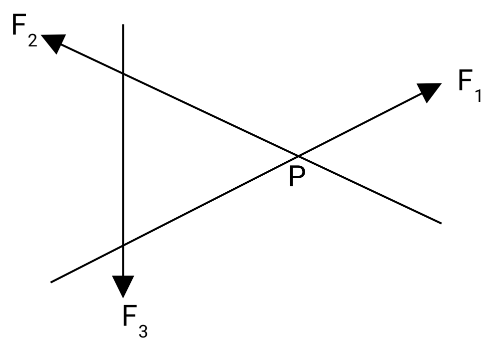
When a body is in equilibrium under the action of three forces, the resultant of the forces must be zero. Hence three non-parallel forces in equilibrium can be represented in magnitude and direction by the sides of a triangle. Also the sum of moments of coplanar forces in equilibrium about any point in their plane is zero.

Therefore one approach to three force problems is to:

(i) take moments about any suitable point, and

(ii) use a triangle of forces.

Consider three forces F_1, F_2 and F_3 shown below.

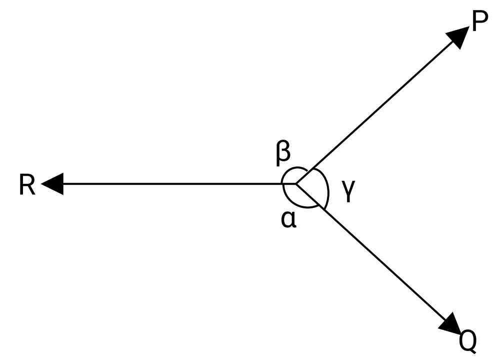


If F_1 and F_2 act along lines which meet at P then the moments of F_1 and F_2 about P are zero. Since F_1, F_2 and F_3 are in equilibrium, the moment of F_3 about P must also be zero. Hence the line of action of F_3 should pass through P.

Hence if a rigid body is in equilibrium under the action of three coplanar forces, then the lines of action of the forces are either parallel or concurrent.

9.2 Lami's theorem

When solving three force problems where angles rather than length are given, Lami's theorem may be applied. It states that for forces P, Q and R in equilibrium, as shown in the diagram.



$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}$$

9.3 General method

The following steps are employed when solving problems involving a rigid body in equilibrium under the action of coplanar forces.

(i) Interpret the information given and draw an appropriate sketch.

(ii) Indicate clearly all forces acting on the body in magnitude and direction.

(iii) Take moments about a suitable point.

(iv) Resolve forces in two perpendicular directions.

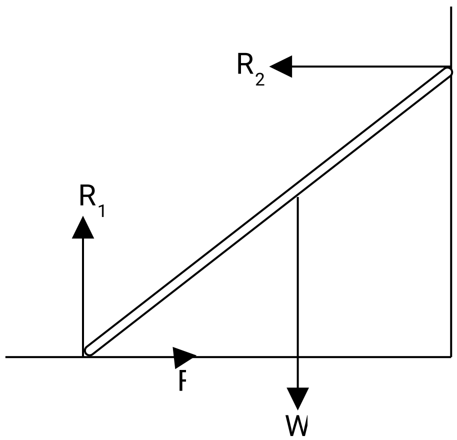
Note:

- At a smooth point of contact there is a reaction normal to the surface.
- At a rough point of contact there is a normal reaction and friction force.
- When a rigid body rests against and over a smooth surface, there is a reaction normal to the rigid body at the point of contact with the surface.
- When a rigid body rests against and over a rough surface, there is friction force and a reaction normal to the rigid body at the point of contact with the surface.
- When a rigid body is hinged on a surface, there is a reaction force having components in two perpendicular directions.

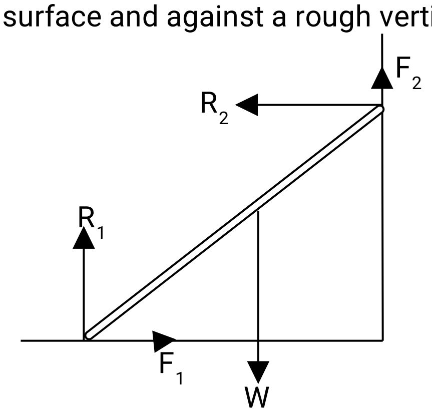
9.4 Ladder problems

Some common cases:

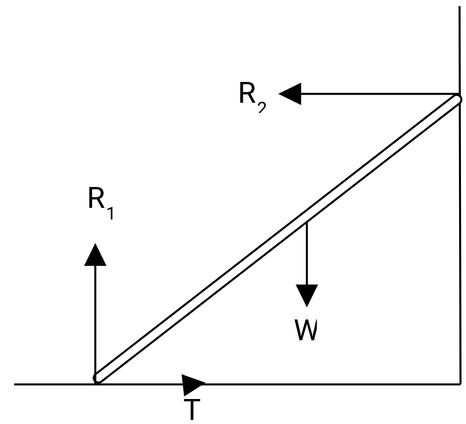
(i) A ladder resting on a rough horizontal surface and the other end on a smooth vertical wall.



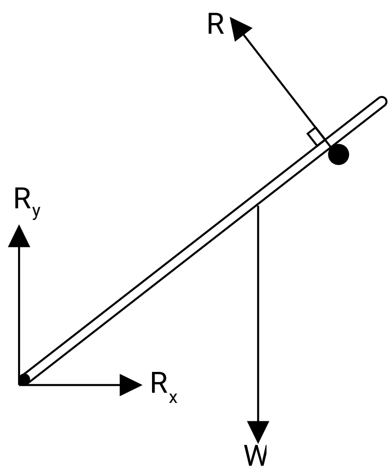
(ii) A ladder resting on a rough horizontal surface and against a rough vertical wall.



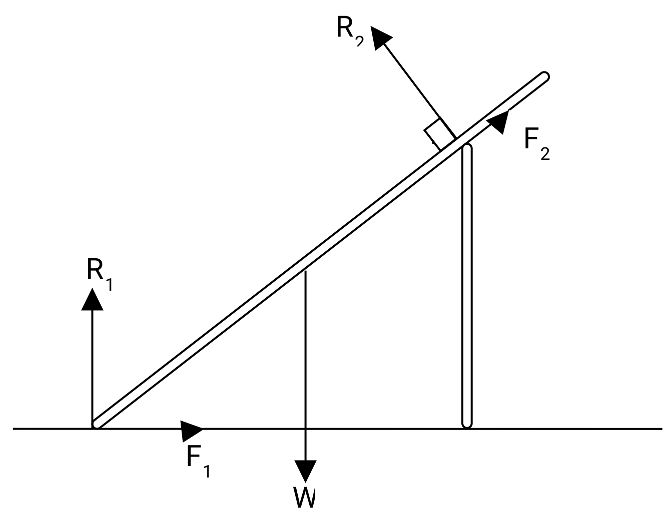
(iii) A ladder resting on a smooth horizontal surface and against a smooth vertical wall with a string tied to the foot of the ladder and fastened to the base of the wall.



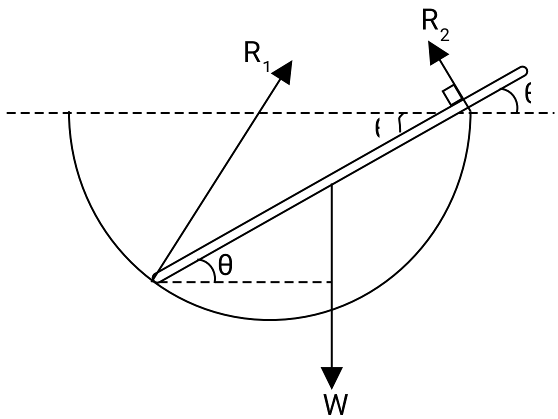
(iv) A ladder hinged on a smooth horizontal surface while its upper end rests against and over a smooth horizontal bar.



(v) A ladder resting on a rough horizontal surface while its upper end rests against and over a rough vertical wall.



(vi) A uniform rod resting partly inside and partly outside a uniform hemispherical bowl whose rim is horizontal.



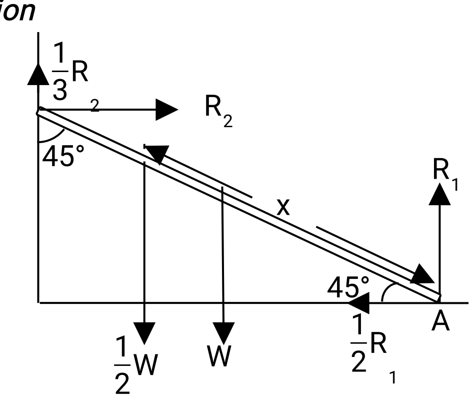
9.4.1 Climbing a ladder

When climbing a ladder, the climber can go as far as limiting equilibrium is not broken. Hence to find the distance that can be climbed before equilibrium is broken, we represent the forces and consider the system being in limiting equilibrium. The usual conditions for equilibrium are applied to solve the system.

Example 1

A uniform ladder of length 7 m rests against a vertical wall with which it makes an angle of 45°, the coefficient of friction between the ladder and the wall and the ladder and horizontal ground being $\frac{1}{3}$ and $\frac{1}{2}$ respectively. How far along the ladder can a man whose weight is half that of the ladder ascend before it slips?

Solution



Resolving horizontally:

$$(\rightarrow):\ \frac{1}{2}R_1 = R_2$$

$$R_1 = 2R_2 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (i)$$

Resolving vertically:

$$R_1 + \frac{1}{3}R_2 = \frac{1}{2}W + W$$

$$2R_2 + \frac{1}{3}R_2 = \frac{3}{2}W$$

$$R_2 = \frac{9}{14}W$$

Taking moments about A:

$$R_2 \times 7\sin 45 + \frac{1}{3}R_2 \times 7\cos 45$$

$$= \frac{1}{2}W \times x\cos 45 + W \times \frac{7}{2}\cos 45$$

$$\frac{9}{14}W \times 7 + \frac{1}{3} \times \frac{9}{14}W \times 7 = \frac{1}{2}Wx + \frac{7}{2}W$$

$$x = 5 \text{ m}$$

Example 2

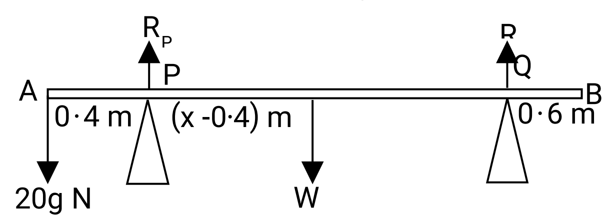
A non-uniform beam AB, 4·5 m long is balanced on two supports P and Q such that AP = 0·4 m and QB = 0·6 m. When a mass of 20 kg is placed at either end, the beam is on the point of toppling. Find the:

(i) distance from A at which the weight of the beam acts.
(ii) weight of the beam.
(iii) distance from A at which the 20 kg mass must be placed for the reactions of the supports to be equal.

Solution

(i) Let W be the weight of the beam acting through a point at a distance x from A.

1st Case: When the 20 kg mass is placed at A:



In equilibrium:

Resolving vertically:

$$R_P + R_Q = 20g + W$$

At the point of toppling, $R_Q = 0$

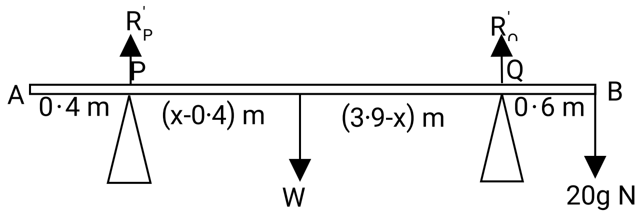
$$R_P = 20g + W \ \ldots\ldots\ldots\ldots\ldots\ldots\ (i)$$

Taking moments about A:

$$R_P \times 0{\cdot}4 = Wx$$

$$\Rightarrow R_P = \frac{5}{2}Wx \text{ ………… (ii)}$$

2nd Case: When the 20 kg mass is placed at B:



In equilibrium, resolving vertically:

$$R'_P + R'_Q = W + 20g$$

At point of toppling, $R'_P = 0$

$$R'_Q = W + 20g \text{ ………………… (iii)}$$

Taking moments about B:

$$0{\cdot}6R'_Q = W(4{\cdot}5 - x)$$

$$\Rightarrow R'_Q = \frac{15}{2}W - \frac{5}{3}Wx \text{……………. (iv)}$$

From equation (i) and equation (ii):

$$\frac{5}{2}Wx = 20g + W \text{ ………………… (v)}$$

From equation (iii) and equation (iv):

$$W + 20g = \frac{15}{2}W - \frac{5}{3}Wx$$

$$\frac{5}{3}Wx = \frac{13}{2}W - 20g \text{ ………………….. (vi)}$$

Adding equation (v) and equation (vi):

$$\frac{25}{6}Wx = \frac{15}{2}W$$

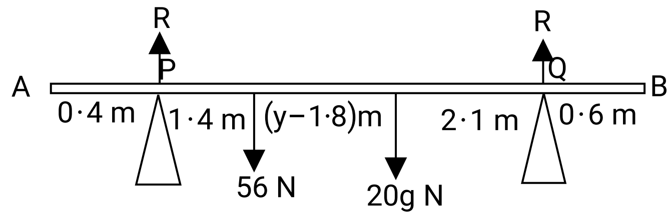
$$x = 1{\cdot}8 \text{ m}$$

(ii) weight of the beam

From equation (v): $\frac{5}{2}W \times 1{\cdot}8 = 20g + W$

$$W = \frac{40}{7}g = \frac{40}{7} \times 9{\cdot}8 \Rightarrow W = 56 \text{ N}$$

(iii) Let the 20 kg mass be placed at a distance y from A.



Resolving vertically:

$$2R = 56 + 20g$$

$$2R = 56 + 20 \times 9{\cdot}8 \Rightarrow R = 126 \text{ N}$$

Taking moments about P:

$$R \times 3{\cdot}5 = 56 \times 1{\cdot}4 + 20g \times (y - 0{\cdot}4)$$

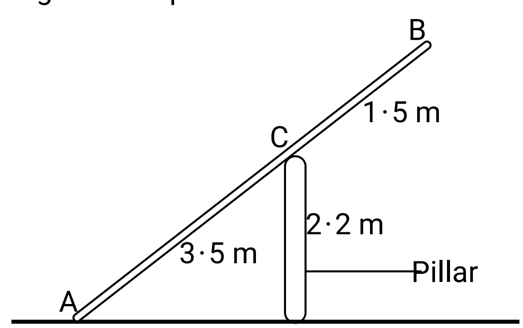
$$3{\cdot}5 \times 126 = 78{\cdot}4 + 20 \times 9{\cdot}8 \times (y - 0{\cdot}4)$$

$$\Rightarrow y = 2{\cdot}25 \text{ m}$$

Example 3

The diagram below shows a uniform wooden plank AB of mass 70 kg and length 5 m. The end A rests on a rough horizontal ground. The plank is in contact with the top of a rough pillar at C.

The height of the pillar is 2·2 m and AC = 3·5 m.



Given that the coefficient of friction at the ground is 0·6 and the plank is just about to slip, find the:

(a) angle the plank makes with the ground at A.

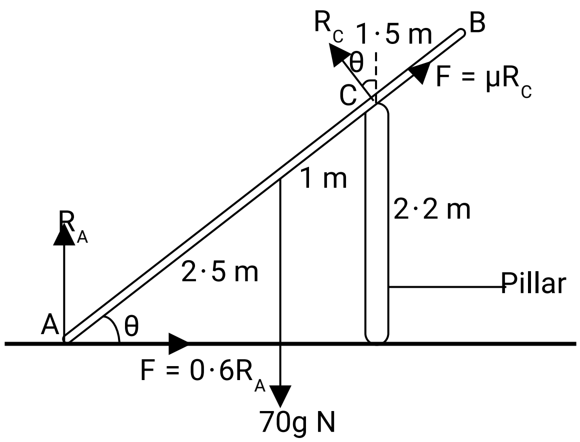
(b) normal reaction at:

(i) A

(ii) C

(c) coefficient of friction at C.

Solution:



(a) $\sin\theta = \frac{2{\cdot}2}{3{\cdot}5}$

$\theta = 38{\cdot}9°$

(b) (ii) Talking moments about A:

$R_C \times 3{\cdot}5 = 70g \times 2{\cdot}5\cos\theta$

$R_C = \frac{70\times9{\cdot}8\times2{\cdot}5\cos 38{\cdot}9}{3{\cdot}5}$

$R_C = 381{\cdot}34$ N

(i) Taking moments about C:

$R_A \times 3{\cdot}5\cos\theta = 0{\cdot}6R_A \times 2{\cdot}2 + 70g \times 1\cos\theta$

$R_A(3{\cdot}5\cos 38.9 - 0{\cdot}6\times2{\cdot}2) = 70\times9.8\cos 38{\cdot}9$

$R_A = 380{\cdot}495$ N

(c) Resolving horizontally:

$(\rightarrow)$: $R_C\sin\theta = \mu R_C\cos\theta + 0{\cdot}6R_A$

$381.34\sin 38.9 = \mu\times381.34\cos 38{\cdot}9 + 0{\cdot}6\times380.495$

$\mu = 0{\cdot}038$

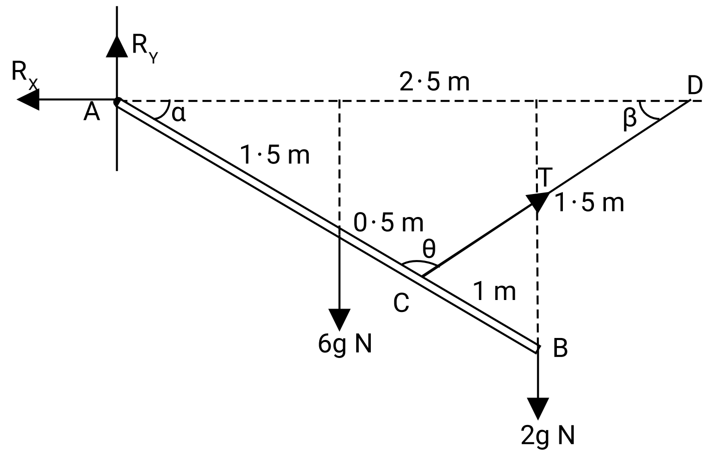
Example 4

A uniform rod AB of mass 6 kg and length 3 m is freely hinged at A. A load of mass 2 kg is attached to the rod at B. The rod is kept in equilibrium by a light inextensible string CD of length 1·5 m, which is attached to a point C on the rod and to a fixed point D on the same horizontal level as A. Given that AC = 2 m and AD = 2·5 m, find the:

(a) tension in the string.

(b) magnitude and direction of the reaction at the hinge.

Solution



$2{\cdot}5^2 = 1{\cdot}5^2 + 2^2 - 2\times1{\cdot}5\times2\cos\theta$

$\Rightarrow \theta = 90°$

(a) Taking moments about A:

$T\times2 = 6g\times1{\cdot}5\cos\alpha + 2g\times3\cos\alpha$

$T = 3\times9{\cdot}8\times1{\cdot}5\times\frac{2}{2{\cdot}5} + 9{\cdot}8\times\frac{2}{2{\cdot}5}$

$T = 58{\cdot}8$ N

(b) Resolving horizontally:

$(\rightarrow)$: $R_X = T\cos\beta$

$= 58{\cdot}8\times\frac{1{\cdot}5}{2{\cdot}5}$

$= 35{\cdot}28$ N

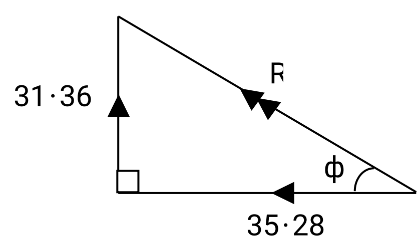
Resolving vertically:

$(\uparrow)$: $R_Y = 8g - T\sin\beta$

$= 8\times9{\cdot}8 - 58{\cdot}8\times\frac{2}{2{\cdot}5}$

$= 31{\cdot}36$ N

Let R be the magnitude of the reaction at A:



$R = \sqrt{31{\cdot}62^2 + 35{\cdot}28^2} \Rightarrow R = 47{\cdot}203$ N

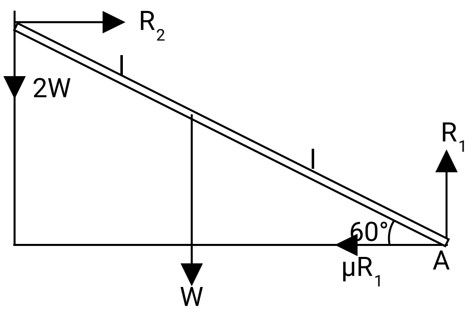
$\tan\phi = \frac{31{\cdot}36}{35{\cdot}28} \Rightarrow \phi = 41{\cdot}6°$

Example 5

A uniform ladder of weight W rests at 60° to the horizontal with its foot on a rough horizontal ground and the other end against a smooth vertical wall. If the ladder is just at a point of slipping when a man of weight 2W stands at the top, find the coefficient of friction between the ladder and the ground. Find the friction force between the ladder and the ground when the man is standing at the midpoint of the ladder.

Solution

When the man stands at the upper end of the ladder:



Resolving vertically:

$$(\uparrow): R_1 = 3W$$

Taking moments about A:

$$R_2 \times 2l\sin 60 = 2W \times 2l\cos 60 + W \times l\cos 60$$

$$R_2 = \frac{5\sqrt{3}}{6}W$$

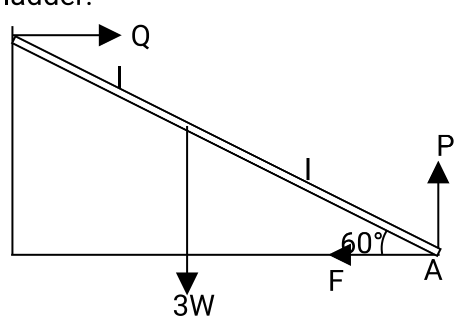
Resolving horizontally:

$$(\rightarrow): \mu R_1 = R_2$$

$$\mu \times 3W = \frac{5\sqrt{3}}{6}W$$

$$\mu = \frac{5\sqrt{3}}{18}$$

When the man stands at the midpoint of the ladder:



Resolving vertically:

$$(\uparrow): P = 3W$$

Taking moments about A:

$$Q \times 2l\sin 60 = 3W \times l\cos 60$$

$$Q = \frac{\sqrt{3}}{2}W$$

Resolving horizontally:

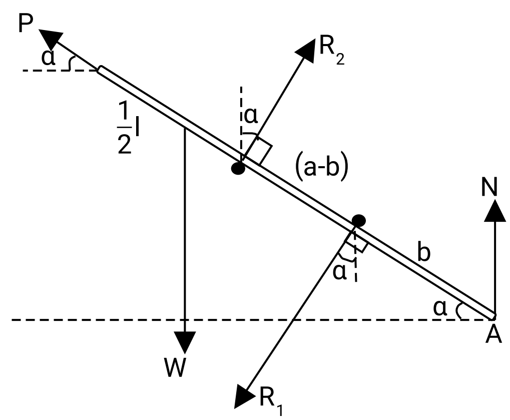
$$(\rightarrow): F = Q$$

$$F = \frac{\sqrt{3}}{2}W$$

Example 6

A uniform rod of weight W rests in contact with two parallel, horizontal, smooth pegs, passing over the higher peg and under the lower one. A force of magnitude P acts on the upper end of the rod in the direction of the rod tending to move it upwards. The length of the rod is l and its inclination to the horizontal is α. If the upper and lower pegs are at distances a and b respectively from the lower end of the rod, prove that as long as the rod remains in equilibrium, the reaction of the upper peg on the rod is $\frac{Wl\cos\alpha - 2Pb\cot\alpha}{2(a-b)}$.

Solution



Taking moments about A:

$$R_2 \times a = R_1 \times b + W \times \frac{1}{2}l\cos\alpha$$

$$R_2 a = R_1 b + \frac{W}{2}l\cos\alpha \text{ .... (i)}$$

Resolving horizontally:

$$R_2\sin\alpha = R_1\sin\alpha + P\cos\alpha$$

$$R_2 = R_1 + P\cot\alpha$$

$$R_1 = \left(R_2 - P\cot\alpha\right) \text{.........................(ii)}$$

Substituting in equation (i):

$$R_2 a = (R_2 - P\cot\alpha)b + \frac{W}{2}l\cos\alpha$$

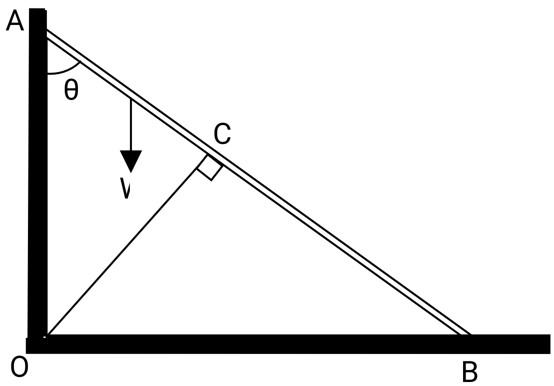
$$2R_2(a-b) = Wl\cos\alpha - 2Pb\cot\alpha$$

$$R_2 = \frac{Wl\cos\alpha - 2Pb\cot\alpha}{2(a-b)}$$

Example 7

The diagram below shows a uniform rod AB of weight W and length l resting at an angle θ against a smooth vertical wall at A. The other end B rests on a smooth horizontal table. The rod is prevented from slipping by an inelastic string OC, C being a point on AB such that OC is perpendicular to AB and O is the point of intersection of the wall and the table. If

$\angle AOB = 90°$

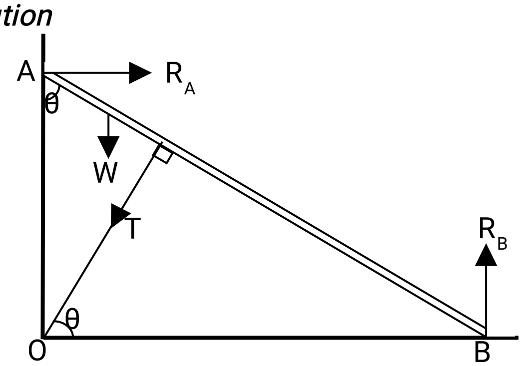


Find the:

(i) tension in the string.

(ii) reactions at A and B in terms of θ and W.

Solution



Resolving horizontally:

$(\rightarrow) : R_A = T\cos\theta$ ………………… (i)

Resolving vertically:

$(\uparrow) : R_B = T\sin\theta + W$ ……………. (ii)

Taking moments about O:

$$R_A \times l\cos\theta = R_B \times l\sin\theta - W \times \frac{1}{2} l\sin\theta$$

$$R_A = \left(R_B - \frac{1}{2}W\right)\tan\theta \text{………………… (iii)}$$

From equations (i), (ii) and (iii):

$$T\cos\theta = (T\sin\theta + W - \frac{1}{2}W)\tan\theta$$

$$T = \frac{W\sin\theta}{2\cos 2\theta}$$

(ii) $R_A = \dfrac{W\sin\theta}{2\cos 2\theta} \times \cos\theta = \dfrac{1}{4}W\tan 2\theta$

$$R_B = \frac{W\sin\theta}{2\cos 2\theta} \times \sin\theta + W$$

$$R_B = \frac{W}{2}\left(\frac{2-\tan^2\theta}{1-\tan^2\theta}\right)$$

Example 8

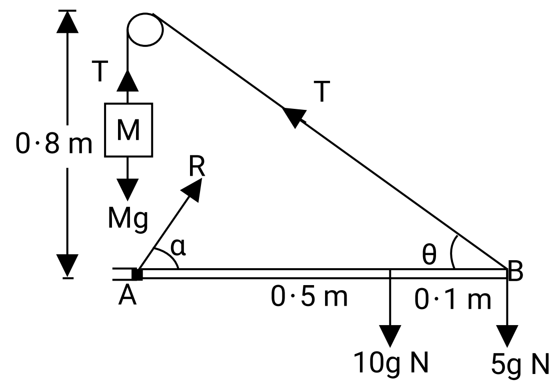
A rod AB of length 0·6 m and mass 10 kg is hinged at A. Its centre of mass is 0·5 m from A. A light inextensible string attached at B passes over a smooth fixed pulley 0·8 m above A and supports a mass M hanging freely. If a mass of 5 kg is attached at B so as to keep the rod in horizontal position, find the:

(a) value of M.

(b) reaction at the hinge.

Solution

(a)



Taking moments about A:

$$T\sin\theta \times 0.6 = 10g \times 0.5 + 5g \times 0.6$$

$$T \times \frac{0{\cdot}8}{1} \times 0{\cdot}6 = 8g \Rightarrow T = \frac{8 \times 9{\cdot}8}{0{\cdot}8 \times 0{\cdot}6} = \frac{490}{3}\text{ N}$$

Equilibrium of M;

Resolving vertically:

$$Mg = T$$

$$M \times 9{\cdot}8 = \frac{490}{3} \Rightarrow M = \frac{50}{3}\text{ kg}$$

$$= 16\frac{2}{3}\text{ kg}$$

(b) Equilibrium of rod AB;

Resolving horizontally:

$$R\cos\alpha = T\cos\theta$$

$$R\cos\alpha = 0{\cdot}6T = 0{\cdot}6 \times \frac{490}{3}$$

$$\Rightarrow R\cos\alpha = 98 \text{…………………(i)}$$

Resolving vertically:

$$R\sin\alpha + T\sin\theta = 10g + 5g$$

$$R\sin\alpha = 15 \times 9{\cdot}8 - \frac{490}{3} \times \frac{0{\cdot}8}{1}$$

$$\Rightarrow R\sin\alpha = \frac{49}{3} \text{………………… (ii)}$$

Dividing equation (ii) by equation (i)

$$\tan\alpha = \frac{1}{6} \Rightarrow \alpha = 9{\cdot}5°$$

From equation (i);

$$R\cos 9{\cdot}5 = 98 \Rightarrow R = 99{\cdot}358\ N$$

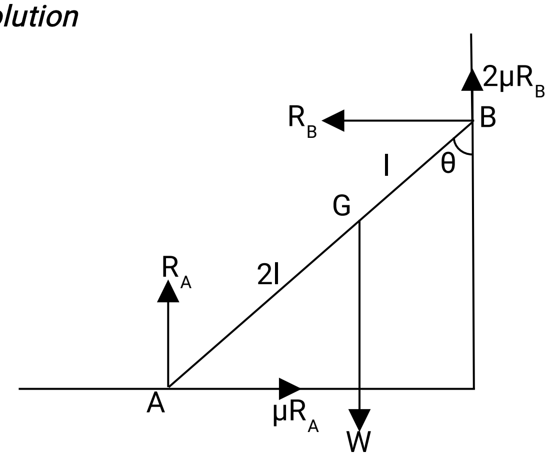
Example 9

A non-uniform ladder AB is in equilibrium with A in contact with a horizontal floor and B in contact with a vertical wall. The ladder is in a vertical plane perpendicular to the wall. The centre of gravity of the ladder is at G where $AG = \frac{2}{3}AB$. The coefficient of friction between the ladder and the wall is twice that between the ladder and the floor.

If the ladder makes an angle θ with the wall and the angle of friction between the ladder and the floor is λ, prove that $4\tan\theta = 3\tan 2\lambda$.

How far can a man of mass m ascend the ladder without the ladder slipping given that θ = 45° and the coefficient of friction between the ladder and the floor is $\frac{1}{2}$.

Solution



Let AB = 3l

$\mu = \tan\lambda$

Resolving horizontally:

$(\rightarrow): R_B = \mu R_A$ …………………… (i)

Resolving vertically:

$(\uparrow): R_A + 2\mu R_B = W$………………(ii)

From equation (i) and equation (ii):

$$R_A + 2\mu^2 R_A = W \Rightarrow R_A = \frac{W}{1+2\mu^2}$$

$$R_B = \frac{\mu W}{1+2\mu^2}$$

Taking moments about B:

$$R_A \times 3l\sin\theta = \mu R_A \times 3l\cos\theta + W \times l\sin\theta$$

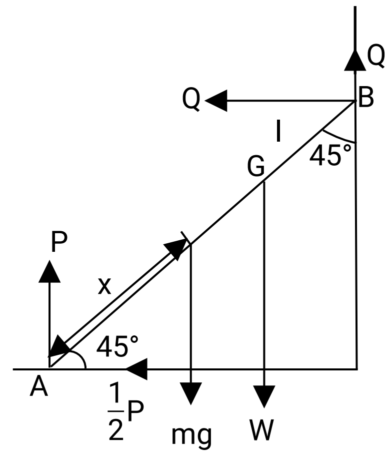
$$\frac{3W}{1+2\mu^2} = \frac{3\mu W}{1+2\mu^2}\cot\theta + W$$

$$\frac{2}{3}\tan\theta = \frac{\mu}{1-\mu^2}$$

But $\tan 2\lambda = \frac{2\tan\lambda}{1-\tan^2\lambda} \Rightarrow \tan 2\lambda = \frac{2\mu}{1-\mu^2}$

$\therefore\ \frac{2}{3}\tan\theta = \frac{1}{2}\tan 2\lambda \Rightarrow 4\tan\theta = 3\tan 2\lambda$

When the man ascends the ladder:



Resolving horizontally:

$$\frac{1}{2}P = Q$$

$P = 2Q$ ………………………. (i)

Resolving vertically:

$P + Q = mg + W$ …………………… (ii)

From equation (i) and equation (ii):

$$2Q + Q = mg + W$$

$$Q = \frac{1}{3}(mg + W)$$

$$P = \frac{2}{3}(mg + W)$$

Taking moments about A:

$Q \times 3l\sin 45 + Q \times 3l\cos 45$

$= mg \times x\cos 45 + W \times 2l\cos 45$

$3Ql + 3Ql = mgx + 2Wl$

$\Rightarrow 6 \times \frac{1}{3}l(mg + W) = mgx + 2Wl$

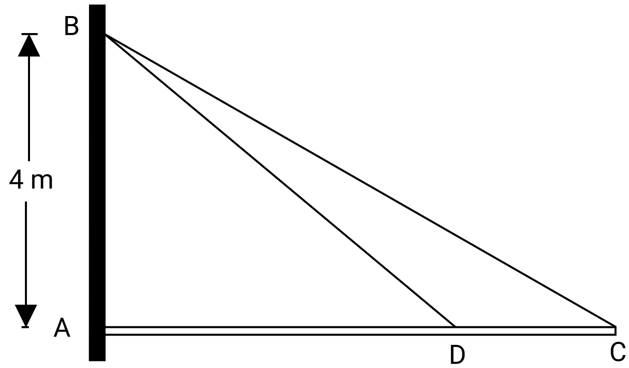
$2mgl = mgx$

$x = 2l$

Hence he climbs up to point G

Example 10

A uniform beam AC of mass 8 kg and length 8 m is hinged at A and is maintained in equilibrium by two strings attached to it at points C and D as shown in the figure below.



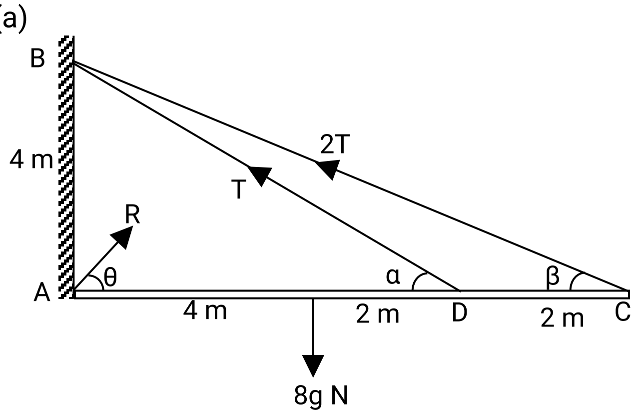
The tension in BC is twice that in BD, AB = 4 m and AD = $\frac{3}{4}$AC. Find the:

(a) tension in string BC.

(b) magnitude and direction of the reaction at the hinge.

Solution

(a)



Taking moments about A:

$$T\sin\alpha\times6 + 2T\sin\beta\times8 = 8g\times4$$

$$T = \frac{8\times9{\cdot}8\times4}{6\sin 33{\cdot}7+16\sin 26{\cdot}6} = 29\cdot9098 \text{ N}$$

Tension in BC = 2×29·9098 = 59·8196 N

(b) Resolving horizontally:

$$R\cos\theta = T\cos\alpha + 2T\cos\beta$$

$$= 29\cdot9098\cos 33.7 + 2\times29.9098\cos 26.6$$

$R\cos\theta = 78.3884$ …………………… (i)

Resolving vertically:

$$R\sin\theta = 8g-(T\sin\alpha + 2T\sin\beta)$$

$$R\sin\theta = 8\times9.8-(29.9098\sin 33.7 + 2\times29.9098\sin 26.6)$$

$R\sin\theta=35.0523$……………………..(ii)

Dividing equation (ii) by equation (i)

$$\tan\theta = \frac{35{\cdot}0523}{78{\cdot}3884}$$

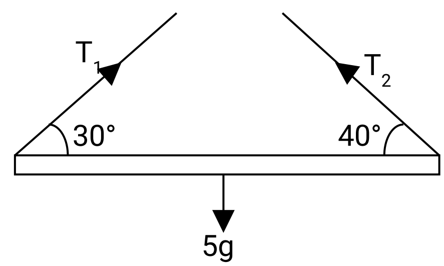
$$\theta = 24\cdot1°$$

From equation (i);

$$R\cos 24{\cdot}1 = 78\cdot3884 \Rightarrow R = 85\cdot8685 \text{ N}$$

Example 11

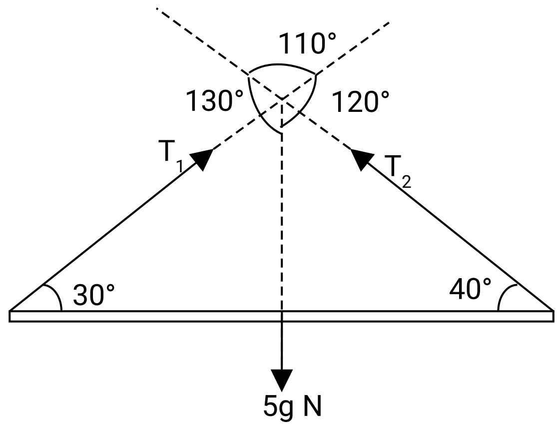
(a) A non-uniform beam of mass 5 kg rests horizontally in equilibrium, supported by two light strings attached at each end of the beam. The tensions in the strings are T_1 and T_2 and the strings make angles of 30° and 40° with the beam as shown in the diagram below. Find T_1 and T_2.



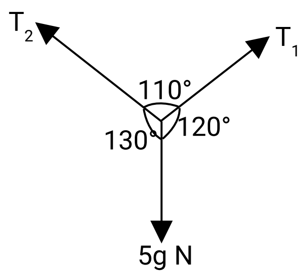
(b) A non-uniform ladder AB of length 6 m has its centre of gravity at a point C on the ladder, such that AC = 4 m. The ladder rests in limiting equilibrium with end A on a rough horizontal ground (coefficient of friction $\frac{1}{3}$) and end B against a rough vertical wall (coefficient of friction $\frac{1}{4}$). If the ladder makes an acute angle θ with the ground, prove that $\tan\theta = \frac{23}{12}$.

Solution

(a)



Using Lami's theorem:



$$\frac{T_1}{\sin 130} = \frac{5g}{\sin 110} = \frac{T_2}{\sin 120}$$

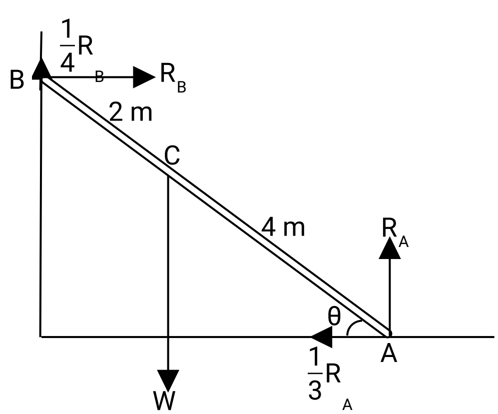
$$T_1 = \frac{5\times9\cdot8\times\sin 130}{\sin 110}$$

$$T_1 = 39\cdot9452 \text{ N}$$

$$T_2 = \frac{5\times9\cdot8\times\sin 120}{\sin 110}$$

$$T_2 = 45\cdot1586 \text{ N}$$

(b)



Resolving horizontally:

$$(\rightarrow): \frac{1}{3}R_A = R_B$$

$$R_A = 3R_B$$

Resolving vertically:

$$(\uparrow): \frac{1}{4}R_B + R_A = W$$

$$\frac{1}{4}R_B + 3R_B = W \Rightarrow R_B = \frac{4}{13}W$$

Taking moments about A:

$$\frac{1}{4}R_B\times6\cos\theta + R_B\times6\sin\theta = W\times4\cos\theta$$

$$\frac{3}{2}\times\frac{4}{13}W + 6\times\frac{4}{13}W\tan\theta = 4W$$

$$\frac{24}{13}\tan\theta = \frac{46}{13} \Rightarrow \tan\theta = \frac{23}{12}$$

Example 12

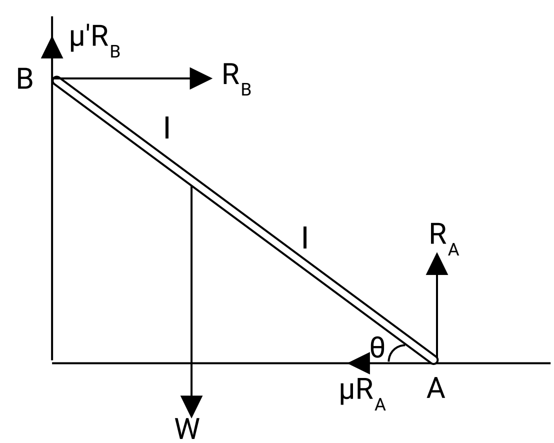
A uniform ladder AB of weight W has end A on a rough horizontal floor and end B leans against a rough vertical wall. Given that the ladder is on the point of slipping and the coefficients of friction at A and B are μ and μ' respectively:

(a) Show that the inclination of the ladder to the horizontal is $\tan^{-1}\left(\frac{1-\mu\mu'}{2\mu}\right)$.

(b) Find the reaction of the ladder on the wall.

Solution

(a)



Resolving horizontally:

$$(\rightarrow): R_B = \mu R_A \quad\text{................(i)}$$

Resolving vertically:

$$(\uparrow): \mu'R_B + R_A = W$$

$$\therefore \mu\mu'R_A + R_A = W$$

$$R_A = \left(\frac{W}{1+\mu\mu'}\right)$$

From equation (i): $R_B = \left(\frac{\mu W}{1+\mu\mu'}\right)$

Taking moments about A:

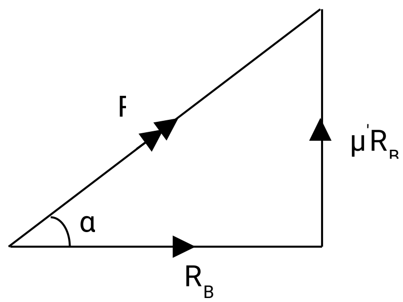
$$\mu' R_B \times 2l\cos\theta + R_B \times 2l\sin\theta = W \times l\cos\theta$$

$$\frac{2\mu\mu' W}{1+\mu\mu'} + \left(\frac{2\mu W}{1+\mu\mu'}\right)\tan\theta = W$$

$$\tan\theta = \frac{1-\mu\mu'}{2\mu}$$

$$\theta = \tan^{-1}\left(\frac{1-\mu\mu'}{2\mu}\right)$$

(b)



$$R = \sqrt{R_B^2 + (\mu' R_B)^2}$$

$$R = \sqrt{\left(\frac{\mu W}{1+\mu\mu'}\right)^2 + \left(\frac{\mu\mu' W}{1+\mu\mu'}\right)^2}$$

$$R = \frac{\mu W}{1+\mu\mu'}\sqrt{1+(\mu')^2}$$

$$\tan\alpha = \frac{\mu' R_B}{R_B}$$

$$\alpha = \tan^{-1}(\mu')$$

Example 13

A uniform rod AB of length 2a and mass m, rests in equilibrium with its lower end on a rough horizontal floor. Equilibrium is maintained by a horizontal elastic string of natural length a and modulus λ. One end of the string is attached to B on the end point to a point vertically above A. Given that $\theta \le \frac{\pi}{3}$ is the inclination of the rod to the horizontal;

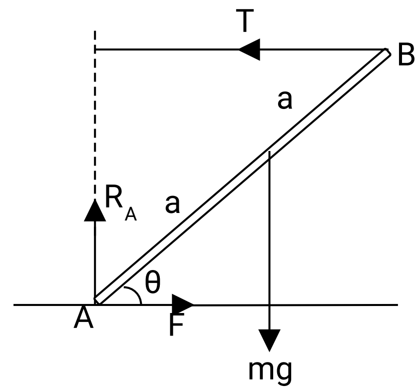
(a) Show that the magnitude of the tension in the string is $\frac{1}{2}mg\cot\theta$.

(b) Prove that $2\lambda = \frac{mg\cot\theta}{2\cos\theta - 1}$.

(c) Given that the system is in limiting equilibrium and the coefficient of friction between the floor and the rod is $\frac{2}{3}$, find tan θ, hence show that $\lambda = \frac{10}{9}mg$.

Solution

(a) The tension in the string:



Taking moments about A:

$$T \times 2a\sin\theta = mg \times a\cos\theta$$

$$T = \frac{1}{2}mg\cot\theta$$

(b) From Hooke's law: $T = \frac{\lambda x}{l_0}$; $l_0 = a$

$$x = 2a\cos\theta - a \Rightarrow x = a(2\cos\theta - 1)$$

$$T = \frac{\lambda a(2\cos\theta - 1)}{a} = \lambda(2\cos\theta - 1)$$

$$\therefore \quad \lambda(2\cos\theta - 1) = \frac{1}{2}mg\cot\theta$$

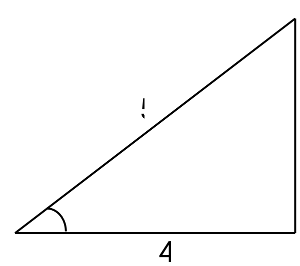
$$2\lambda = \frac{mg\cot\theta}{2\cos\theta - 1}$$

(c) Resolving vertically, $R_A = mg$

Resolving horizontally:

$$T = F, \text{ but } F = \frac{2}{3}mg$$

$$\therefore \quad T = \frac{2}{3}mg$$



$$\frac{1}{2}mg\cot\theta = \frac{2}{3}mg \Rightarrow \cot\theta = \frac{4}{3} \Rightarrow \tan\theta = \frac{3}{4}$$

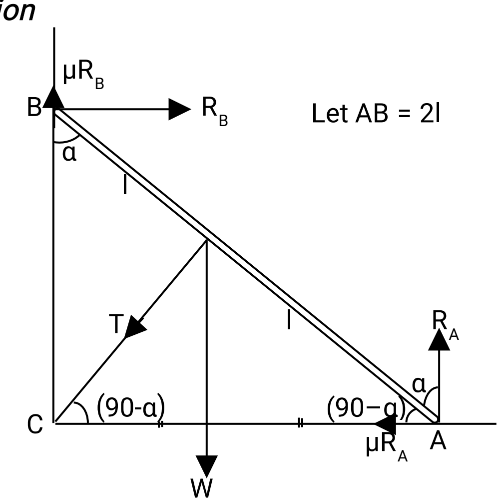
$$\therefore \quad 2\lambda = \frac{mg\times\frac{4}{3}}{2\times\frac{4}{5}-1}$$

$$\lambda = \frac{10}{9}mg$$

Example 14

A uniform ladder AB of weight W, leans at an angle α to the vertical against a rough vertical wall BC, with its other end on a rough horizontal floor AC, the coefficient of friction at each point of contact being μ. The midpoint of the ladder is attached to point C by means of a taut inextensible string. If the ladder is at the point of slipping, prove that the tension in the string is $\frac{W}{2\mu}\left\{(1-\mu^2)\sin\alpha - 2\mu\cos\alpha\right\}$ and find the reactions at A and B, taking α = 45° and $\mu = \frac{1}{4}$.

Solution



Resolving horizontally:

$$(\rightarrow): R_B = \mu R_A + T\cos(90-\alpha)$$

$$R_B = \mu R_A + T\sin\alpha \text{ .... (i)}$$

Resolving vertically:

$$(\uparrow): R_A + \mu R_B = W + T\sin(90-\alpha)$$

$$\Rightarrow R_A + \mu(\mu R_A + T\sin\alpha) = W + T\cos\alpha$$

$$R_A = \frac{W+T(\cos\alpha-\mu\sin\alpha)}{1+\mu^2} \text{ .... (ii)}$$

From equation (i);

$$R_B = \frac{\mu[W+T(\cos\alpha-\mu\sin\alpha)]}{1+\mu^2} + T\sin\alpha$$

$$R_B = \frac{\mu W+T(\mu\cos\alpha+\sin\alpha)}{1+\mu^2} \text{ .... (iii)}$$

Taking moments about C:

$$R_A\times 2l\sin\alpha - W\times l\sin\alpha = R_B\times 2l\cos\alpha$$

$$2R_A\sin\alpha - W\sin\alpha = 2R_B\cos\alpha \text{ .... (iv)}$$

Substituting for R_A and R_B in equation (iv):

$$\frac{2\sin\alpha[W+T(\cos\alpha-\mu\sin\alpha)]}{1+\mu^2} - W\sin\alpha = \frac{2\cos\alpha[\mu W+T(\mu\cos\alpha+\sin\alpha)]}{1+\mu^2}$$

$$T = \frac{W}{2\mu}\left\{(1-\mu^2)\sin\alpha - 2\mu\cos\alpha\right\}$$

When $\alpha = 45^0$ and $\mu = \frac{1}{4}$

$$T = \frac{W}{2\times\frac{1}{4}}\left\{\left(1-\frac{1}{16}\right)\sin 45 - 2\times\frac{1}{4}\times\cos 45\right\}$$

$$= 2W\left\{\frac{15\sqrt{2}}{32} - \frac{\sqrt{2}}{4}\right\}$$

$$T = \frac{7\sqrt{2}}{16}W$$

From equation (i): $R_B = \frac{1}{4}R_A + \frac{7\sqrt{2}W}{16}\times\frac{\sqrt{2}}{2}$

$$R_B = \frac{1}{4}R_A + \frac{7}{16}W \text{ ....(v)}$$

From equation (iv): $2R_A\times\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}W = 2R_B\times\frac{\sqrt{2}}{2}$

$$R_B = R_A - \frac{1}{2}W \text{ ....(vi)}$$

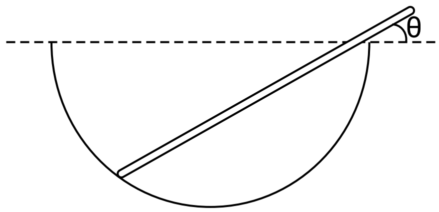
From equation (v) and equation (vi)

$$R_A - \frac{1}{2}W = \frac{1}{4}R_A + \frac{7}{16}W \Rightarrow R_A = \frac{5}{4}W$$

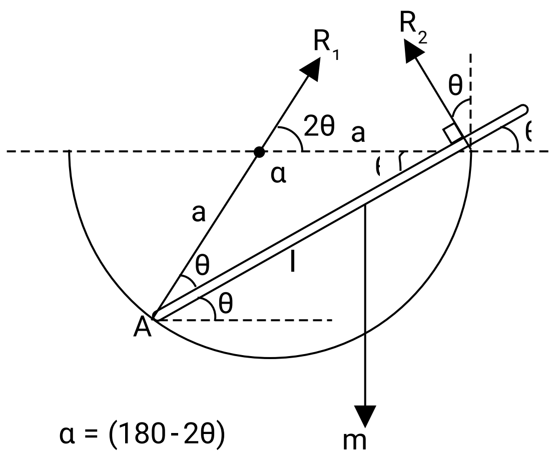
$$R_B = \frac{5}{4}W - \frac{1}{2}W \Rightarrow R_B = \frac{3}{4}W$$

Example 15

A uniform smooth rod of mass m and length 2l rests partly inside and partly outside a fixed smooth hemispherical bowl of radius a. The rim of the bowl is horizontal and one point of the rod is in contact with the rim. Prove that the inclination θ of the rod to the horizontal is given by $2a\cos 2\theta = l\cos\theta$. Find the reaction between the rod and bowl at the rim.



Solution



$\alpha = (180 - 2\theta)$

Taking moments about A:

$$R_2 \times \frac{a\sin(180-2\theta)}{\sin\theta} = mgl\cos\theta$$

$$R_2 a \times 2\cos\theta = mgl\cos\theta \Rightarrow R_2 = \frac{mgl}{2a}$$

Resolving horizontally:

$$(\rightarrow): R_1\cos 2\theta = R_2\sin\theta$$

$$R_1 = \frac{mgl\sin\theta}{2a\cos 2\theta}$$

Resolving vertically:

$$(\uparrow): R_1\sin 2\theta + R_2\cos\theta = mg$$

$$\therefore \quad \frac{mgl\sin\theta}{2a\cos 2\theta}\times\sin 2\theta + \frac{mgl\cos\theta}{2a} = mg$$

$$l(\sin 2\theta\sin\theta + \cos 2\theta\cos\theta) = 2a\cos 2\theta$$

$$2a\cos 2\theta = l\cos(2\theta - \theta)$$

$$2a\cos 2\theta = l\cos\theta$$

The reaction at the rim, $R_2 = \frac{mgl}{2a}$ normal to the rod.

Exercises

Exercise: 9A

1. A heavy uniform rod of length 2a rests in equilibrium against a smooth horizontal peg with one end of the rod on a rough horizontal floor. If the height of the peg above the floor is b and friction is limiting when the rod makes an angle of 30° with the floor, show that the coefficient of friction between the floor and the rod is $\frac{a\sqrt{3}}{(8b-3a)}$.
2. A uniform ladder of length 6·5 m rests with one end against a smooth vertical wall and the other end on a rough horizontal ground at a distance of 2·5 m from the wall. If the ladder is in limiting equilibrium, find the coefficient of friction between the ladder and the ground.
3. A uniform ladder of weight W and length l rests with its upper end against a smooth vertical wall. Its lower end rests on a rough horizontal ground, and the coefficient of friction between the ladder and the ground is $\frac{1}{2}$. The ladder is in limiting equilibrium. Find the angle the ladder makes with the horizontal and the reaction at the wall. A man of weight W climbs the ladder. Find how far up he can go before the ladder will slip. Find also how far up the ladder he can go when a load of weight W is placed on the foot of the ladder.
4. A uniform ladder of length 2l and weight W rests in a vertical plane with one end against a rough vertical wall and the other against a rough horizontal surface, the angles of friction at each end being $\tan^{-1}\frac{1}{3}$ and $\tan^{-1}\frac{1}{2}$ respectively.
   (a) If the ladder is in limiting equilibrium at either end, find θ, the inclination of the ladder to the horizontal.
   (b) A man of weight 10 times that of the ladder begins to ascend it, how far will he climb before the ladder slips?
5. A non-uniform ladder AB, 10 m long and mass 8 kg lies in limiting equilibrium with its lower end A resting on a rough horizontal ground

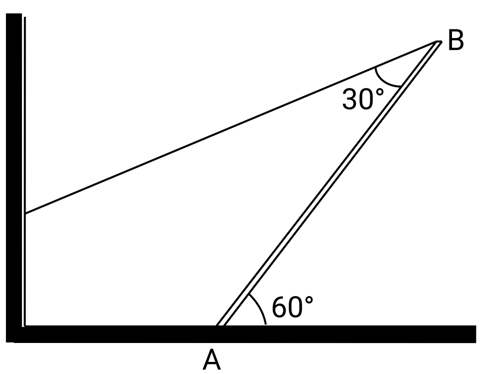
and the upper end B resting against a smooth vertical wall. If the centre of gravity of the ladder is 3 m from the foot of the ladder and the ladder makes an angle of 30° with the horizontal, find the:

(a) Coefficient of friction between the ladder and the ground.
(b) Reaction at the wall.

6. A uniform ladder rests in equilibrium with its top end against a smooth vertical wall and its base on a smooth inclined plane. The plane makes an angle α with the horizontal and the ladder makes an angle β with the wall. Prove that tan β = 2tan α.

7. A uniform rod of length 2 m and weight 20 N rests horizontally on smooth supports A and B. A load of 10 N is attached to the rod at a distance of 40 cm from A. Find the reactions on the supports at A and B.

8. A light inextensible string has one end fixed to a wall and the other end tied to the top end of a uniform beam AB of mass 10 kg as shown below.



If the string makes an angle of 30° with the beam which is inclined at 60° to the horizontal. Find the tension in the string and the reaction at A.

9. A uniform rod of length l rests in a vertical plane against (and over) a smooth horizontal bar at a height h, the lower end of the rod being on level ground. Show that if the rod is on the point of slipping when its inclination to the horizontal is θ, the coefficient of friction between the rod and the ground is $\frac{l\sin 2\theta \sin\theta}{4h - l\sin 2\theta\cos\theta}$.

10. A uniform rod AB of mass 1·5 kg and length 4 m is smoothly hinged at A and has a particle of mass 3 kg attached at B. A light inextensible string is attached to the rod at C, where AC = 2·5 m and to D vertically above A, keeps the rod in horizontal position. If the angle between the rod and the string is 30°, find the:

(i) tension in the string.
(ii) magnitude and direction of the reaction at the hinge.

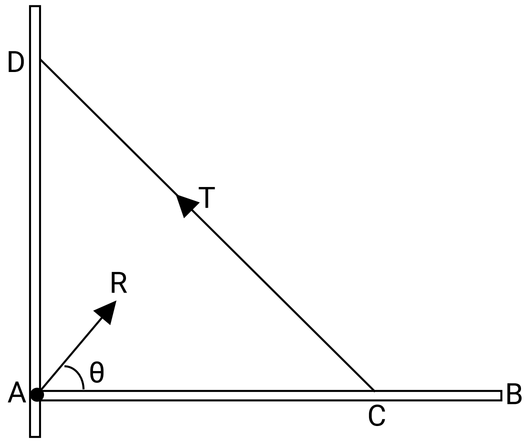
Exercise: 9B

1. A uniform rod AB of weight W and length 2l is smoothly hinged to a wall at A. One end of a light string is tied to the rod at B and the other end is tied to the wall at C, where C is vertically above A and AC = 2l. The rod is in equilibrium when AB makes an angle 2θ with the downward vertical (θ<45°). Determine in terms of W and θ the:

(i) tension in the string.
(ii) magnitude and direction of the reaction at the hinge.
(iii) Given that the string BC is elastic with natural length 2l and modulus of elasticity 1·5W, find the value of θ.

2. A uniform ladder AB rests on a rough horizontal ground and against a smooth vertical wall. The ladder stands at an angle α to the horizontal where tan α = $\frac{3}{2}$. How far up the ladder will a man whose weight is thrice that of the ladder climb before equilibrium is broken if the coefficient of friction is $\frac{1}{2}$?

3. A uniform rod AB of mass 6 kg and length 4 m is freely hinged at A to a vertical wall. The rod is horizontal and kept in equilibrium by a light inextensible string CD with D being vertically above A as shown below.



If AB = AD = 4 m and AC = 3 m, find the:

(i) tension in the string.

(ii) reaction R and the value of θ.

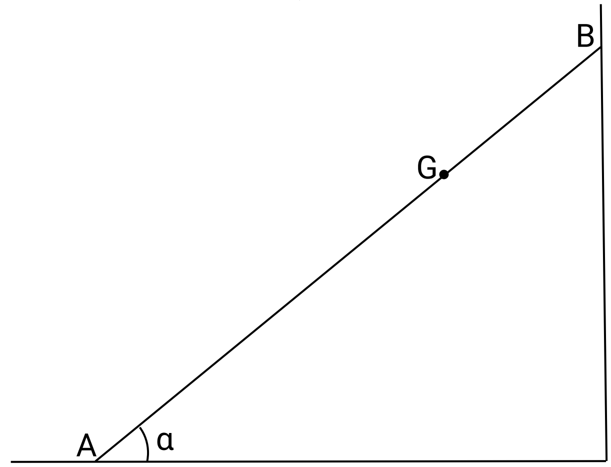
4. A uniform ladder of weight W_1 has one end resting on a smooth vertical wall and the other on a rough horizontal ground. If the ladder makes an angle α with the ground, prove that a man of weight W_2 will be able to climb up to the top of the ladder without having it slip, if the coefficient of friction is at least $\left(\frac{\frac{1}{2}W_1+W_2}{W_1+W_2}\right)\cot\alpha$.

5. A uniform ladder of length 2a rests in limiting equilibrium in a vertical plane with its lower end on a rough horizontal ground and its upper end on a smooth vertical wall. If the ladder makes an angle of 30° with the ground and the coefficient of friction is $\frac{\sqrt{3}}{6}$. How far up the ladder can a man whose weight is four times that of the ladder climb before it slips.

6. A uniform rod AB of length 2l and weight W has one end A on a rough horizontal ground, coefficient of friction μ. The rod is tied to a smooth peg at P where AP = a(l < a < 2l). If the rod is in limiting equilibrium inclined at an angle α to the horizontal and the peg is normal to the rod, prove that $\mu = \frac{l\sin\alpha\cos\alpha}{a-l\cos^2\alpha}$.

7. A uniform ladder rests with its lower end on a rough horizontal ground and the upper end on an equally rough vertical wall, the angle of friction being λ. If the ladder is in limiting equilibrium at either end, show that the angle of inclination to the vertical is 2λ.

8. A non-uniform ladder AB of length 3a has its centre of mass at G, where AG = 2a.



The ladder rests in limiting equilibrium with end B against a smooth vertical wall and end A resting on a rough horizontal ground as shown in the diagram above. If the angle AB makes with the horizontal is α, where $\tan\alpha=\frac{14}{9}$, calculate the coefficient of friction between the ladder and the ground.

9. A uniform bar AB of weight W and length 4a can turn freely in a vertical plane about a hinge A. The bar is supported by a light chain of length 5a, one end of which is fastened to the bar at C, where AC = 3a and the other to a point D vertically above A, where AD = 4a. The bar carries a load of weight 10W, suspended from B. Find the tension in the chain and reaction at the hinge.

10. A uniform rod AB of weight W leans against a smooth vertical wall at A and rests at B on a rough ground that slopes downwards at an angle of 30° to the horizontal.
Given that the coefficient of friction between the rod and the ground is $\frac{3}{4}$ and the rod is in limiting equilibrium, find the:
    (i) normal reactions at A and B.
    (ii) angle which AB makes with the vertical.
    (iii) resultant reaction at B.

Exercise: 9C

1. A uniform rod AB of mass 10 kg rests with its lower end A on a rough horizontal floor,

coefficient of friction μ. One end of a string is attached to end B of the rod and the other end is fixed so that the string is perpendicular to the rod. If the rod rests in equilibrium at an angle of 30° to the horizontal, find the:

(i) tension T in the string.

(ii) normal reaction R and friction force F.

(iii) least possible value of μ for equilibrium to be possible.

2. A uniform ladder of length 2a rests in limiting equilibrium with its lower end on a rough horizontal ground and its upper end against a smooth vertical wall. If the ladder makes an angle of 60° with the ground, find the coefficient of friction.

3. A uniform rod AB of length 4 m and mass 5 kg is freely hinged at A to a wall. It is kept in horizontal position by means of a force P acting in a direction making an angle of 30° with BA. Given that the reaction at A makes an angle α with BA, find the:
   (i) value of P.
   (ii) reaction at A.

4. A uniform ladder of length 5 m and weight 45 N leans against a smooth vertical wall and stands on a smooth horizontal ground, kept in equilibrium by an inelastic string of length 4 m attached to the foot of the ladder and the point of intersection of the wall and ground. Find the reactions at the wall and on the ground.

5. A uniform bar AB of weight 2W and length 5 m is free to turn about a smooth hinge at its upper end A, a horizontal force is applied to the end B so that the bar is in equilibrium with B at a distance 4 m from the vertical through A. Show that the reaction at the hinge is equal to $\frac{2\sqrt{13}}{3}$W.

6. A uniform ladder of mass 30 kg and length 8 m rests against a smooth vertical wall and on a rough horizontal ground, coefficient of friction 0·3. The ladder makes an angle of 40° with the ground. A man of mass 40 kg carrying a box of mass 5 kg climbs the ladder. When the ladder is about to slip:
   (i) calculate the normal reaction at the wall.
   (ii) show that the man has climbed approximately 0·69 m.

7. A uniform beam AB of length 2l rests with end A in contact with a rough horizontal ground. A point C on the beam rests against a smooth support. AC is of length $\frac{3}{2}$l with C higher than A and AC makes an angle of 60° with the horizontal. If the beam is in limiting equilibrium, find the coefficient of friction between the beam and the ground.

8. A uniform beam AB is supported at angle θ to the horizontal by a light string attached to end B, and with end A resting on a rough horizontal ground (angle of friction λ). The beam and the string lie in the same vertical plane and the beam rests in limiting equilibrium with the string at right angles to the beam. Prove that $\tan \lambda = \frac{\sin 2\theta}{3-\cos 2\theta}$.

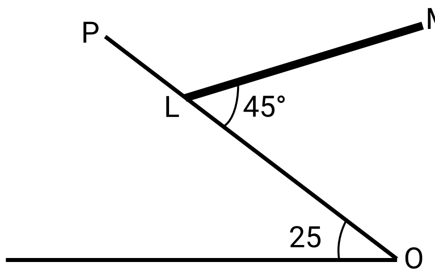
9. A uniform ladder of mass 10 kg and length 4 m rests with one end on a smooth horizontal floor and the other end against a smooth vertical wall. The ladder is kept in equilibrium at an angle $\tan^{-1} 2$ to the horizontal, by a light horizontal string attached to the base of the wall, at a point vertically below the top of the ladder.

   A man of mass 100 kg ascends the ladder. If the string will break when the tension exceeds 490 N, find how far up the ladder the man can go before this occurs. What tension must the string be capable of withstanding if the man is to reach the top of the ladder?

10. A non-uniform ladder AB of length 12 m and mass 30 kg has its centre of gravity at the point of trisection of its length, nearer to A. The ladder rests with end A on a rough horizontal ground(coefficient of friction $\frac{1}{4}$) and end B against a rough vertical wall (coefficient of friction $\frac{1}{5}$). The ladder makes an angle θ with the horizontal such that $\tan \theta=\frac{9}{4}$. A straight horizontal string connects A to a point at the base of the wall vertically below B. A man of mass 90 kg begins to climb the ladder. How far up the ladder can he go without causing tension in the string? What tension must the string be capable of withstanding if the man is to reach the top of the ladder safely?

11. (a) A rod AB, 1 m long has a weight of 20 N at its centre of gravity which is 60cm from A. It rests horizontally with A against a rough vertical wall. A string BCis fastened to the wall at C, 75 cm vertically above A. Find the:
   (i) normal reaction and frictional force at A. If friction is limiting, find the coefficient of friction.
   (ii) tension in the string.

(b) A uniform rod LM of weight W rests with L on a smooth plane PO of inclination 25° as shown below.



What force parallel to PO applied at M will keep the rod in equilibrium? (Give your answer in terms of W)

Answers to Exercises

Exercise: 9A

1. 2. $\frac{5}{24}$ 3. 45° ; $\frac{1}{2}$W ; half the way up, that is, $\frac{1}{2}$l ; all the way

4. (a) 39·8° (b) half way, that is, a distance l from the bottom

5. (a) 0·5196 (b) 40·74 N 7. 18 N; 12 N

8. 49 N; 129.6418 N at 19.1° to the vertical **10.** (i) 117·6 N **(ii)** 102·9 N at 8·2° below AB

Exercise: 9B

1. (i) Wcos θ (ii) Wsin θ at $\tan^{-1}(\cot\theta)$ to the upward vertical (iii) 41•4°

2. $\frac{5}{6}$AB **3.** (i) 49 N (ii) 35.3344 N ; 33.7°

4. 5. $\frac{1}{6}$a from the bottom

8. $\frac{3}{7}$ **9.** 17.5W ; 10·92W at 15·9° below AB

10. (i) $\frac{W}{39}(48+25\sqrt{3})$; $\frac{8W}{39}(4\sqrt{3}+3)$ (ii) 77.9° (iii) $\frac{10W}{39}(4\sqrt{3}+3)$ at 53·1° to the surface or at 11° above BA.

Exercise: 9C

1. **(i)** $\frac{49\sqrt{3}}{2}$ N (ii) 61.25 N ; $\frac{49\sqrt{3}}{4}$ N (iii) $\frac{\sqrt{3}}{5}$

2. $\frac{\sqrt{3}}{6}$ **3.** (i) 49 N **(ii)** 49 N at 30° to AB

4. 30 N ; 45 N **6.** (i) 220·5 N (ii)

7. $\frac{\sqrt{3}}{5}$ **9.** 3.8 m; 514.5 N **10.** 8 m; 126 N

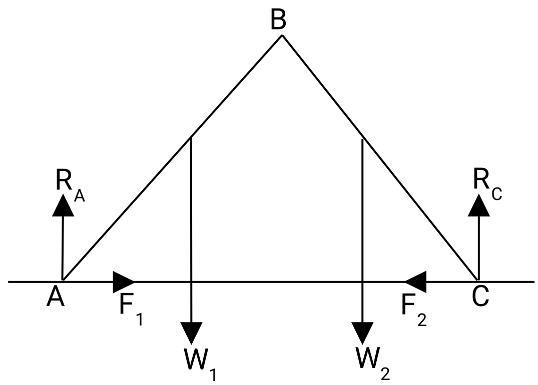
11 (a) (i) 16 N; 8 N ; $\frac{1}{2}$ (ii) 20 N (b) 0.664W

10. JOINTED RODS

For jointed rods, we consider the equilibrium of the system before separation and then the equilibrium of one of the rods after separation.

10.1 Equilibrium before seperation

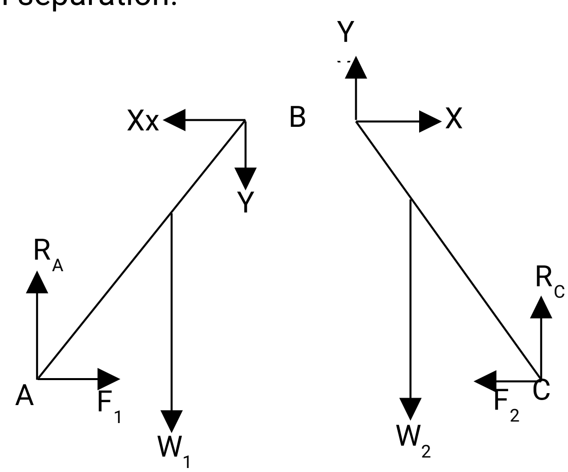
Consider two uniform rods AB and BC of weights W_1 and W_2 respectively resting on a rough horizontal surface and jointed at B.



- By taking moments about A or C one of the reactions is obtained.
- By resolving vertically the other reaction force is also obtained.

10.2 Equilibrium after seperation

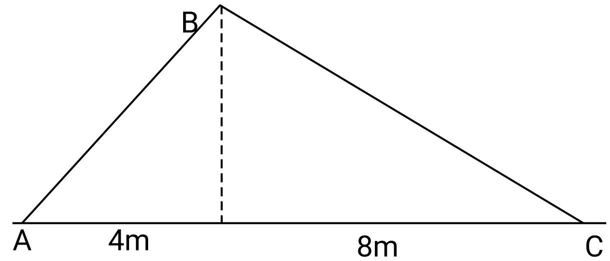
On separation:



The other unknowns are obtained by considering the equilibrium of one of the rods
(preferably the least congested).

Example 1

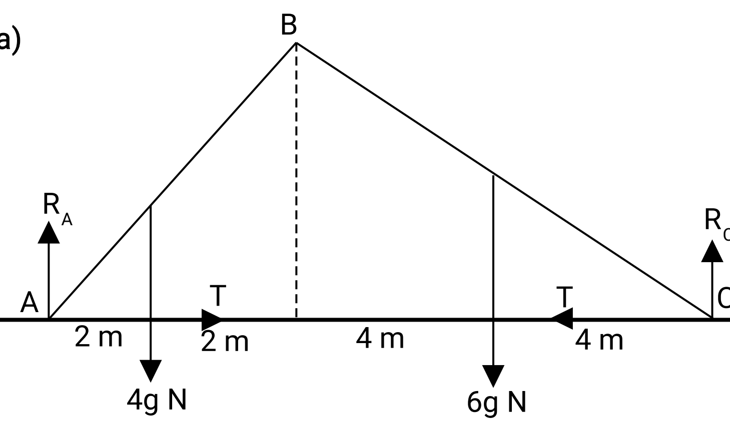
Two uniform rods AB, BC of masses 4 kg and 6 kg are hinged at B and rest in a vertical position on a smooth floor as shown below. A and C are connected by a rope.



(a) Find the reactions between the rods and the floor at A and C when the rope is taut.
(b) If now a body is attached a quarter of the way up CB and the reactions are equal, find the mass of the body.

Solution

(a)



Taking moments about A:

$$R_C \times 12 = 4g \times 2 + 6g \times 8$$

$$R_C = \frac{8 \times 9 \cdot 8 + 48 \times 9 \cdot 8}{12} = \frac{686}{15} = 45 \cdot 7333 \text{ N}$$

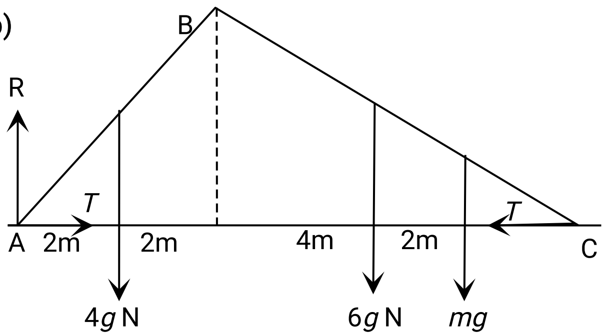
Resolving vertically:

$$(\uparrow): R_A + \frac{686}{15} = 4g + 6g$$

$$R_A = 10 \times 9.8 - \frac{686}{15} = \frac{784}{15}$$

$$\Rightarrow R_A = 52 \cdot 2667 \text{ N}$$

(b)



Resolving vertically:

$$(\uparrow):\ 2R = 4g + 6g + mg$$

$$R = \frac{1}{2}g(10 + m) \ \ldots\ldots\ldots\ldots\ldots\ldots\ (i)$$

Taking moments about A:

$$R\times12 = 4g\times2 + 6g\times8 + mg\times10$$

$$12R = g(56 + 10m)$$

$$\Rightarrow 12\times\frac{1}{2}g(10 + m) = g(56 + 10m)$$

$$m = 1 \text{ kg}$$

Example 2

Two uniform rods AB, AC each of weight W and length 10 cm are smoothly hinged at A. The ends B and C rest on a smooth horizontal plane. An inextensible string joins B and C and the system is kept in equilibrium in a vertical plane with the string taut. Another object of weight 2W climbs the rod AC to a point E such that AE = 8 cm. Given that $\angle BAC = 2\theta$. Determine in terms of W and θ;

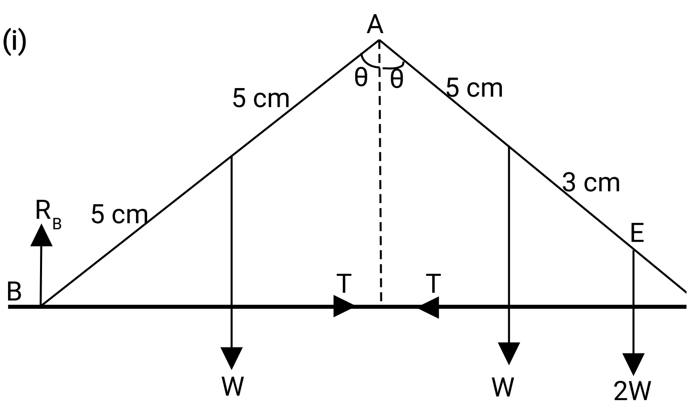
(i) The reaction at ends B and C.

(ii) The tension in the string.

Hence show that the reaction at the hinge A is given by $\frac{W}{10}\sqrt{(49\tan^2\theta+4)}$.

Solution

(i)



Taking moments about B:

$$R_C\times20\sin\theta = W\times5\sin\theta + W\times15\sin\theta + 2W\times18\sin\theta$$

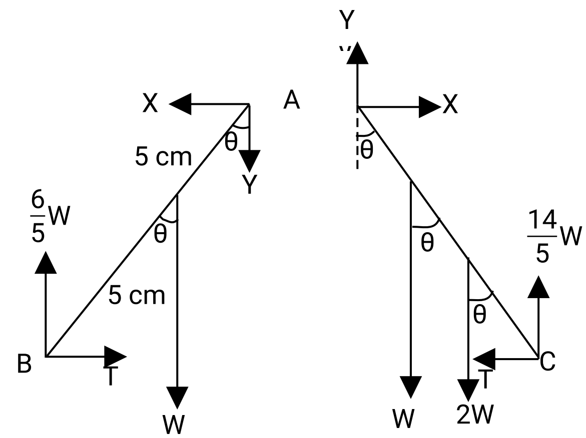
$$20R_C = 56W \Rightarrow R_C = \frac{14}{5}W$$

Resolving vertically:

$$(\uparrow):\quad R_B + R_C = W + W + 2W$$

$$R_B = 4W - \frac{14}{5}W \Rightarrow R_B = \frac{6}{5}W$$

(ii) On separation



Consider equilibrium of rod AB:

Resolving vertically:

$$(\uparrow):\ Y + W = \frac{6}{5}W \Rightarrow Y = \frac{W}{5}$$

Taking moments about B:

$$X\times10\cos\theta = W\times5\sin\theta + Y\times10\sin\theta$$

$$10X = (5W + 10\times\frac{W}{5})\tan\theta$$

$$X = \frac{7}{10}W\tan\theta$$

Resolving horizontally:

$$T = X$$

$$T = \frac{7}{10}W\tan\theta$$

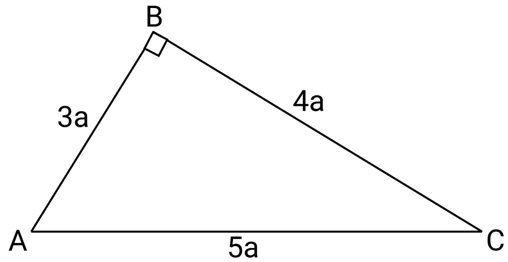
Reaction at A: $R = \sqrt{X^2+Y^2}$

$$R = \sqrt{\left(\frac{7}{10}W\tan\theta\right)^2+\left(\frac{W}{5}\right)^2}$$

$$R = \frac{W}{10}\sqrt{(49\tan^2\theta+4)}$$

Example 3

Two uniform rods AB, BC of length 3a, 4a and weights W, 2W respectively are smoothly joined at B. The rods rest with A and C on a rough horizontal surface, 5a apart as shown below.



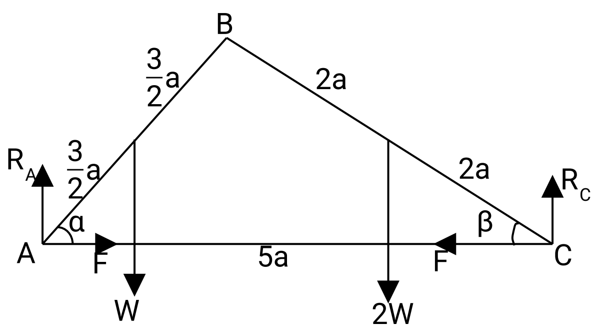
If the coefficient of friction at each point of contact is μ and one rod is about to slip.

(a) Find the value of μ.

(b) Determine the magnitude of the reaction at joint B and angle it makes with the horizontal.

Solution

(a)



Taking moments about A:

$$R_C \times 5a = W\times\frac{3}{2}a\cos\alpha + 2W\times(5a-2a\cos\beta)$$

$$5R_C = \frac{3}{2}W\times\frac{3}{5} + 2W\times\left(5-2\times\frac{4}{5}\right)$$

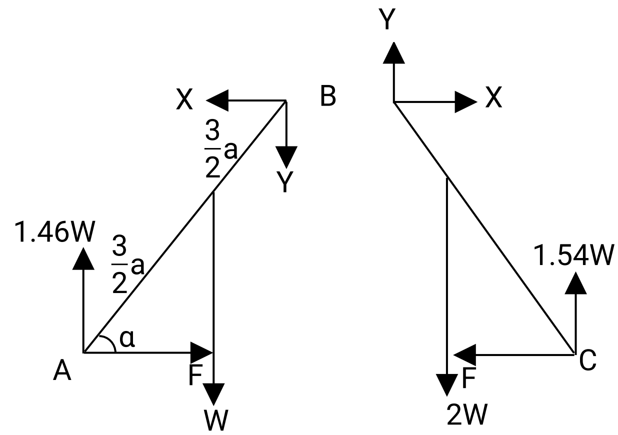
$$R_C = 1.54W$$

Resolving vertically:

$$(\uparrow): R_A + R_C = W + 2W$$

$$R_A = 3W - 1\cdot54W \Rightarrow R_A = 1.46W$$

On separation:



Consider equilibrium of rod AB;

Resolving vertically:

$$(\uparrow): Y = 1\cdot46W - W$$

$$Y = 0.46W$$

Taking moments about A:

$$X\times 3a\sin\alpha = Y\times 3a\cos\alpha + W\times\frac{3}{2}a\cos\alpha$$

$$X\tan\alpha = Y + \frac{1}{2}W$$

$$\frac{4}{3}X = 0\cdot46W + \frac{1}{2}W \Rightarrow X = 0\cdot72W$$

Resolving horizontally:

$$(\rightarrow): F = X$$

$$F = 0.72W$$

At A, $F^A_{max} = \mu R_A = (1.46W)\times\mu = 1.46\mu W$

At C, $F^C_{max} = \mu R_C = (1.54W)\times\mu = 1.54\mu W$

Since $F^A_{max} < F^C_{max}$, the rod AB is about to slip; hence at A, $F = \mu R_A$

$$\mu\times 1\cdot46W = 0.72W$$

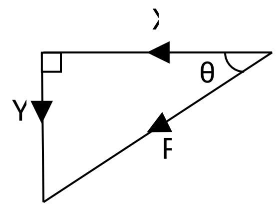
$$\mu = \frac{36}{73} = 0\cdot493$$

(b) Magnitude of reaction at joint B:

$$R = \sqrt{X^2+Y^2}$$

$$= \sqrt{(0\cdot72W)^2+(0\cdot46W)^2}$$

$$= 0.8544W$$



$$\tan\theta = \frac{Y}{X}$$

$$= \frac{0{\cdot}46W}{0{\cdot}72W}$$

$$\theta = 32.6°$$

Example 4

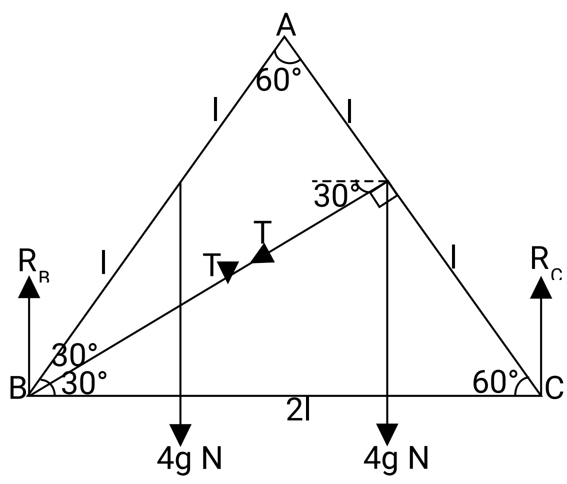
Two uniform metal bars AB and AC are smoothly jointed at A in order to make a frame for a tent. B is joined by a light rope to the midpoint of AC. The bars rest with B and C on a smooth plane. If the ∠BAC=60° and the bars each are of mass 4 kg and length 2l. Determine the:

(a) reactions at B and C.

(b) tension in the string.

(c) magnitude and direction of the reaction at joint A.

Solution



(a) Taking moments about B:

$$R_C \times 2l = 4g \times l\cos 60 + 4g \times 3l\cos 60$$

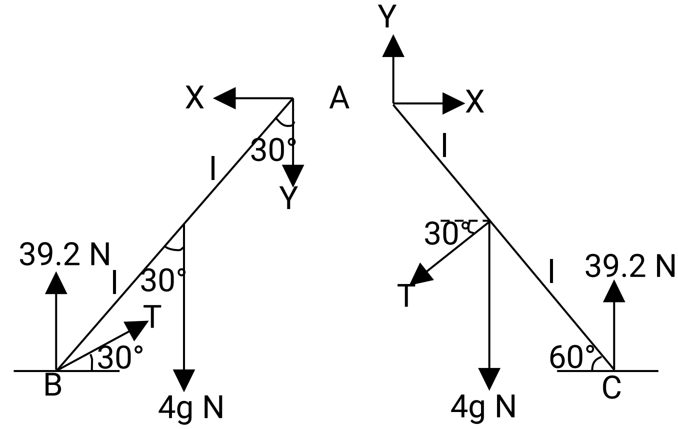
$$R_C = 39.2 \text{ N}$$

Resolving vertically:

$$R_B + R_C = 4g + 4g$$

$$R_B = 8 \times 9.8 - 39.2 \Rightarrow R_B = 39.2 \text{ N}$$

(b) On separation:



Consider equilibrium of rod BA;

Taking moments about B:

$$X \times 2l\cos 30 = Y \times 2l\sin 30 + 4g \times l\sin 30$$

$$X = (Y + 2g)\tan 30$$

$$X = \frac{\sqrt{3}}{3}Y + \frac{2\sqrt{3}}{3}g \text{ .... (i)}$$

Resolving horizontally:

$$(\rightarrow): X = T\cos 30$$

$$T \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{3}(Y+2g)$$

$$T = \frac{2}{3}Y + \frac{4}{3}g \text{ .... (ii)}$$

Resolving vertically:

$$Y + 4g = T\sin 30 + 39.2$$

$$Y = \frac{1}{2}T + 39.2 - 4 \times 9.8$$

$$\Rightarrow Y = \frac{1}{2}T \text{ .... (iii)}$$

From Eqn (ii) and Eqn (iii):

$$T = \frac{1}{2} \times \frac{2}{3}T + \frac{4}{3} \times 9 \cdot 8 \Rightarrow T = 19.6 \text{ N}$$

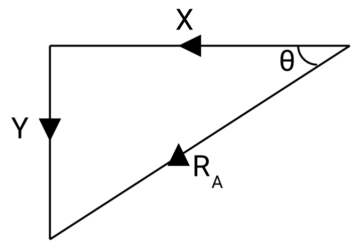
From equation (iii):

$$Y = \frac{1}{2} \times 19.6 \Rightarrow Y = 9.8 \text{ N}$$

From X = Tcos 30

$$X = 19.6 \times \frac{\sqrt{3}}{2} = 9.8 \times \sqrt{3} = \frac{49\sqrt{3}}{5} \text{ N}$$

(c)



$$R_A = \sqrt{X^2+Y^2}$$

$$= \sqrt{\left(\frac{49\sqrt{3}}{5}\right)^2 + 9{\cdot}8^2}$$

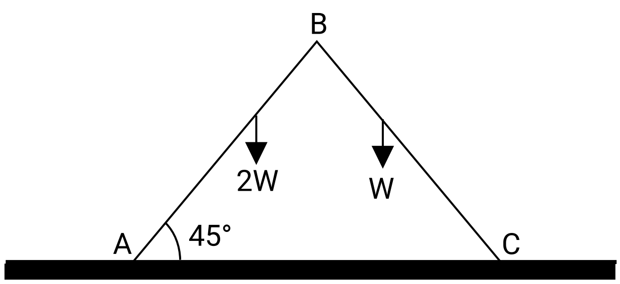
$$R_A = 19 \cdot 6 \text{ N}$$

$$\theta = \tan^{-1}\left(\frac{Y}{X}\right)$$

$$= \tan^{-1}\left(\frac{9{\cdot}8}{9{\cdot}8\sqrt{3}}\right)$$

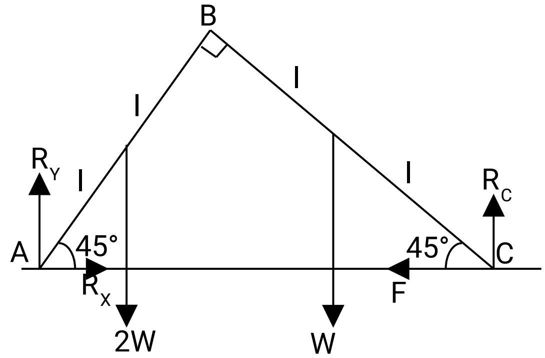
$$\theta = 30°$$

Example 5



The diagram shows two uniform rods AB and BC of equal length, smoothly jointed together at B. End A is freely hinged to a rough horizontal surface and C stands on the same surface coefficient of friction μ. A, B and C all lie in the same vertical plane. AB and BC have weights 2W and W respectively and ⟨BAC=45°. Show that slipping will not occur provided $\mu \geq 0 \cdot 6$. Find the horizontal and vertical components of the reaction at B.

Solution



Taking moments about A:

$$R_C \times 4l\cos 45 = 2W \times l\cos 45 + W \times 3l\cos 45$$

$$R_C = \frac{5}{4}W$$

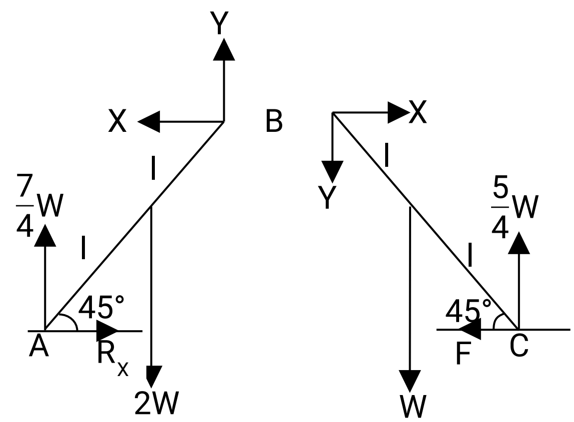
Resolving vertically:

$$(\uparrow): R_Y + R_C = 2W + W$$

$$R_Y = 3W - \frac{5}{4}W$$

$$R_Y = \frac{7}{4}W$$

On separation:



Consider equilibrium of rod BC;

Resolving vertically:

$$(\uparrow): Y = \frac{5}{4}W - W$$

$$Y = \frac{W}{4}$$

Taking moments about C:

$$X \times 2l\sin 45 - Y \times 2l\cos 45 = W \times l\cos 45$$

$$2X - 2 \times \frac{W}{4} = W$$

$$X = \frac{3}{4}W$$

Resolving horizontally:

$$F = \frac{3}{4}W$$

But $F \le F_{max}$ for no slipping

$$\frac{3}{4}W \le \mu \times \frac{5}{4}W$$

$$\mu \ge 0 \cdot 6$$

Components of reaction at B;

Horizontal component: $X = \frac{3}{4}W$

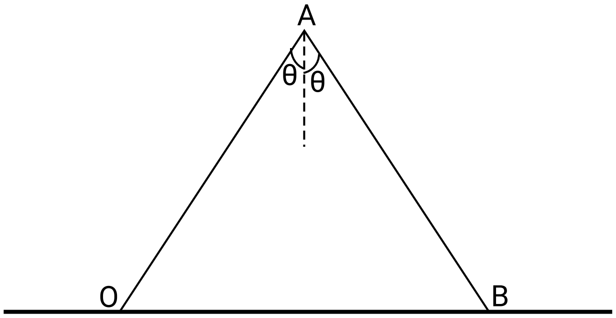
Vertical component: $Y = \frac{W}{4}$

Exercises

Exercise: 10A

1. Uniform rods AB, BC of weights 2W, W and length 3 m and 4 m respectively are smoothly jointed at B. Rod AB is smoothly hinged on a rough horizontal surface at A, end C of rod BC is in limiting equilibrium on this surface when ⟨ABC=90°. Calculate the:
   (i) normal reactions at A and C.
   (ii) coefficient of friction at C.
   (iii) reactions at A and B.

2. The diagram shows a step ladder modeled as two rods OA, AB each of length 2l. The angle each rod makes with the vertical is θ.

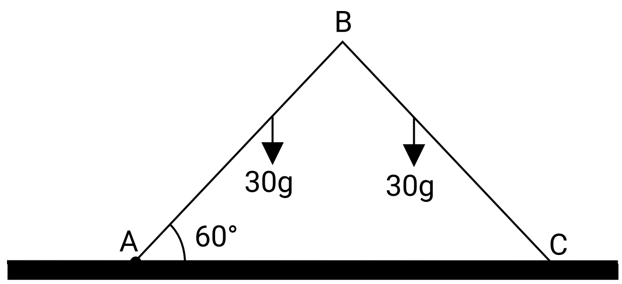


The mass of AB is 4m and that of OA is m. If the system rests in equilibrium on a rough horizontal surface;

   (a) Show that the magnitudes of the normal reactions at O and B are $\frac{7}{4}mg$ and $\frac{13}{4}mg$ respectively.
   (b) Find the reaction at A and also show that the magnitude of the friction force is $\frac{5}{4}mg\tan\theta$.

3. Two uniform rods AB, AC of weights W_1, W_2 and of equal length, are smoothly hinged at A and rest with B, C on a smooth horizontal plane being kept in equilibrium by an inextensible string BC.
   A weight W is suspended from a point in AC at a distance $\frac{3}{4}$AC from A. Prove that the tension in the strin $\frac{1}{4}\left(W_1+W_2+\frac{1}{2}W\right)\tan\left(\frac{1}{2}A\right)$.

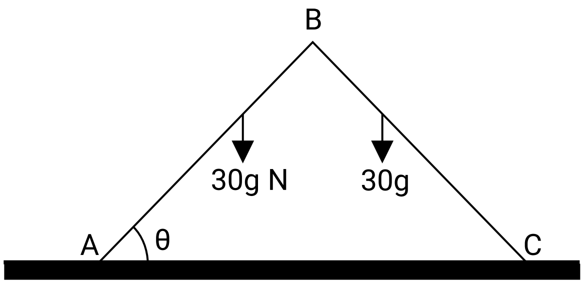
4.



The diagram shows two identical uniform rods AB and BC smoothly jointed together at B. End A is freely hinged to a rough horizontal surface and end C stands on the same surface, coefficient of friction μ. A, B and C all lie in the same vertical plane. Each rod is of mass 30 kg and ⟨BAC=60°. If C is just on the point of slipping, find the:
   (a) reaction at B.
   (b) horizontal and vertical components of the reaction at A.
   (c) value of μ.

5. Two identical uniform heavy rods, AB and BC are smoothly jointed together at B and have ends A and C resting on a rough horizontal ground, coefficient of friction μ.
   The points A, B and C lie in the same vertical plane and ⟨BAC=α. Show that slipping will not occur provided $\mu \ge \frac{1}{2}\cot\alpha$.

6.



The diagram shows two identical uniform rods AB and BC, each of mass 30 kg and lying in the same vertical plane. The rods are smoothly jointed together at B and ends A and C rest on a rough horizontal ground, coefficient of friction μ.

If ⟨BAC=θ, where tan θ = $\frac{3}{2}$ and the rods are just on the point of slipping, find the value of μ and the magnitude of the reaction at B.

7. Two uniform rods each of length 2a and weight W are hinged at A. The ends B and C rest on a smooth horizontal plane. They are joined by an inextensible string and the system is in equilibrium in a vertical plane with the string taut. The string is attached to points B and C. A weight W is suspended from a point D on AC such that AD:DC = 4:1. Given that angle BAC = 2α, find an expression for the tension in the string. Show that the reaction at the hinge is $\frac{W}{10}\sqrt{(36\tan^2\alpha+1)}$.

Exercise: 10B

1. Two uniform rods AB and BC are smoothly hinged at B, with ends A and C resting on a frictionless horizontal surface. The rods are kept in equilibrium in a vertical plane by a light inextensible string attached to the midpoints of AB and BC. The weights of AB and BC are 4W and 7W respectively. AB = BC = 2 m and ⟨ABC= 90°. Find the:
   (i) reactions at A and C.
   (ii) tension in the string.
   (iii) reaction at B.

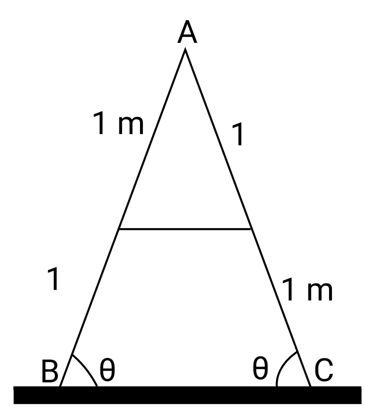
2. Two uniform rods AB and BC of equal length are smoothly jointed together at B. The mass of AB is 6 kg and the mass of BC is 8 kg. The end C is freely hinged to a rough horizontal surface and end A rests on the same surface. The coefficient of friction between rod AB and the surface is μ. The points A, B and C lie in the same vertical plane and ∠ BAC = 30°. If AB is in limiting equilibrium, find the:
   (i) horizontal and vertical components of the reaction at C.
   (ii) reaction at B.
   (iii) value of μ.

3. A pair of stairs can be regarded as a pair of uniform rods AB and BC of weight 2W and W respectively and of the same length 2a, are freely jointed at B. They stand in a vertical plane with their feet A and C on a smooth horizontal floor and joined by a light inextensible string of length 4b, a man of weight W climbs halfway up AB.

   Find in terms of W the reactions at A and C, and the horizontal and vertical components of the reactions at the hinge and its resultant and direction. Find also the tension in the string in terms of α, the angle AB makes with the vertical.

4. Two uniform rods AB and BC are of the same length and have masses 3m and m respectively. They are smoothly jointed at B and stand in a vertical plane with A and C on a rough horizontal plane. The coefficient of friction between the rods and plane is $\frac{2}{3}$. Equilibrium is about to be broken by one of the rods slipping on the plane. Find:
   (a) which rod will slip first and the angle each rod makes with the plane.
   (b) the reaction at the hinge B in magnitude and direction.

5.



A pair of steps can be modeled as a uniform rod AB, of mass 24 kg and length 2 m, freely hinged at A to a uniform rod AC of mass 6 kg and length 2 m. The midpoints of the rods are joined by a light inextensible string. The rods rest in a vertical plane on a smooth horizontal ground, with each rod inclined to the horizontal at an angle $\theta = \tan^{-1}\left(\frac{5}{4}\right)$. Find the tension in the string and the horizontal and vertical components of the force acting on AB at A.

6. Two uniform ladders each of weight W and length 2b are smoothly hinged at their upper ends and stand on a smooth horizontal plane. A weight W is hung on one of the ladders at a distance d, from its lower end and the ladders are prevented from slipping by means of a rope of length 2a attached to their lower ends. Prove that the tension in the string is given by $T = \frac{aW(2b+d)}{4b(4b^2-a^2)^{\frac{1}{2}}}$.

7. Two equal uniform rods AB, BC smoothly jointed at B are in equilibrium with the end C resting on a rough horizontal plane and end A freely pivoted at a point above the plane.

Prove that if α and β are the inclinations of CB and BA to the horizontal, and BC is in limiting equilibrium then $\mu = \frac{2}{\tan\beta + 3\tan\alpha}$ where μ is the coefficient of friction between BC and the plane.

Answers to exercises

Exercise: 10A

1. (i) $\frac{49}{25}W$; $\frac{26}{25}W$ (ii) $\frac{9}{13}$ (iii) $\frac{\sqrt{109}}{5}W$ at 69·8° to the horizontal; $\frac{\sqrt{13}}{5}W$ at 3.2° to the horizontal 2.(a) (b) $\frac{1}{4}mg\sqrt{(25\tan^2\theta+9)}$ at $\tan^{-1}\left(\frac{3}{5}\cot\theta\right)$ to the horizontal 3. 4. (a) $49\sqrt{3}$ N horizontally (b) $49\sqrt{3}$ N ; 294 N (c) $\frac{\sqrt{3}}{6}$ 5. 6. $\frac{1}{3}$; 98 N 7. $T = \frac{3}{5}W\tan\alpha$

Exercise: 10B

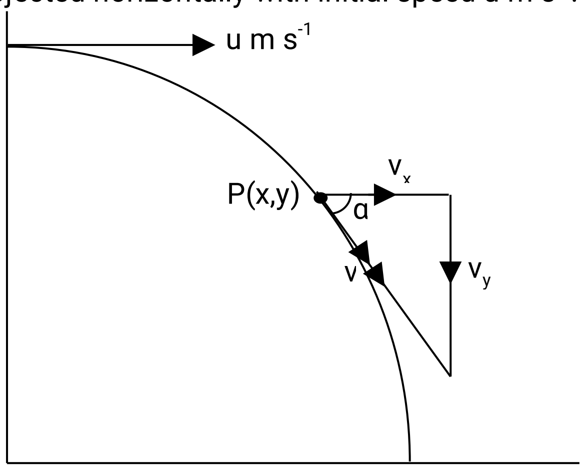
1. (i) $\frac{19}{4}W$; $\frac{25}{4}W$ (ii) $\frac{11}{2}W$ (iii) 5·551W at 7·8° to horizontal
2. (i) 59·41 N ; 73·5 N (ii) 59·611 N at 4·7° to horizontal (iii) 0·9326
3. $R_A = \frac{5}{2}W$; $R_C = \frac{3}{2}W$; $X = \frac{bW}{\sqrt{a^2-b^2}}$; $Y = \frac{W}{2}$; $R_B = \frac{W}{2}\sqrt{\frac{a^2+3b^2}{a^2-b^2}}$; at $\tan^{-1}\left(\frac{\sqrt{a^2-b^2}}{2b}\right)$ to the horizontal ; T = Wtan α 4. (a) Rod BC ; 45° (b) $\frac{\sqrt{5}}{2}mg$ at 26·6° to the horizontal 5. 117·6 N ; 117·6 N ; 44·1 N 6. 7.

11. PROJECTILES

In projectile motion, the horizontal and vertical motion of a projectile can be considered separately. When this is done each kind of motion is linear and equations of linear motion can be applied.

11.1 Horizontal projection

This is when the initial direction of motion of the projectile is horizontal. Consider a particle projected horizontally with initial speed u m s^{-1}.



If after time t, particle passes through a point P(x,y):

From v = u + at

$$v_x = u_x + a_x t,\ a_x = 0$$

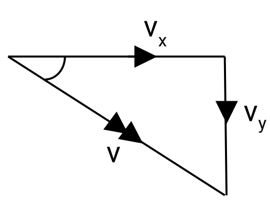
$$v_x = u_x = u$$

$$v_y = u_y + a_y t,\ u_y = 0,\ a_y = g$$

$$v_y = gt$$

The speed of the particle at time t is $v = \sqrt{v_x^2 + v_y^2}$.

The direction α of the particle at time t, is obtained from the velocity of the particle.



$$\tan \alpha = \frac{v_y}{v_x}$$

$$\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

From $s = ut + \frac{1}{2}at^2$

$$s_x = u_x t + \frac{1}{2}a_x t^2,\ a_x = 0,\ u_x = u$$

$$x = ut$$

$$s_y = u_y t + \frac{1}{2}a_y t^2,\ u_y = 0,\ a_y = g$$

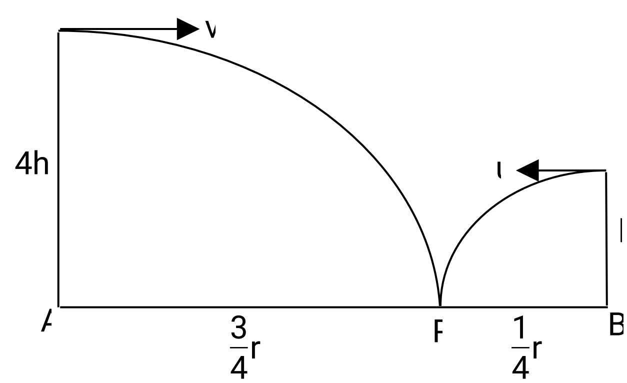
$$y = \frac{1}{2}gt^2$$

Example 1

A and B are two points on level ground. A vertical tower of height 4h has its base at A and a vertical tower of height h has its base at B. When a stone is thrown horizontally with speed v from the top of the taller tower towards the smaller tower, it lands at a point P where AP = $\frac{3}{4}$AB. When a stone is thrown horizontally with speed u from the top of the smaller tower towards the taller tower, it also lands at the point P. Show that 3u = 2v.

Solution

Let AB = r



From taller tower:

From $x = u_x t$

$$\frac{3}{4}r = vt_1 \Rightarrow t_1 = \frac{3r}{4v}$$

From $y = \frac{1}{2}gt^2$

$$4h = \frac{1}{2}gt_1^2 \Rightarrow 4h = \frac{1}{2}g\left(\frac{3r}{4v}\right)^2$$

$$128hv^2 = 9gr^2 \text{………………………………… (i)}$$

From smaller tower:

From $x = u_x t$

$$\frac{1}{4}r = ut_2 \Rightarrow t_2 = \frac{r}{4u}$$

From $y = \frac{1}{2}gt^2$

$$h = \frac{1}{2}gt_2^2 \Rightarrow h = \frac{gr^2}{32u^2}$$

$$32hu^2 = gr^2 \text{ ……………………. (ii)}$$

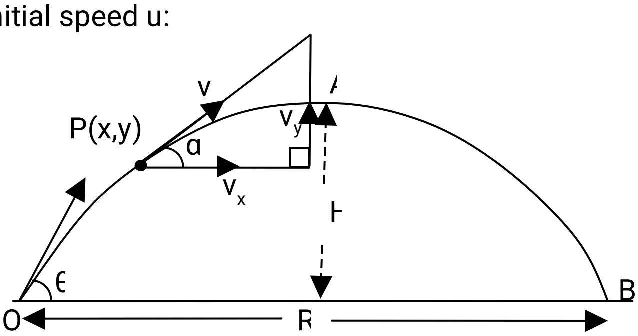
Dividing equation (ii) by equation (i)

$$\frac{u^2}{4v^2} = \frac{1}{9} \Rightarrow 9u^2 = 4v^2$$

$$3u = 2v$$

11.2 Projection from level ground

Consider a particle projected at an angle θ with initial speed u:



If the particle passes through a point P(x,y) after time t.

From v = u + at

$$v_x = u_x = u\cos\theta$$

$$v_y = u_y + a_y t,\ a_y = -g$$

$$v_y = u\sin\theta - gt$$

The vertical component of velocity reduces with time and becomes zero at point A (maximum height) and then increases in the downward direction. Since the vertical component of velocity is zero at A, the motion of the particle is said to be horizontal.

Hence at A, $v_y = 0$

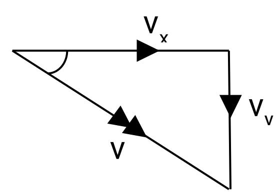
$$0 = u\sin\theta - gt$$

$$t = \frac{u\sin\theta}{g}$$

This is the time taken to reach the maximum height. The speed of the particle after time, t is $v = \sqrt{v_x^2 + v_y^2}$.

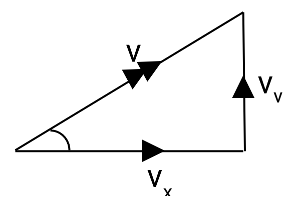
The direction of the particle is the angle between the horizontal and velocity of the particle. The angle is above the horizontal when the particle is ascending and below the horizontal when the particle is descending.

Ascent



$$\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

Descent



$$\beta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

From $s = ut + \frac{1}{2}at^2$

$$s_x = u_x t + \frac{1}{2}a_x t^2,\ a_x = 0$$

$$x = (u\cos\theta)t \text{ ……………………. (i)}$$

$$s_y = u_y t + \frac{1}{2}a_y t^2,\ u_y = u\sin\theta,\ a_y = -g$$

$$y = (u\sin\theta)t - \frac{1}{2}gt^2 \text{……………….. (ii)}$$

11.2.1 Equation of trajectory

This is the equation of the path taken by a projectile.

From equation (i): $x = (u\cos\theta)t \Rightarrow t = \frac{x}{u\cos\theta}$

Substituting in equation (ii)

$$y = (u\sin\theta)\times\frac{x}{u\cos\theta} - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$$

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$

$$y = x\tan\theta - \frac{gx^2\sec^2\theta}{2u^2}$$

$$y = x\tan\theta - \frac{gx^2\left(1+\tan^2\theta\right)}{2u^2}$$

11.2.2 Maximum height (H)

At maximum height $v_y = 0 \Rightarrow u\sin\theta - gt = 0$

$$\Rightarrow t = \frac{u\sin\theta}{g}$$

From $y = (u\sin\theta)t - \frac{1}{2}gt^2$

$$H = u\sin\theta\times\frac{u\sin\theta}{g} - \frac{1}{2}g\left(\frac{u\sin\theta}{g}\right)^2$$

$$H = \frac{u^2\sin^2\theta}{2g}$$

Alternatively; from $v^2 = u^2 + 2as$

$$v_y^2 = u_y^2 - 2gy, \text{ when } y = H, v_y = 0$$

$$0 = (u\sin\theta)^2 - 2gH$$

$$H = \frac{u^2\sin^2\theta}{2g}$$

11.2.3 Time of Flight (T)

This is the time taken by a particle to return to the level of projection.

From $y = (u\sin\theta)t - \frac{1}{2}gt^2$

When the particle returns to the level of projection, $y = 0$

Therefore $0 = (u\sin\theta)t - \frac{1}{2}gt^2$

Either $t = 0$ or $t = \frac{2u\sin\theta}{g}$

Hence time of flight, $T = \frac{2u\sin\theta}{g}$

11.2.4 Range (R)

This is the horizontal displacement covered by a particle to return to the level of projection.

From $x = (u\sin\theta)t$

When $x = R$, $t = T = \frac{2u\sin\theta}{g}$

$$R = (u\cos\theta)\times\frac{2u\sin\theta}{g}$$

$$R = \frac{u^2\sin 2\theta}{g}$$

For any speed of projection, the range R, varies with the angle of projection. The maximum range,

$$R_{max} = \frac{\left(u^2\sin 2\theta\right)_{max}}{g} = \frac{u^2}{g}\times\left(\sin 2\theta\right)_{max}$$

but $\left(\sin 2\theta\right)_{max} = 1$ when $2\theta = 90° \Rightarrow \theta = 45°$

hence $R_{max} = \frac{u^2}{g}$.

Note: If a particle is projected with a certain initial speed, there are always two possible angles of projection for which it hits the plane at the same horizontal displacement from the point of projection. In particular if the projection is on level ground, then the two angles of projection for a given range are complimentary. That is, if the angles are α and β then $\alpha + \beta = 90°$.

Verification:

From $R = \frac{u^2\sin 2\theta}{g} \Rightarrow 2\theta = \sin^{-1}\left(\frac{Rg}{u^2}\right)$

Let $\sin^{-1}\left(\frac{Rg}{u^2}\right) = \phi$, then $2\theta = \phi, 180 - \phi$

$$\therefore\ \theta = \frac{1}{2}\phi, 90 - \frac{1}{2}\phi \Rightarrow \alpha = \frac{1}{2}\phi,\ \beta = 90 - \frac{1}{2}\phi$$

Hence $\alpha + \beta = 90°$.

Example 2

A particle is projected with velocity of 40 m s^{-1} at an angle of 60° to the horizontal. Find the maximum height and range of the particle.

Solution

$u = 40$ m s^{-1}, $\theta = 60°$

$$H = \frac{u^2\sin^2\theta}{2g}$$

$$H = \frac{40^2(\sin 60)^2}{2\times9\cdot8}$$

$$H = 61\cdot224 \text{ m}$$

$$R = \frac{u^2\sin 2\theta}{g}$$

$$R = \frac{40^2\sin 120}{9\cdot8}$$

$$R = 141\cdot392 \text{ m}$$

Example 3

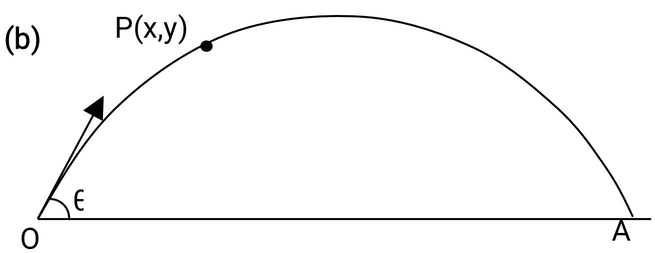
(a) Derive an equation of path of a particle projected from origin O at an angle α to the horizontal with initial speed u m s^{-1}.

(b) A particle projected from a point O on a horizontal ground moves freely under gravity and hits the ground at A. Taking O as the origin, the equation of trajectory of the particle is $60y = 20\sqrt{3}x - x^2$, where x and y are measured in metres. Determine the:
   (i) initial speed and angle of projection.
   (ii) distance OA. (Take g as 10 m s^{-2})

Solution

(a) See introduction.

(b)



(i) $60y = 20\sqrt{3}x - x^2 \Rightarrow y = \frac{\sqrt{3}}{3}x - \frac{x^2}{60}$

From $y = x\tan\theta - \frac{gx^2(1+\tan^2\theta)}{2u^2}$

Comparing coefficients:

$$\tan\theta = \frac{\sqrt{3}}{3} \Rightarrow \theta = 30°$$

$$\frac{g(1+\tan^2\theta)}{2u^2} = \frac{1}{60}$$

$$\frac{10\left(1+\frac{1}{3}\right)}{2u^2} = \frac{1}{60}$$

$$u = 20 \text{ m s}^{-1}$$

(ii) Along OA, y = 0

$$60\times0 = x(20\sqrt{3}-x)$$

Either $x = 0$ or $x = 20\sqrt{3}$

Hence $AB = 20\sqrt{3}$ m

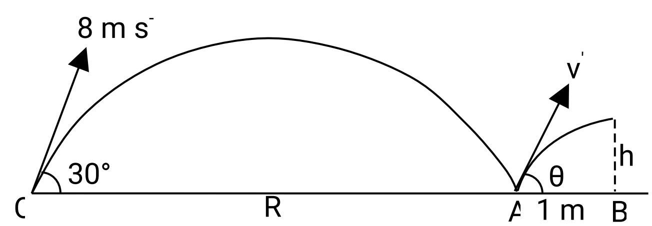
Example 4

A football player projects a ball at a speed of 8 m s^{-1} at an angle of 30° with the ground. The ball strikes the ground at a point which is level with the point of projection. After impact with the ground, the ball bounces and the horizontal component of velocity remains the same but the vertical component is reversed in direction and halved in magnitude. The player running after the ball kicks it again at a point which is a horizontal distance of 1 m from the point where it bounced, so that the ball continues in the same direction. Find the:

(a) horizontal distance between the points of projection and the point at which the ball strikes the ground.

(b) (i) time interval between the ball striking the ground and the player kicking it again.
   (ii) height of the ball above the ground when it is kicked again. (Take *g* = 10 m s^{-2})

Solution



(a) From $y = (u\sin\theta)t - \frac{1}{2}gt^2$

At A, y = 0

$$0 = (u\sin\theta)t - \frac{1}{2}gt^2$$

Either $t = 0$ or $t = \frac{2u\sin\theta}{g}$

$\therefore\ t = T = \frac{2u\sin\theta}{g}$

From $x = (u\cos\theta)t$

When x = R, t = T

$$R = u\cos\theta\times\frac{2u\sin\theta}{g} = \frac{u^2\sin 2\theta}{g}$$

$$R = \frac{8^2\sin 60}{10}$$

$$R = 5\cdot5426 \text{ m}$$

(b) (i) $T = \frac{2u\sin\theta}{g} = \frac{2\times 8\sin 30}{10}$

$T = 0{\cdot}8$ s

$v_x = u\cos\theta$

$= 8\cos 30$

$v_x = 4\sqrt{3}$ m s^{-1}

$v_y = u\sin\theta - gt$

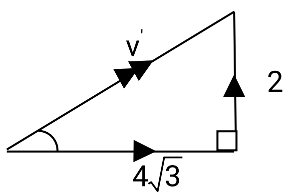
$= 8\sin 30 - 10\times 0{\cdot}8$

$v_y = -4$ m s^{-1}

On bouncing from ground:

$v_x' = v_x = 4\sqrt{3}$ m s^{-1}

$v_y' = 2$ m s^{-1}



$\tan\theta = \frac{2}{4\sqrt{3}}$

$\theta = 16{\cdot}1°$

$v' = \sqrt{2^2 + (4\sqrt{3})^2}$

$= 7{\cdot}211$ m s^{-1}

From $x = u_x t$

$= (8\cos 30)t$

$1 = 4\sqrt{3}t$

$t = 0{\cdot}1443$ s

(ii) From $y = v_y' t - \frac{1}{2}gt^2$

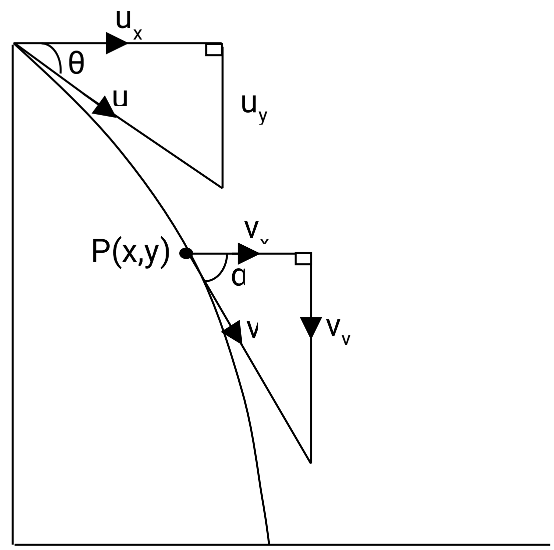
$h = 2\times 0{\cdot}1443 - \frac{1}{2}\times 10\times 0{\cdot}1443^2$

$h = 0{\cdot}1845$ m

11.3 Projection from a height above level ground

11.3.1 Projection at an angle below the horizontal

Consider a particle projected from a height h, above level ground at an angle θ below the horizontal.



If the particle passes through a point P(x,y) after time t:

From $v = u + at$

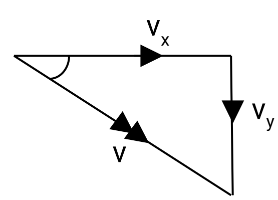
$v_x = u_x = u\cos\theta$

$v_y = u_y + a_y t,\ a_y = g$

$v_y = u\sin\theta + gt$

The speed of the particle after time t is $v = \sqrt{v_x^2 + v_y^2}$.

The direction of the particle at this time:



$\tan\alpha = \frac{v_y}{v_x} \Rightarrow \alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right)$

From $s = ut + \frac{1}{2}at^2$

$x = u_x t + \frac{1}{2}a_x t^2,\ a_x = 0$

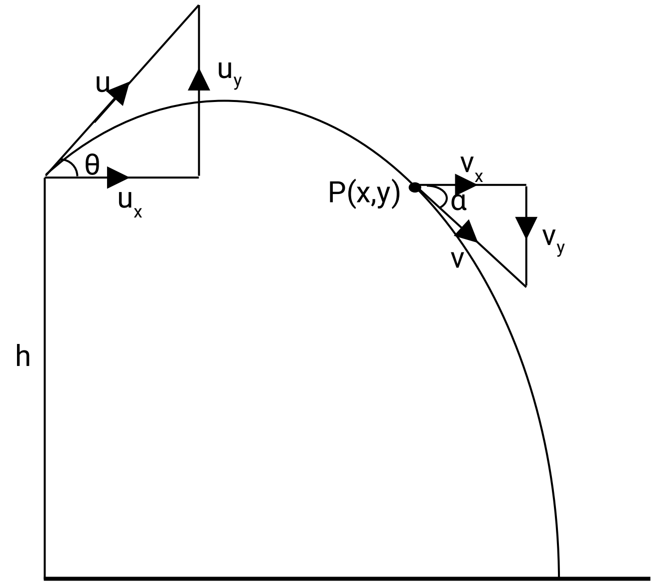
$x = (u\cos\theta)t$

$y = u_y t + \frac{1}{2}a_y t^2,\ a_y = g$

$y = (u\sin\theta)t + \frac{1}{2}gt^2$

11.3.2 Projection at an angle above the horizontal

Consider a particle projected from a height h, above level ground at an angle θ above the horizontal.



Vertical motion goes above the level of projection and then below the level of projection. The sign of g depends on the initial direction of motion. If the initial direction is upwards, we use a = -g and if the initial direction is downwards, we use a = g. Positive vertical displacements are those above the level of projection, zero the at level of projection and negative below the level of projection.

If the particle passes through a point *P*(*x*, *y*) after time *t*:

From v = u + at

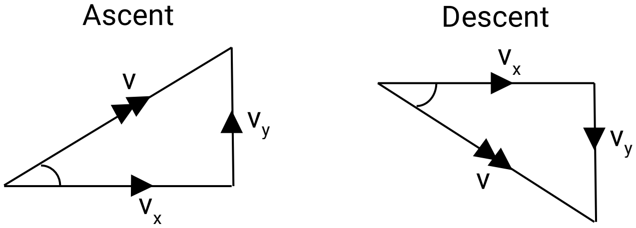
$$v_x = u_x = u\cos\theta$$

$$v_y = u_y - gt$$

$$v_y = u\sin\theta - gt$$

Speed of particle; $v = \sqrt{v_x^2 + v_y^2}$.

Direction of the particle:



$\tan\alpha = \dfrac{v_y}{v_x}$ $\qquad$ $\tan\beta = \dfrac{v_y}{v_x}$

$$\Rightarrow \alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right) \qquad \Rightarrow \beta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

From $s = ut + \frac{1}{2}at^2$

$$x = u_x t + \frac{1}{2}a_x t^2,\ a_x = 0$$

$$x = (u\cos\theta)t$$

$$y = u_y t + \frac{1}{2}a_y t^2,\ a_y = -g$$

$$y = (u\sin\theta)t - \frac{1}{2}gt^2$$

From equation (i): $x = (u\cos\theta)t \Rightarrow t = \dfrac{x}{u\cos\theta}$

Substituting in equation (ii)

$$y = (u\sin\theta)\times\frac{x}{u\cos\theta} - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$$

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$

$$y = x\tan\theta - \frac{gx^2\sec^2\theta}{2u^2}$$

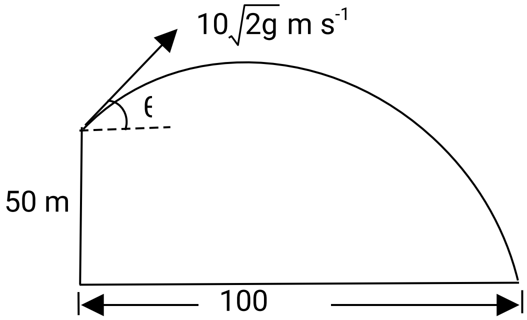
$$y = x\tan\theta - \frac{gx^2(1+\tan^2\theta)}{2u^2}$$

Note: $-h = x\tan\theta - \dfrac{gx^2(1+\tan^2\theta)}{2u^2}$

Example 5

A particle is projected with a speed of $10\sqrt{2g}$ m s^{-1} from the top of a cliff 50 m high. The particle hits the sea at a distance of 100 m from the vertical through the point of projection. Show that there are two possible directions of projection which are perpendicular. Determine the time taken from the point of projection in each case.

Solution



From $y = x\tan\theta - \dfrac{gx^2(1+\tan^2\theta)}{2u^2}$

$$-50 = 100\tan\theta - \frac{g\times 100^2(1+\tan^2\theta)}{2\left(10\sqrt{2g}\right)^2}$$

$$-50 = 100\tan\theta - \frac{10000g\left(1+\tan^2\theta\right)}{2\times 200g}$$

$$-2 = 4\tan\theta-(1+\tan^2\theta)$$

$$\tan^2\theta - 4\tan\theta - 1 = 0$$

$$(\tan\theta - 2)^2 = 5$$

$$\tan\theta = 2 \pm \sqrt{5}$$

If the angles are θ_1 and θ_2 then; $\tan\theta_1 = 2+\sqrt{5}$ and $\tan\theta_2 = 2-\sqrt{5}$

The directions of projection are perpendicular if $\tan\theta_1 \times \tan\theta_2 = -1$

From $\tan\theta_1 \times \tan\theta_2 = \left(2+\sqrt{5}\right)\left(2-\sqrt{5}\right)$

$$= 4 - 5 = -1$$

Hence the two directions are perpendicular.

When $\tan\theta_1 = 2+\sqrt{5}$

$$\theta_1 = 76\cdot 7°$$

From $x = u_x t$

$$100 = (10\sqrt{2\times 9\cdot 8}\cos 76\cdot 7)t_1$$

$$t_1 = 9\cdot 8313\text{ s}$$

When $\tan\theta_2 = 2-\sqrt{5}$

$$\theta_2 = -13\cdot 3°$$

$= 13\cdot 3°$ below the horizontal

From $x = u_x t$

$$100 = \left(10\sqrt{2\times 9\cdot 8}\cos 13\cdot 3\right)t_2$$

$$t_2 = 2\cdot 3209\text{ s}$$

Note: Angle between directions of projection can also be obtained from,

$$\theta_1 - \theta_2 = 76\cdot 7° - -13\cdot 3° = 90°$$

Example 6

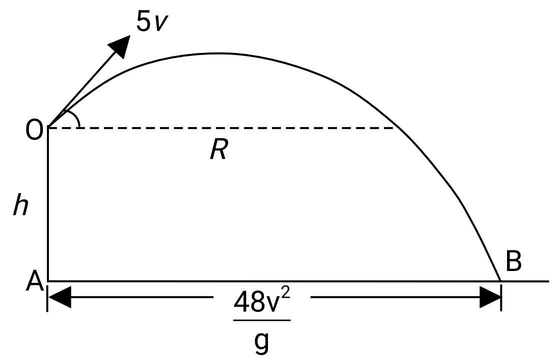
A point O is directly above point A of a horizontal plane. A particle P is projected from O with a speed of 5v at an angle $\cos^{-1}\left(\dfrac{3}{5}\right)$ above the horizontal and hits the plane at point B at a distance $\dfrac{48v^2}{g}$ from A.

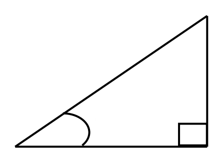
(i) Show that the height of O above A is $\dfrac{64v^2}{g}$.

(ii) Find the distance of P from O when it is directly on level with it.
- A second particle is now projected with a speed of 24w from O at an angle α above the horizontal and also hits the plane at B. Find the equation involving v, w and α.
- Given that the value of α is 45°, find w in terms of v and show that the other value occurs such that $7\tan^2\alpha - 6\tan\alpha - 1 = 0$.

Solution

(i)





From $y = x\tan\theta - \dfrac{gx^2}{2u^2\cos^2\theta}$

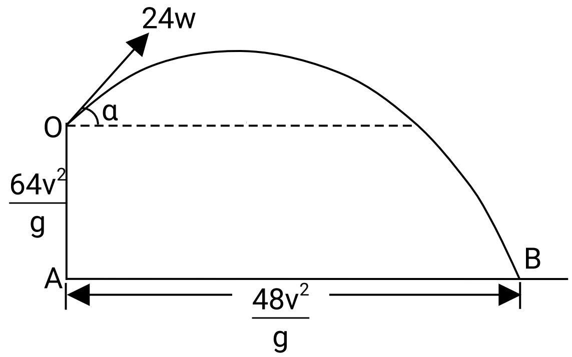
$$-h = \frac{48v^2}{g}\times\frac{4}{3} - \frac{g\times\left(\frac{48v^2}{g}\right)^2}{2\times(5v)^2\times\left(\frac{3}{5}\right)^2}$$

$$-h = \frac{64v^2}{g} - \frac{128v^2}{g} \Rightarrow h = \frac{64v^2}{g}$$

(ii) $$R = \frac{u^2 \sin 2\theta}{g}$$

$$R = \frac{2\times(5v)^2\times \sin\theta\cos\theta}{g}$$

$$R = \frac{50v^2}{g}\times\frac{4}{5}\times\frac{3}{5} = \frac{24v^2}{g}$$



From $y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2u^2}$

$$\frac{-64v^2}{g} = \frac{48v^2}{g}\tan\alpha - g\left(\frac{48v^2}{g}\right)^2 \times \frac{\sec^2\alpha}{2\times(24w)^2}$$

$$\frac{2v^2\sec^2\alpha}{w^2} = 48\tan\alpha + 64$$

$$v^2\sec^2\alpha = 4w^2(6\tan\alpha + 8)$$

When $\alpha = 45°$

$$2v^2 = 4w^2(6 + 8)$$

$$w^2 = \frac{v^2}{28}$$

Substituting for w^2;

$$v^2\sec^2\alpha = 4\times\frac{v^2}{28}(6\tan\alpha + 8)$$

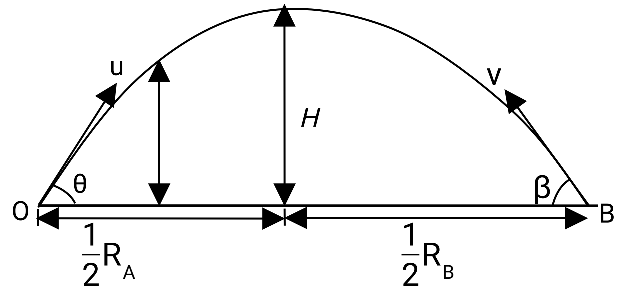
$$7(1 + \tan^2\alpha) = 6\tan\alpha + 8$$

$$7\tan^2\alpha - 6\tan\alpha - 1 = 0$$

Example 7

Two equal particles are projected at the same instant from points A and B on horizontal ground, the first from A with speed u at an angle of elevation α and the second from B with speed v at an angle of elevation β. They collide directly when they are moving horizontally in opposite directions. Find v in terms of u, α and β, and show that $AB = \frac{u^2\sin\alpha\sin(\alpha+\beta)}{g\sin\beta}$.

Solution



$$H = \frac{u^2\sin^2\alpha}{2g} = \frac{v^2\sin^2\beta}{2g}$$

$$v = \frac{u\sin\alpha}{\sin\beta}$$

$$R_A = \frac{u^2\sin 2\alpha}{g}$$

$$R_B = \frac{v^2\sin 2\beta}{g} = \left(\frac{u\sin\alpha}{\sin\beta}\right)^2\times\frac{\sin 2\beta}{g} =$$

$$\frac{2u^2\sin^2\alpha\cos\beta}{g\sin\beta}$$

$$AB = \frac{1}{2}(R_A + R_B)$$

$$= \frac{1}{2}\left(\frac{u^2\sin 2\alpha}{g} + \frac{2u^2\sin^2\alpha\cos\beta}{g\sin\beta}\right)$$

$$= \frac{u^2\sin\alpha}{g}\left(\frac{\sin\beta\cos\alpha + \cos\beta\sin\alpha}{\sin\beta}\right)$$

$$AB = \frac{u^2\sin\alpha\sin(\alpha+\beta)}{g\sin\beta}$$

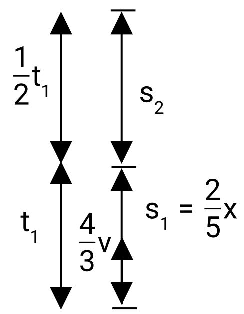
Example 8

(a) A particle is projected vertically upwards from a point O with speed $\frac{4}{3}v$. After it has travelled a distance $\frac{2}{5}x$ above O, on its upward motion, a second particle is projected vertically upwards from the same point with the same initial speed.
Given that the particles collide at a height $\frac{2}{5}x$ above O, x and v being constant, show that:
(i) at maximum height H, $8v^2 = 9gH$.
(ii) when the particles collide $9x = 20H$.

(b) A stone projected at an angle α to the horizontal with speed, $u\ m\ s^{-1}$ just clears a vertical wall 4 m high and 10 m from the point of projection when travelling horizontally. Find the angle of projection.

Solution

(a)



(ii) (i) At maximum height, $v_y = 0$

From $v = u - gt$

$$0 = \frac{4}{3}v - g\left(\frac{3}{2}t_1\right)$$

$$t_1 = \frac{8v}{9g}$$

From $s = ut - \frac{1}{2}gt^2$

$$H = \frac{4}{3}v \times \left(\frac{3}{2} \times \frac{8v}{9g}\right) - \frac{1}{2}g \times \left(\frac{3}{2} \times \frac{8v}{9g}\right)^2$$

$$H = \frac{8v^2}{9g}$$

$$8v^2 = 9gH$$

(iii) For 1st particle:

$t = 2t_1 = \frac{16v}{9g}$ and $s = \frac{2}{5}x$ when the particles collide

$$s = \frac{4}{3}v \times \frac{16v}{9g} - \frac{1}{2}g\left(\frac{16v}{9g}\right)^2$$

$$\frac{2}{5}x = \frac{64v^2}{27g} - \frac{128v^2}{81g}$$

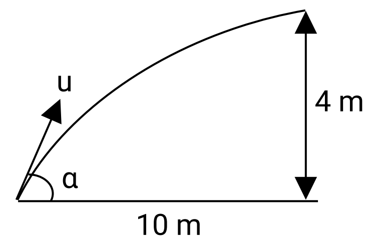
$$81gx = 160v^2$$

From $8v^2 = 9gH \Rightarrow v^2 = \frac{9gH}{8}$

Hence $81gx = 160 \times \frac{9gH}{8}$

$$9x = 20H$$

(b)



A particle travels horizontally when $v_y = 0$, that is, at maximum height.

$H = 4$ m and $\frac{1}{2}R = 10$ m $\Rightarrow R = 20$ m

From $H = \frac{u^2\sin^2\alpha}{2g}$

$$4 = \frac{u^2\sin^2\alpha}{2g}$$

$u^2\sin^2\alpha = 8g$ ……………………… (i)

From $R = \frac{u^2\sin 2\alpha}{g}$

$$20 = \frac{u^2\sin 2\alpha}{g}$$

$10g = u^2\sin\alpha\cos\alpha$ …………………. (ii)

Dividing equation (i) by equation (ii)

$$\tan\alpha = \frac{8}{10}$$

$$\alpha = 38{\cdot}7°$$

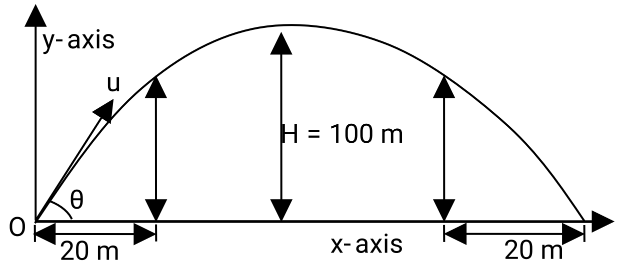
Example 9

The horizontal and vertical components of the initial velocity of a particle projected from point O on a horizontal plane are p and q respectively.

(a) Express the vertical distance y, travelled in terms of horizontal distance x, and the components p and q.

(b) Find the greatest height H, attained and range R, on the horizontal plane through O. Hence show that $y = \frac{4Hx(R-x)}{R^2}$. Given that the particle passes through the point (20,80) and H = 100 m, find the velocity of projection.

Solution



$u_x = p$, $u_y = q$

(a) From $y = u_y t - \frac{1}{2}gt^2$

$$y = qt - \frac{1}{2}gt^2 \quad \text{.......................... (i)}$$

From $x = u_x t$

$$x = pt \quad \text{.................................. (ii)}$$

(b) At maximum height, $v_y = 0$

From $v_y = u_y - gt$

$$v_y = q - gt \Rightarrow 0 = q - gt \Rightarrow t = \frac{q}{g}$$

From equation (i)

$$H = q \times \frac{q}{g} - \frac{1}{2}g\left(\frac{q}{g}\right)^2$$

$$H = \frac{q^2}{2g} \quad \text{.............................. (iii)}$$

When x = R, y = 0

From equation (i)

$$O = qt - \frac{1}{2}gt^2 \Rightarrow t = \frac{2q}{g}$$

From equation (ii)

$$R = p \times \frac{2q}{g}$$

$$R = \frac{2pq}{g} \quad \text{................................ (iv)}$$

From equation (ii) $t = \frac{x}{p}$

Substituting for t in equation (i)

$$y = \frac{q}{p}x - \frac{gx^2}{2p^2} \text{............................(v)}$$

Dividing Eqn (iii) by Eqn (iv)

$$\frac{H}{R} = \frac{q}{4p} \Rightarrow \frac{q}{p} = \frac{4H}{R}$$

Squaring Eqn (iv) and dividing by Eqn (iii)

$$\frac{R^2}{H} = \frac{8p^2}{g} \Rightarrow p^2 = \frac{R^2 g}{8H}$$

Hence from equation (v)

$$y = \frac{4Hx}{R} - \frac{gx^2}{2} \times \frac{8H}{R^2 g}$$

$$y = \frac{4Hx(R-x)}{R^2}$$

The particle passes through (20,80) and H = 100 m

$$80 = \frac{4 \times 100 \times 20(R-20)}{R^2}$$

$$R^2 - 100R + 2000 = 0$$

$$(R-50)^2 = 50^2 - 2000$$

$$R = 50 \pm 10\sqrt{5}$$

Either R = 72.36 m or R = 27.64 m

Since R > 40 m $\Rightarrow$ R = 72.36 m

From equation (iii) : $100 = \frac{q^2}{2 \times 9.8}$

$$\Rightarrow q = 44.272$$

From equation (iv) : $72.36 = \frac{2p \times 44{\cdot}272}{9{\cdot}8}$

$$\Rightarrow p = 8.0088$$

Speed of projection:

$$u = \sqrt{8.0088^2 + 44.272^2} = 44.9 \text{ m s}^{-1}$$

Angle of projection:

From $\tan\theta = \frac{q}{p} \Rightarrow \tan\theta = \frac{44{\cdot}272}{8{\cdot}0088} \Rightarrow \theta = 79.7°$

Example 10

(a) A particle is projected from a point O with initial velocity 3**i** + 4**j**. Find in vector form the position vector of the particle at any time t.

(b) A particle P is projected from a point A with an initial velocity of 60 m s^{-1} at an angle of 30° to the horizontal. At the same instant a particle Q is projected in opposite direction with an initial speed of 50 m s^{-1} from a point at the same level with A and 100 m from A. Given that the particles collide, find the:

(i) angle of projection of Q.

(ii) time when the collision occurs.

Solution

(a) $u = (3i + 4j)$ m s^{-1}

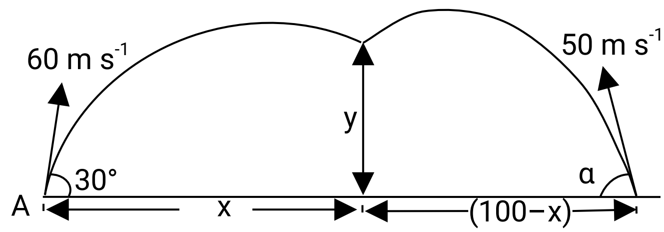
$$x = u_x t$$

$$x = 3t$$

$$y = u_y t - \frac{1}{2}gt^2$$

$$= 4t - \frac{1}{2} \times 9 \cdot 8t^2$$
$$= 4t - 4 \cdot 9t^2$$
$$\mathbf{r}(t) = x\mathbf{i} + y\mathbf{j}$$
$$\mathbf{r}(t) = 3t\mathbf{i} + (4t - 4 \cdot 9t^2)\mathbf{j}$$

(b) $u_P = 60 \text{ m s}^{-1}$, $\theta = 30°$ and $u_Q = 50 \text{ m s}^{-1}$



For P:

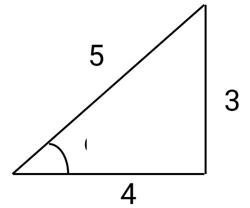
$$x = (60\cos 30)t$$
$$x = \left(30\sqrt{3}\right)t \text{ .... (i)}$$
$$y = (60\sin 30)t - \frac{1}{2}gt^2$$
$$y = 30t - \frac{1}{2}gt^2 \text{ .... (ii)}$$

For Q:

$$(100 - x) = (50\cos\alpha)t \text{ .... (iii)}$$
$$y = (50\sin\alpha)t - \frac{1}{2}gt^2 \text{ .... (iv)}$$

(i) From equation (ii) and equation (iv):

$$(50\sin\alpha)t - \frac{1}{2}gt^2 = 30t - \frac{1}{2}gt^2$$
$$50\sin\alpha = 30 \Rightarrow \sin\alpha = \frac{3}{5}$$



$$\alpha = 36.9°$$

(ii) From equation (i) and equation (iii):

$$100 - 30\sqrt{3}t = (50\cos\alpha)t$$
$$100 = \left(30\sqrt{3} + 50 \times \frac{4}{5}\right)t$$
$$t = 1 \cdot 0874 \text{ s}$$

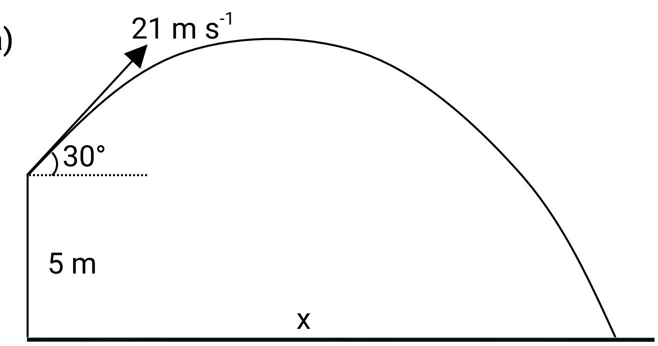
Example 11

(a) A particle is projected at an angle of elevation 30° with a speed of 21 m s^{-1}. If the point of projection is 5 m above the horizontal ground find the horizontal distance that the particle travels before striking the ground.

(b) A boy throws a ball at an initial speed of 40 m s^{-1} at an angle of elevation α. Show that the times of flight corresponding to a horizontal range of 80 m are positive roots of the equation $T^4 - 64T^2 + 256 = 0$. (Take g = 10 m s^{-2})

Solution

(a)



From $y = u_y t - \frac{1}{2}gt^2$

$$-5 = (21\sin 30)t - \frac{1}{2} \times 10t^2$$
$$-5 = \frac{21}{2}t - 5t^2 \Rightarrow t^2 - \frac{21}{10}t - 1 = 0$$
$$\left(t - \frac{21}{20}\right)^2 - \left(\frac{21}{20}\right)^2 - 1 = 0 \Rightarrow t = \frac{21}{20} \pm \frac{29}{20}$$

Either $= \frac{21}{20} - \frac{29}{20} = -\frac{2}{5}$ or $t = \frac{21}{20} + \frac{29}{20} = \frac{5}{2}$

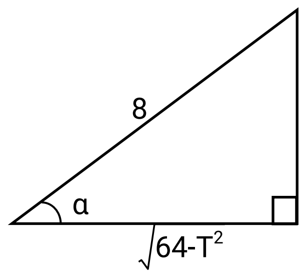
Hence $t = \frac{5}{2}$ s

$$x = u_x t$$
$$x = (21\cos 30) \times \frac{5}{2} = 45 \cdot 466 \text{ m}$$

(b) $u = 40 \text{ m s}^{-1}$, $g = 10 \text{ m s}^{-2}$, $R = 80$ m

$$T = \frac{2u\sin\alpha}{g}$$

$$T = \frac{2\times 40\sin\alpha}{10} \Rightarrow \sin\alpha = \frac{T}{8}$$



$$R = \frac{u^2\sin 2\alpha}{g}$$

$$80 = \frac{40^2\times 2\sin\alpha\cos\alpha}{10} \Rightarrow \frac{1}{4} = \sin\alpha\cos\alpha$$

$$\frac{1}{4} = \frac{T}{8}\times\frac{\sqrt{64-T^2}}{8}$$

$$16^2 = \left(T\sqrt{64-T^2}\right)^2 \Rightarrow 256 = T^2\left(64-T^2\right)$$

$$T^4 - 64T^2 + 256 = 0$$

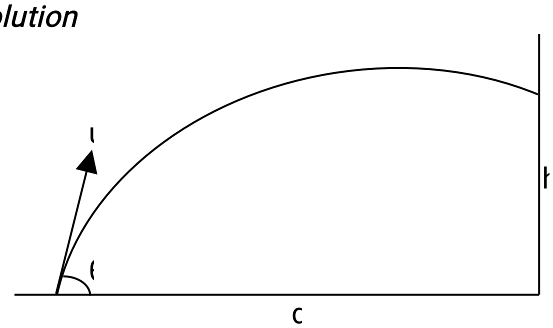
Example 12

A boy throws a stone at a vertical wall a distance d away. Given that R is the maximum range on the horizontal through the point of projection that can be attained by the speed of projection, show that:

(a) the height above the point of projection of highest point on that wall that he can hit is $\left(\frac{R^2-d^2}{2R}\right)$.

(b) in this case the angle of projection is $\tan^{-1}\left(\frac{R}{d}\right)$.

Solution



$$R = \frac{u^2}{g} \Rightarrow u = \sqrt{Rg}$$

From $x = u_x t$

$$d = (u\cos\theta)t \text{ .... (i)}$$

From $y = u_y t - \frac{1}{2}gt^2$

$$y = (u\sin\theta)t - \frac{1}{2}gt^2 \text{ ....(ii)}$$

From equation (i) $t = \frac{d}{u\cos\theta}$

Substituting in equation (ii)

$$y = u\sin\theta\times\frac{d}{u\cos\theta} - \frac{1}{2}g\left(\frac{d}{u\cos\theta}\right)^2$$

$$y = d\tan\theta - \frac{gd^2\left(1+\tan^2\theta\right)}{2u^2}$$

But $u = \sqrt{Rg}$, $y = h$

$$\therefore h = d\tan\theta - \frac{d^2\left(1+\tan^2\theta\right)}{2R}$$

For h_{max}; $\frac{dh}{d\theta} = 0$

$$\frac{dh}{d\theta} = d\sec^2\theta - \frac{d^2}{2R}\left(2\sec^2\theta\tan\theta\right)$$

$$\therefore d\sec^2\theta\left(1-\frac{d}{R}\tan\theta\right) = 0$$

$$1 - \frac{d}{R}\tan\theta = 0 \Rightarrow \tan\theta = \frac{R}{d}$$

(b) $\theta = \tan^{-1}\left(\frac{R}{d}\right)$

(a) $h_{max} = d\times\frac{R}{d} - \frac{d^2}{2R}\left(1+\frac{R^2}{d^2}\right)$

$$h_{max} = R - \frac{d^2+R^2}{2R}$$

$$h_{max} = \left(\frac{R^2-d^2}{2R}\right)$$

Example 13

A stone is thrown from a height of 1•5 m above a level ground with a speed of 10 m s^{-1} and hits a bottle standing on a wall 4 m high and 5 m away.

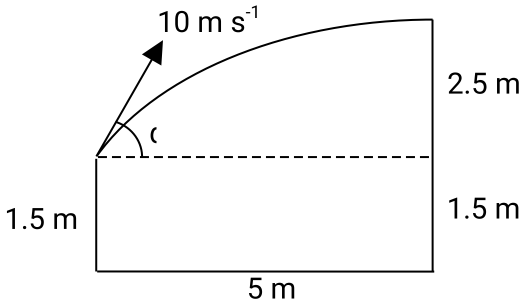
(i) Show that if α is the angle of projection of the stone then $\tan^2\alpha - 4\tan\alpha + 3 = 0$.

(ii) The horizontal component of the stones velocity has to be at least 6 m s^{-1} for the bottle to be knocked off. By solving the above equation, or otherwise, show that α has to be 45° for the bottle to be knocked off.

(iii) If α is 45°, find the direction in which the stone is moving when it hits the bottle.

(iv) If the bottle has a velocity of 3 m s^{-1} after being struck find where it hits the ground. (Use $g = 10\text{ m s}^{-2}$)

Solution



(i) After time t:

$x = (10\cos\alpha)t$ .... (i)

$y = (10\sin\alpha)t - \frac{1}{2}gt^2$ .... (ii)

From equation (i): $t = \frac{x}{10\cos\alpha}$

Substituting in equation (ii)

$$y = 10\sin\alpha \times \frac{x}{10\cos\alpha} - \frac{1}{2}g\left(\frac{x}{10\cos\alpha}\right)^2$$

$$y = x\tan\alpha - \frac{gx^2\left(1+\tan^2\alpha\right)}{200}$$

When x = 5 m, y = 2.5 m

$$2.5 = 5\tan\alpha - \frac{10\times5^2\left(1+\tan^2\alpha\right)}{200}$$

$$2.5 = 5\tan\alpha - \frac{5}{4}\left(1+\tan^2\alpha\right)$$

$$5\tan^2\alpha - 20\tan\alpha + 15 = 0$$

$$\tan^2\alpha - 4\tan\alpha + 3 = 0$$

(ii) $\tan^2\alpha - 4\tan\alpha + 3 = 0$

$$\left(\tan\alpha - 2\right)^2 - 4 + 3 = 0$$

$$\tan\alpha = 2 \pm 1$$

Either $\tan\alpha = 2 \Rightarrow \alpha = 63\cdot4°$ or $\tan\alpha = 1$

$$\Rightarrow \alpha = 45°$$

When $\alpha = 63\cdot4°$;

$$v_x = u_x = 10\cos 63\cdot4 = 4\cdot472\text{ m s}^{-1}$$

When $\alpha = 45°$;

$$v_x = u_x = 10\cos 45 = 5\sqrt{2}\text{ m s}^{-1}$$

$$= 7.071\text{ m s}^{-1}$$

Hence for the bottle to be knocked off $\alpha = 45°$ since in this case $u_x \geq 6\text{ m s}^{-1}$

(iii) $v_x = u_x = 5\sqrt{2}\text{ m s}^{-1}$

From $x = u_x t$

When x = 5 m

$$5 = \left(10\cos 45\right)t \Rightarrow t = \frac{\sqrt{2}}{2}\text{s}$$

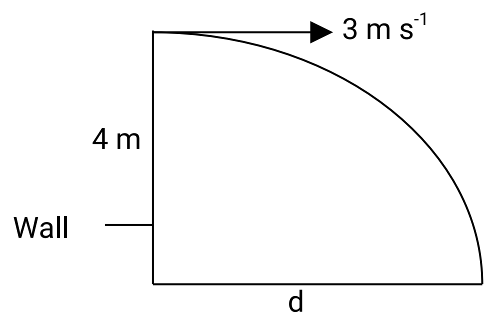
$$v_y = u_y - gt$$

$$v_y = 10\sin 45 - 10t$$

$$\Rightarrow v_y = 5\sqrt{2} - 10\times\frac{\sqrt{2}}{2} = 0$$

Hence it will be moving horizontally

(iv)



From $y = \frac{1}{2}gt^2$

$$4 = \frac{1}{2}\times10t^2 \Rightarrow t = 0\cdot8944\text{ s}$$

From x = ut

$$d = 3\times0\cdot8944 = 2\cdot68\text{ m}$$

Exercises

Exercise: 11A

1. A stone is thrown horizontally with speed u from the edge of a vertical cliff of height h. The stone hits the ground at a point which is a distance d horizontally from the base of the cliff. Show that $2hu^2 = gd^2$.
2. (a) A particle is projected with speed u at an

angle of elevation θ from O on level ground. Show that the equation of its trajectory is

$$y = x\tan\theta - \frac{gx^2(1+\tan^2\theta)}{2u^2}.$$

(b) A particle is projected with speed $10\sqrt{g}$ m s^{-1} from a point O on the ground at an elevation θ. If the particle must clear a vertical tower of height 40 m and at a horizontal distance 40 m from O, prove that $2 \leq \tan\theta \leq 3$.

3. A particle is projected with a speed of 28 m s^{-1} at an angle θ to the horizontal where $\tan\theta = \frac{4}{3}$. Find the:
   (i) speed and direction of motion after 2 s.
   (ii) horizontal and vertical distance travelled in this time.

4. A particle is projected from a level ground towards a vertical pole 4 m high and 30 m away from the point of projection. It just clears the pole in one second, find:
   (a) its initial speed and angle of projection.
   (b) the distance beyond the pole where the particle will fall.

5. A particle is projected from a point O, 24•5 m above a horizontal plane. After 5 seconds it hits the plane at a point whose horizontal distance from O is 100 m. Find the horizontal and vertical components of the initial velocity of the particle and the greatest height reached above the plane.

6. If the horizontal range of a particle projected with velocity u is R, show that the maximum height H attained is given by the equation $16gH^2 - 8u^2H + gR^2 = 0$.

7. (a) A stone thrown at an angle θ to the horizontal takes T seconds in its flight and moves R metres on the horizontal range, show that $2R\tan\theta = gT^2$.
   (b) If the stone in (a) above is now thrown from a point O with an initial speed of 30 m s^{-1} so as to pass through a point 40 m from O horizontally and 10 m above O. Show that there are two possible angles of projection for which this is possible. If these angles are θ_1 and θ_2, show that $\tan(\theta_1+\theta_2) + 4 = 0$.
   (Take g = 10 m s^{-2}).

8. A particle P, projected from a point A on horizontal ground, moves freely under gravity and hits the ground again at B. Taking A as the origin, AB as the x-axis and the upward vertical at A as the y-axis, the equation of path of P is $y = x - \frac{x^2}{40}$, where x and y are measured in metres. Calculate the:
   (i) distance AB.
   (ii) greatest height above AB attained by P.
   (iii) magnitude and direction of the velocity of P at A.
   (iv) time taken by P to reach B from A.

9. A ball is projected from a point A and falls at a point B which is in level with A and at a distance of 160 m from A. The greatest height attained by the ball is 40 m. Find the:
   (a) angle and speed at which the ball is projected.
   (b) time taken for the ball to attain its greatest height. [Use g = 10 m s^{-2}]

10. A particle is projected from a point O on horizontal ground with an initial velocity whose horizontal and vertical components are 3u and 5u m s^{-1} respectively. Find the equation of the trajectory of the particle. Given that it just clears an obstacle 5 m high and 9 m from O, find the value of u and the distance from O of the point at which the particle strikes the ground.

11. A ball is projected from a horizontal ground and has an initial velocity of 20 m s^{-1} at an angle of elevation $\tan^{-1}\left(\frac{7}{24}\right)$. When the ball is travelling horizontally, it strikes a vertical wall. How high above the ground does the impact occur?

12. A stone is thrown from the top of a vertical cliff, 100 m above sea level. The initial velocity of the stone is 13 m s^{-1} at an angle of elevation of $\tan^{-1}\left(\frac{5}{12}\right)$. Find the time taken for the stone to reach the sea and its horizontal distance from the cliff at that time. (Take g = 10 m s^{-2})

Exercise: 11B

1. A particle projected from point A with speed u at an angle α to the horizontal hits the horizontal plane through A at B. Show that if the particle is to be projected from A with the same angle of elevation to the horizontal so as to hit a target at a height h above B, the speed of projection must be

$$\frac{u^2\sin\alpha}{\left(u^2\sin^2\alpha-\frac{1}{2}gh\right)^{\frac{1}{2}}}$$

2. A particle that is projected from a point on level ground and attains a maximum height H, just clears two vertical walls each of height h. Prove that the time taken by the particle to fly between the walls is $\sqrt{\frac{8(H-h)}{g}}$.

3. A particle is projected upwards with a speed of 20 m s^{-1} at an angle θ to the horizontal from a point h metres above a horizontal plane. The particle takes t seconds to hit the plane.
   (i) Show that t is a positive root of the equation $5t^2-20t\sin\theta-h=0$.
   (ii) Given that the particle is moving horizontally after 1 second and the total time of flight is 3 seconds, calculate the value of θ and show that h = 15 m.
   (iii) Determine the horizontal distance travelled and the greatest height attained by the particle above the horizontal plane.
   (iv) Show that the angle at which the particle strikes the plane is $\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$. (Take g = 10 m s^{-2})

4. (a) A particle is projected with a speed of $2\sqrt{70}$ m s^{-1} at an angle of elevation θ. The particle just clears a wall 5 m high and 20 m away from the point of projection. Find the possible values of θ.
   (b) A particle is projected from the top of a cliff H metres above the ground at an angle α above the horizontal. If the particle hits the horizontal plane through the bottom of the cliff, at a distance D from the base of the cliff. Show that the maximum height attained by the particle above the ground is given by

$$\frac{4H(H+D\tan\alpha)+D^2\tan^2\alpha}{4(H+D\tan\alpha)}$$

5. A particle is projected with a speed of $2\sqrt{gh}$ m s^{-1} at an angle α to the horizontal. If it clears a vertical pole of height h which is at a horizontal distance 2h from the point of projection prove that $1 < \tan\alpha < 3$.

6. A particle is projected from a point O on a level ground at an elevation α, and while still rising it passes through a point P with speed v in a direction β to the horizontal. Prove that the time it takes to reach point P is $\frac{v\sin(\alpha-\beta)}{g\cos\alpha}$.

7. (a) A particle is fired with a velocity of 35 m s^{-1} from the edge of a vertical cliff of height 20 m and hits the sea at a distance 50 m from the foot of the cliff. Show that the two possible angles of projection are perpendicular and find the angles.
   (b) Two points P and Q on a horizontal ground are 30 m apart. A ball is thrown from P with a velocity of 20 m s^{-1} at 30° to PQ. A second ball is thrown from Q at 60° to QP at the same time. Find the speed of projection of the second ball if they collide. Also find the time taken for the collision to occur.

8. A boy of height 1·5 m throws a ball at 20 m s^{-1} from the level of his head to land at a point which is 25 m from his feet, on a level ground. Find the:
   (i) two possible angles of projection.
   (ii) velocity of the ball as it passes through a point 1 m above the ground if he used the smaller angle of projection.

9. Two particles P and Q are projected simultaneously from a point O with the same speed but at different angles of elevation and they both pass through a point C which is at a horizontal distance 2h from O and at a height h above the level of O. The particle P is projected at an angle $\tan^{-1}2$ above the horizontal. Show that:
   (i) the speed of projection is $\sqrt{\frac{10gh}{3}}$.
   (ii) Q is projected at an angle $\tan^{-1}\left(\frac{4}{3}\right)$ to the horizontal.
   (iii) the time interval between the times of

arrival of the two particles at C is $(3-\sqrt{5})\sqrt{\left(\frac{2h}{3g}\right)}$.

10. Two particles are projected at the same instant from points A and B at the same level, the first from A towards B with a velocity u at 45° above AB towards B and a second from B towards A at 60° above BA with speed v. If the particles collide when each reaches its maximum height find the ratio $v^2:u^2$ and given that AB = a, prove that $u^2 = ga(3-\sqrt{3})$.

11. Two particles A and B are projected simultaneously, A from the top of a vertical cliff and B from the base. Particle A is projected horizontally with speed $3u\ \text{m s}^{-1}$ and B is projected at an angle θ above the horizontal with speed $5u\ \text{m s}^{-1}$. The height of the cliff is 56 m and the particles collide after 2 seconds. Find the vertical and horizontal distances from the point of collision to the base of the cliff and the values of u and θ.

Exercise: 11C

1. Initially a particle is at an origin O and is projected with a velocity of $a\mathbf{i}\ \text{m s}^{-1}$. After t seconds, the particle is at a point with position vector $(30\mathbf{i}-10\mathbf{j})$ m. Find the values of t and a.

2. A stone is thrown with an initial velocity of $(24\mathbf{i}+10\mathbf{j})\ \text{m s}^{-1}$ from the edge of a vertical cliff. The stone hits the sea at a point level with the base of the cliff and at a distance of 72 m from it. Find the:
   (i) time for which the stone is in air.
   (ii) height of the cliff.
   (iii) maximum height reached by the stone.
   (iv) velocity with which the stone hits the sea.

3. (a) A particle is projected from the top of a cliff 87·5 m high. It takes 5 s to hit the ground. If it covers a horizontal distance of 120 m, calculate the speed and direction of projection.
   (b) A particle is projected vertically upwards from ground level with a speed of $u\ \text{m s}^{-1}$ and clears the top of a pole H metres high in t s and returns to the top of the pole after $\frac{1}{2}t$ s. Show that:
   (i) $12u^2 = 25gH$.
   (ii) the speed at the top of the pole is $\frac{1}{5}u$.

4. A particle is projected from the origin and has an initial velocity of $(7\mathbf{i}+5\mathbf{j})\ \text{m s}^{-1}$. Given that the particle passes through the point P, position vector $(x\mathbf{i}-30\mathbf{j})$ m, find the time taken for this to occur and the value of x. (Take $g = 10\ \text{m s}^{-2}$)

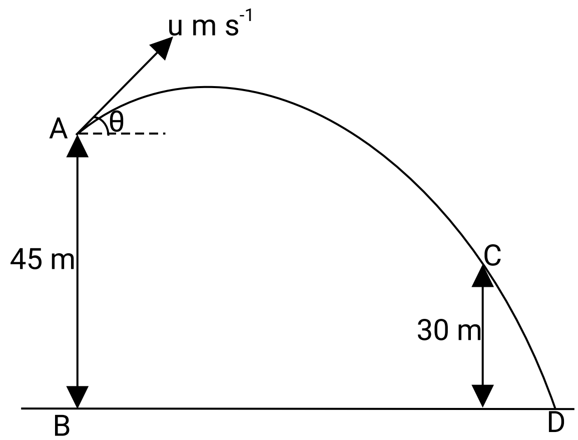
5. A particle projected from origin O with initial velocity $u\mathbf{i} + v\mathbf{j}$ passes through points A and B with position vectors $20(\mathbf{i} + \mathbf{j})$ and $25\mathbf{i}$ respectively. Find the values of u and v. Show that a particle projected from O with velocity $v\mathbf{i} + u\mathbf{j}$ also passes through B. (Use $g = 10\ \text{m s}^{-2}$)

6. (a) Initially a particle is projected with a velocity $\begin{pmatrix}20\\0\end{pmatrix}\ \text{m s}^{-1}$ from a point with position vector $\begin{pmatrix}10\\90\end{pmatrix}$ m. Find the distance of the particle from the origin after 4 seconds.
   (b) A footballer kicks a ball with a velocity of $52\ \text{m s}^{-1}$ at an angle $\tan^{-1}\left(\frac{5}{12}\right)$ to the horizontal. Determine the:
   (i) time for which the ball is at least 12 m above the ground level.
   (ii) maximum height and time taken to reach it.

7. Two particles A and B are projected simultaneously under gravity, A from O on horizontal ground and B from a point 40 m vertically above O. B is projected horizontally with speed $28\ \text{m s}^{-1}$. If the particles hit the ground simultaneously at the same time; determine the:
   (a) time taken for B to reach the ground and horizontal distance travelled.
   (b) magnitude and direction of the velocity with which A is projected. Show that just before hitting the ground, the direction of the motion of A and B differ by approximately 18·4°.

8. A particle is projected from a point A with speed u m s^{-1} at an angle θ, where $\cos\theta = \frac{4}{5}$.

The point B, on horizontal ground, is vertically below A and AB = 45 m. After projection, the particle moves freely under gravity passing through point C, 30 m above the ground and lands at point D, as shown in the diagram below.



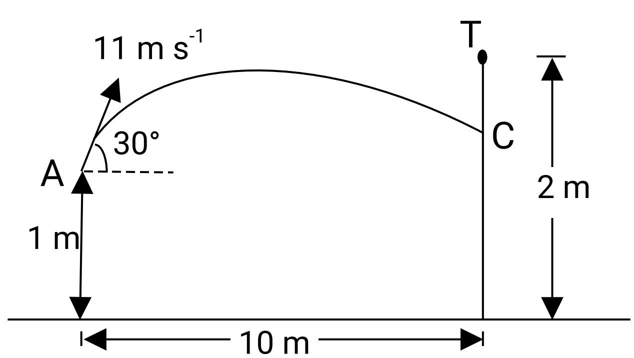
Given that the particle passes through C with speed 24·5 m s^{-1}, find the:

(a) value of speed u.

(b) direction of the particle at C.

(c) distance BD.

9. The aim of a game is to throw a ball B from a point A to hit a target T which is placed at the top of a vertical pole as shown in the diagram below.



The point A is 1 m above the horizontal ground and the height of pole is 2 m. If the ball hits the pole at C, where the pole is at a horizontal distance of 10 m from A and the ball is projected from A with a speed of 11 m s^{-1} at an angle of 30°.

(a) (i) Calculate the time taken by B to move from A to C.

(ii) Find the distance CT.

(b) The ball is thrown again from A with the speed of projection of B increasing to v m s^{-1} and the angle of elevation remaining 30°. Given that B hits T, calculate the value of v.

10. A particle is fired with speed u at an angle θ to the horizontal at a height h, above the horizontal ground. It takes a time T, to reach the horizontal ground. Prove that

$$T = \frac{u\sin\theta}{g}\left\{1 \pm \left(1 + \frac{2gh}{u^2\sin^2\theta}\right)^{\frac{1}{2}}\right\}$$

Answers to exercises

Exercise: 11A

1. 2. (a) (b) 3. (i) 17•032 m s^{-1} ; 9•5° above horizontal (ii) 33•6 m ; 25•2 m

4. (a) 31•292 m s^{-1} ; 16•5° to the horizontal (b) 24•42 m

5. 20 m s^{-1}; 19•6 m s^{-1}; 44•1 m

8. (i) 40 m (ii) 10 m (iii) 20 m s^{-1} at 45° to horizontal (iv) $2\sqrt{2}$ s

9. (a) 45° ; 40 m s^{-1} (b) $2\sqrt{2}$ seconds

10. $y = \frac{5}{3}x - \frac{49x^2}{90u^2}$; 2·1 m s^{-1} ; 13•5 m 11. 1•6 m 12. 5 s ; 60 m

Exercise: 11B

1. 2. 3. (i) (ii) 30° (iii) $30\sqrt{3}$ m; 20 m (iv)

4. (a) 40.6° ; 63•4° 7.(a) 10.9° below horizontal; 79.1° above horizontal (b) 11•547 m s^{-1} ; 1.299 s 8. (i) 15.0° ; 71·5° (ii) 20.24 m s^{-1} at 17•4° below horizontal 9. (i) (ii) (iii)

10. 2:3 11. 36.4 m ; 42 m ; 7 m s^{-1} ; 53.1°

Exercise: 11C

1. $\frac{10}{7}$; 21 2. (i) 3 s (ii) $14 \cdot 1$ m (iii) $19 \cdot 2$ m (above the sea level) ; (iv) $(24i - 19 \cdot 4j)$ m s^{-1}
3. (a) 25 m s^{-1} ; $16 \cdot 3°$ above horizontal (b) (i) (ii)
4. 3 s ; 21 m 5. 5 ; 25
6. (a) $90 \cdot 744$ m (b) (i) $2 \cdot 62$ s (ii) $20 \cdot 4$ m ; $2 \cdot 04$ s
7. (a) $\frac{20}{7}$ s ; 80 m (b) $14\sqrt{5}$ m s^{-1} at $26 \cdot 6°$ to the horizontal
8. (a) $17 \cdot 5$ m s^{-1} (b) $55 \cdot 2°$ below horizontal (c) 60 m
9. (a) (i) $1 \cdot 05$ s (ii) $0 \cdot 626$ m (b) $11 \cdot 699$ m s^{-1} 10.

12. IMPULSE AND MOMENTUM

12.1 The impulse of a force

When a constant force **F** acts for a time t, then the impulse of the force is defined as the vector quantity **F**t. Units of impulse are (N s).

When a variable force **F** acts from time t_1 to time t_2 then the impulse of the force is defined as $\int_{t_1}^{t_2}\mathbf{F}dt$.

Example 1

A particle moving under the action of a force $\mathbf{F} = \mathbf{i}-3t^2\mathbf{j} + 2t\mathbf{k}$ at time t, find the impulse given to the particle in the interval $1 \le t \le 4$.

Solution

$$\text{Impulse} = \int_{t_1}^{t_2}\mathbf{F}dt$$

$$= \int_1^4\left(\mathbf{i}-3t^2\mathbf{j}+2t\mathbf{k}\right)dt$$

$$= \left[t\mathbf{i}-t^3\mathbf{j}+t^2\mathbf{k}\right]_1^4$$

$$= (4\mathbf{i}-64\mathbf{j} + 16\mathbf{k})-(\mathbf{i}-\mathbf{j} + \mathbf{k})$$

$$= 3\mathbf{i}-63\mathbf{j} + 15\mathbf{k}$$

Suppose that at time t, a particle of mass m is moving with velocity **v**, under the action of a force **F**, let **I**, be the impulse given to the particle by **F** in the time interval $t_1 \le t \le t_2$. Whether **F** is variable or constant, $\mathbf{I} = \int_{t_1}^{t_2}\mathbf{F}dt$.

From Newton's second law; $\mathbf{F} = \frac{d(m\mathbf{v})}{dt}$ where m**v** is the momentum of the particle.

$$\mathbf{I} = \int_{t_1}^{t_2}\frac{d(m\mathbf{v})}{dt}\times dt$$

$$\mathbf{I} = [m\mathbf{v}]_{t_1}^{t_2}$$

$$\mathbf{I} = (m\mathbf{v})_{\text{at } t_2}-(m\mathbf{v})_{\text{at } t_1}$$

Hence Impulse = change in momentum

Example 2

A stone of mass 2 kg is falling at a speed of 18 m s^{-1} when it hits the ground. If the stone is brought to rest by the impact, find the impulse exerted by the ground.

Solution

Impulse, I = Change in momentum

$$I = 2\times18 = 36 \text{ N s}$$

Example 3

A ball of mass m kg is moving with a velocity of $(5\mathbf{i}-3\mathbf{j})$ m s^{-1} when it receives an impulse of $(-2\mathbf{i}-4\mathbf{j})$ N s. Immediately after the impulse is applied, the ball's new velocity is $(3\mathbf{i}+\lambda\mathbf{j})$ m s^{-1}. Find the values of λ and m.

Solution

Impulse = Change in momentum

$$-2\mathbf{i}-4\mathbf{j} = m[(3\mathbf{i} + \lambda\mathbf{j})-(5\mathbf{i}-3\mathbf{j})]$$

$$-2\mathbf{i}-4\mathbf{j} = m[-2\mathbf{i} + (\lambda + 3)\mathbf{j}]$$

$$\mathbf{i}:\ -2m = -2 \Rightarrow m = 1$$

$$\mathbf{j}:\ -4 = m(\lambda + 3)$$

$$-4 = 1(\lambda + 3) \Rightarrow \lambda = -7$$

Example 4

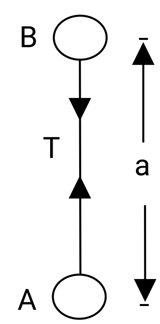
Two equal particles each of mass m, are placed close together and are attached to the ends of an inextensible string of length a and are on a horizontal plane. If one particle is projected vertically upwards with a velocity $\sqrt{2gh}$ where h > a,

(a) show that the other particle will rise a distance $\frac{1}{4}$(h-a) before coming to rest,

(b) determine the loss in kinetic energy when the string becomes taut if a = 20 m, h = 54 m and m = 4•8 kg.

Solution

(a)



From $v^2 = u^2 + 2as$

$$v^2 = \left(\sqrt{2gh}\right)^2-2ga$$

$$v = \sqrt{2g(h-a)}$$

Let v_1 be the velocity of the system due to jerking of the string as a result of impulsive tension in the string.

Impulse, I = Change in momentum

For *B*: $I = m\sqrt{2g(h-a)} - mv_1$............ (i)

For *A*: $I = mv_1$ .... (ii)

From Eqn (i) and Eqn (ii)

$$mv_1 = m\sqrt{2g(h-a)} - mv_1$$

$$v_1 = \frac{1}{2}\sqrt{2g(h-a)}$$

From $v^2 = u^2 - 2gs$

When A reaches greatest height, $v_A = 0$

$$0 = \frac{1}{4}\times 2g(h-a) - 2gs$$

$$s = \frac{1}{4}(h-a)$$

(b) Loss in kinetic energy due to jerking

$$= \frac{1}{2}m\left(\sqrt{2g(h-a)}\right)^2 - \left[2\times\frac{1}{2}m\times\left(\frac{1}{2}\sqrt{2g(h-a)}\right)^2\right]$$

$$= mg(h-a) - \frac{1}{2}mg(h-a)$$

$$= \frac{1}{2}mg(h-a)$$

$$= \frac{1}{2}\times 4\cdot 8\times 9\cdot 8(54-20)$$

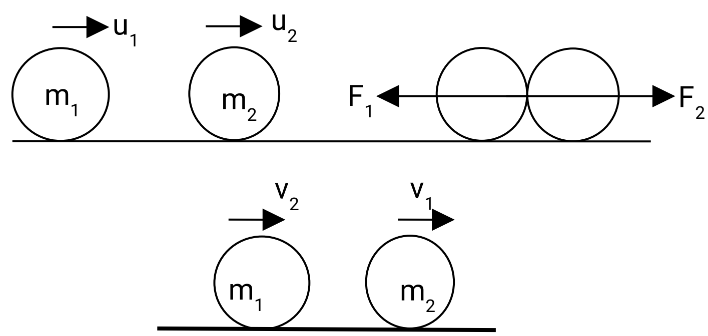
$$= 799\cdot 68 \text{ J}$$

12.2 Momentum

Momentum of a body is a product of its mass and velocity. If a body of mass m kg moves with a velocity **v** then its momentum is mv.

12.2.1 Conservation of momentum

Consider a body of mass m_1 moving with velocity u_1 colliding with a body of mass m_2 moving with velocity u_2. If after collision their velocities are v_1 and v_2 respectively.



If the collision lasts t seconds, from Newton's 2nd law;

$$F_1 = \frac{m_1(v_1 - u_1)}{t}$$

$$F_2 = \frac{m_2(v_2 - u_2)}{t}$$

Where t is the time taken during collision.

From Newton's third law; $F_2 = -F_1$

$$\frac{m_2(v_2 - u_2)}{t} = \frac{-m_1(v_1 - u_1)}{t}$$

$$m_2v_2 - m_2u_2 = -m_1v_1 + m_1u_1$$

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

Momentum before impact = Momentum after impact.

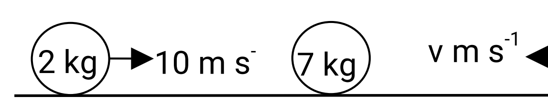
Note:

- In any collision, momentum is conserved.
- In an elastic collision both momentum and kinetic energy are conserved.
- Momentum is a vector quantity.

Example 5

A particle of mass 2 kg moving with speed 10 m s^{-1} collides with a stationary particle of mass 7 kg. Immediately after impact the particles move with the same speed but in opposite directions. Find the loss in kinetic energy during collision.

Solution



By conservation of momentum:

$$2\times10 = -2\times v + 7\times v$$

$$v=4\text{ m s}^{-1}$$

Kinetic energy before collision = $\frac{1}{2}\times2\times10^2 = 100$ J

Kinetic energy after collision

$$= \frac{1}{2}\times2\times4^2 + \frac{1}{2}\times7\times4^2 = 72\text{ J}$$

Loss in kinetic energy = 100−72 = 28 J

Example 6

A ball of mass 0·2 kg strikes a wall when moving horizontally at 12 m s^{-1}. If the ball rebounds horizontally at 8 m s^{-1}, find the impulse exerted by the wall.

Solution

Impulse exerted by wall = Loss in momentum

$$I = m(\mathbf{u}-\mathbf{v})$$

$$= 0\cdot2(12--8)$$

$$= 4\text{ N s}$$

Example 7

A bullet of mass 0·03 kg is fired into a block of wood at a speed of 400 m s^{-1}. If the bullet is brought to rest in 0·02 s, find the average resistance exerted by the wood.

Solution

Let F be the average resistance exerted by the wood.

$$Ft = m(\mathbf{u}-\mathbf{v})$$

$$F\times0\cdot02 = 0\cdot03(400-0)$$

$$F = 600\text{ N}$$

Example 8

A particle of mass 0·5 kg is moving with velocity **u** = 6i−7j when it is given an impulse **I** = 3i + 11j. Find the new velocity of the particle.

Solution

Impulse = Change in momentum

$$m(\mathbf{v}-\mathbf{u}) = \mathbf{I}$$

$$0\cdot5[\mathbf{v}-(6i-7j)] = 3i + 11j$$

$$\mathbf{v} = (6i-7j) + (6i + 22j)$$

$$v=(12i+15j)\text{ m s}^{-1}$$

Example 9

A stone of mass 5 kg is dropped from a height of 10 m above a horizontal ground. Find the impulse exerted by the ground on the stone if it:

(i) comes to rest without rebounding.

(ii) raises to a maximum height of 0·1 m after impact.

Solution

(i) From $v^2 = \sqrt{2gh}$

$$v = \sqrt{2\times9\cdot8\times10}$$

$$=14\text{ m s}^{-1}$$

$$I = m(u-v)$$

$$= 5(14-0)$$

$$= 70\text{ N s}$$

(ii) $v_1=14\text{ m s}^{-1}$, v_2 = ?

From $v^2 = u^2-2gh$

$$O = v_2^2-2\times9\cdot8\times0\cdot1$$

$$v_2=1\cdot4\text{ m s}^{-1}$$

$$I = m(v_1-v_2)$$

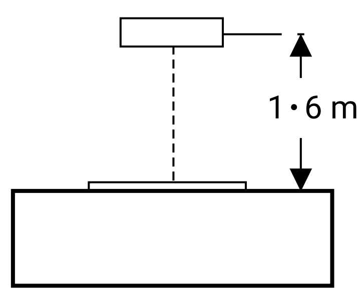
$$= 5(14--1\cdot4)$$

$$= 77\text{ N s}$$

Example 10

A pile-driver of mass 1500 kg falls from a height of 1·6 m and strikes a pile of mass 500 kg. After the blow the pile and driver move on together. If the pile is driven a distance of 0·3 m into the ground, find the speed at which the pile starts to move into the ground and the average resistance of the ground to penetration.

Solution



From $v^2 = u^2 + 2gh$, u = 0

$$v = \sqrt{2\times9\cdot8\times1\cdot6} = 5\cdot6\text{ m s}^{-1}$$

Let v_1 be the Initial speed of pile into ground, by conservation of momentum:

$1500 \times 5 \cdot 6 = (1500 + 500)v_1$

$v_1 = 4 \cdot 2 \text{ m s}^{-1}$

From $v^2 = u^2 + 2as$

$0 = 4 \cdot 2^2 + 2 \times a \times 0 \cdot 3$

$a = -29 \cdot 4 \text{ m s}^{-2}$

$2000g - R = 2000 \times -29 \cdot 4$

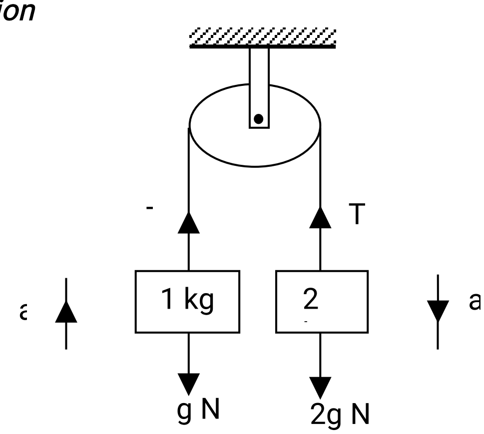
$R = 2000(29 \cdot 4 + 9 \cdot 8)$

$R = 78\,400 \text{ N}$

Example 11

Particles P and Q of masses 1 kg and 2 kg respectively are connected by a light inextensible string which passes over a smooth light pulley. The system is released from rest with the string taut and the hanging parts vertical. After 4•5 s, P picks up a stationary particle of mass 3 kg. Find the velocity of the system immediately after impact and the further time which elapses before the system first comes to instantaneous rest.

Solution



For 1 kg mass:

$T - g = 1 \times a$

$T - g = a$ ……………………………… (i)

For 2 kg mass:

$2g - T = 2a$ ………………………. (ii)

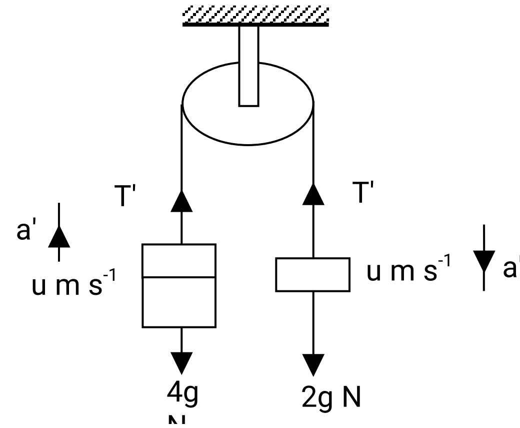
Adding equation (i) and equation (ii):

$g = 3a \Rightarrow a = \frac{1}{3}g$

From $v = u + at$, $u = 0$

$v = \frac{1}{3} \times 9 \cdot 8 \times 4 \cdot 5 \Rightarrow v = 14 \cdot 7 \text{ m s}^{-1}$

When P picks a mass of 3 kg:



Let u be the immediate velocity after impact;

By conservation of momentum during impact:

$(1 \times 14 \cdot 7 + 2 \times 14 \cdot 7) = 4 \times u + 2 \times u$

$u = 7 \cdot 35 \text{ m s}^{-1}$

For combined mass:

$T' - 4g = 4a'$……………………… (iii)

For 2 kg mass:

$2g - T' = 2a'$………………………… (iv)

Adding equation (iii) and equation (iv)

$-2g = 6a'$

$a' = -\frac{1}{3}g \text{ m s}^{-2}$

From $v = u + at$

$v = 7 \cdot 35 - \frac{1}{3}gt$, but $v = 0$

$t = \frac{7 \cdot 35 \times 3}{9 \cdot 8}$

$t = 2 \cdot 25 \text{ s}$

Exercises

Exercise: 12A

1. A particle of mass 3 kg is moving with a speed of 6 m s^{-1} when it receives an impulse of magnitude 24 N s. Find the speed of the particle immediately after the impulse and the change in kinetic energy of the particle if the direction of impulse and original direction of motion are:

(a) the same.
(b) oppose.
(c) perpendicular.

2. A car of mass 12 tonnes is moving along a straight level track at $14\,m\,s^{-1}$ collides with another car of mass 8 tonnes moving in the same direction at $4\,m\,s^{-1}$. After collision, the vehicles move on together. Find their common speed.

3. A pile-driver of mass 1200 kg falls freely from a height of 3·6 m and strikes without rebounding a pile of mass 800 kg. The blow drives the pile a distance of 0·36 m into the ground. Find the average resistance of the ground and the time for which the pile is in motion.

4. A particle A of mass 5 kg is connected to a particle B of mass 2 kg by means of a light inextensible string which passes over a smooth fixed pulley. The system is released from rest with both parts of the string taut and vertical. After 3·5 s particle A hits the ground without rebounding. Assuming that particle B does not hit the pulley, find the further time which elapses before the string is again taut. Find the speed with which A leaves the ground.

5. A particle of mass m_1 moving with velocity u_1 collides head-on with a particle of mass m_2 moving with velocity u_2. If the particles form a composite particle on collision, find the speed of this particle. Show that the collision leads to a loss in kinetic energy equal $\frac{1}{2}\left(\frac{m_1m_2}{m_1+m_2}\right)(u_1+u_2)^2$.

6. A bullet of mass 200 g is fired into a fixed block of wood with a velocity of $300\,m\,s^{-1}$ and is brought to rest in 2 s. Find the resistance exerted by the wood and the distance it penetrates.

7. Two particles A and B of masses 2m and 3m respectively are attached to the ends of a light inextensible string of length c and are placed close together on a horizontal table. The particle A is projected vertically upwards with speed $\sqrt{(6gc)}$.
   (i) Show that, at the instant immediately after the string tightens B is moving with velocity $\frac{4}{5}\sqrt{(gc)}$.
   (ii) State the impulse of the tension in the string.
   (iii) Find the height to which A rises above the table before it comes to instantaneous rest.
   (iv) Calculate the loss in kinetic energy due to tightening of the string.

8. Two particles of masses 4 kg and 3 kg are lying on a smooth horizontal table and are connected by a slack string. The first particle is projected along the table with a velocity of $21\,m\,s^{-1}$ in a direction away from the second particle. Find the velocity of each particle after the string has become taut and also find the difference between the kinetic energies of the system when the string is slack and when it is taut. If the second particle is attached to a third particle of unknown mass by another slack string, and if the velocity of the system after both strings have become taut is $8\,m\,s^{-1}$, find the magnitude of the unknown mass.

9. A particle of mass 2 kg, initially moving with constant velocity $(3i+11j)\,m\,s^{-1}$, has a constant force $(5i+2j)$ N applied to it for 3 s. Find the magnitude of the impulse and speed of the particle after 3 seconds.

10. A bullet of mass 50 g fired horizontally at $500\,m\,s^{-1}$ passes through a wooden block of thickness 0·05 m. The bullet leaves the block at $50\,m\,s^{-1}$. If the block offers uniform resistance, calculate the:
   (a) time taken by the bullet to pass through the block.
   (b) resistance of the block.
   (c) loss in kinetic energy of the bullet.

Exercise: 12B

1. A particle P of mass 4 kg is connected to a particle Q of mass 3 kg by means of a light inextensible string which passes over a smooth fixed pulley. The system is released from rest with both strings taut and vertical. After 3·5 seconds particle P hits the ground without rebounding. If particle Q does not hit the pulley find the:
   (a) further time which elapses before the string is again taut.

(b) speed with which P leaves the ground.

2. A force $F = (4+t^2)i + 2tj$ acts on a particle of mass 0·5 kg from time t = 0 to t = 2 s. Find the:
   (a) power developed when t = 1 second.
   (b) impulse of the force during the interval t = 0 to t = 2 s.

3. P and Q are two particles of mass 4 kg and 8 kg respectively lying in contact on a smooth horizontal table, and connected by a string 3 m long. Q is 7 m from the smooth edge of the table and is connected by a taut string passing over the edge to a particle R of mass 4 kg hanging freely. If the system is released from rest, find the:
   (a) speed with which P begins to move.
   (b) new acceleration of the system after P is in motion.
   (c) tensions in the strings when they are both taut.

4. A particle of mass 0·4 kg is moving so that its position vector **r** metres at time t is given by $\mathbf{r} = (t^2+4t)i + (3t-t^3)j$.
   (a) Calculate the speed of the particle when t = 3.
   (b) When t = 3, the particle is given an impulse $(8i-12j)$ N s. Determine the velocity of the particle immediately after impulse.

5. A hammer of mass 6 kg moving vertically downwards with a speed of 36 km h^{-1} strikes the top of a stationary vertical post of mass 4 kg without rebounding. If the two move together for 1 second before coming to rest, find the average resistance to motion.

6. The net force acting on a particle of mass 10 kg increases uniformly from 56 N to 84 N in the first 5 seconds of its motion. During the next 10 seconds it remains constant at 84 N and decreases uniformly to 0 N in the last 5 seconds. Find the velocity acquired by the particle at the end of the 20 seconds given it was initially at rest.

7. A bullet of mass 50 g, moving horizontally, strikes a stationary target at 486 m s^{-1} and becomes embedded in it. The target is on a smooth horizontal surface and is of mass 4 kg. Calculate the:
   (i) speed at which the target and embedded bullet move.
   (ii) impulse imparted to the target by the bullet.
   (iii) kinetic energy lost in the impact.

8. A pile driver of mass 6 tonnes falls from a height of 4 metres onto a pile of mass 2 tonnes. If the average resistance of the ground is 2×10^3 kN, find the distance in meters the pile penetrates.

9. Two particles P and Q, having masses 3m and m respectively are connected by a light inelastic string and are free to move on a smooth horizontal table. P is initially at rest and Q is projected with speed u in the direction PQ. The diagram below shows the system just before the string is taut. Find the speed of the particles when the string first becomes taut, the impulse and the loss in kinetic energy due to the impact.

10. A brick of mass 2 kg falls freely from rest down a well. After falling 98 m from rest, it strikes the surface of water. It then continues to descend through the water accelerating at $\frac{1}{3}g$ m s^{-2} until it reaches the bottom of the well, 37·5 m below the surface of the water. Given that the impulse of the water on the brick would completely destroy the momentum of a body of mass 0·5 kg which has fallen 32 m from rest. Calculate the:
   (a) time the brick takes from rest to the bottom of the well.
   (b) momentum of the brick when it reaches the bottom of the well.

Answers to exercises

Exercise: 12A

1. (a) 14 m s^{-1} ; 240 J gain (b) 2 m s^{-1}; 48 J loss (c) 10 m s^{-1} ; 96 J gain

2. 10 m s^{-1} 3. 90 160 N ; $\frac{1}{7}$ s 4. 3 s;

4•2 m s^{-1} 5. $\left(\frac{m_2u_2-m_1u_1}{m_1+m_2}\right)$ 6. 30 N; 300 m

7. (i) (ii) $\frac{12}{5}m\sqrt{gc}$ (iii) $\frac{33}{25}c$ (iv) $\frac{12}{5}mgc$ 8. 12 m s^{-1} ; 378 J ; 3•5 kg

9. 16•155 N s ; 17•5 m s^{-1} 10. (a) $1{\cdot}82\times10^{-4}$ s (b) 123 750 N (c) 6187•5 J

3. (a) 3·32 m s^{-1} (b) 2·45 m s^{-2} (c) PQ:9·8 N ; QR:29·4 N

4. (a) 26 m s^{-1} (b) $(30i-54j)$ m s^{-1} 5. 158 N 6. 140 m s^{-1}

7 (i) 6 m s^{-1} (ii) 24 N s (iii) 5832 J 8. 0·0918 m

9. $\frac{1}{4}u$; $\frac{3}{4}mu$; $\frac{3}{8}mu^2$ 10. (a) 5·4 s (b) 81·4 kg m s^{-1}

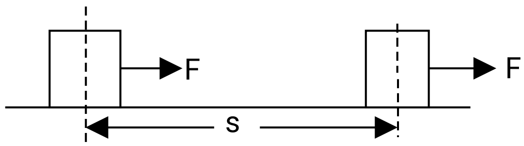
Exercise: 12B

1. (a) 1 s (b) 2·1 m s^{-1} 2. (a) $47\frac{1}{3}W$ (b) $\left(\frac{32}{3}i+4j\right)$ N s

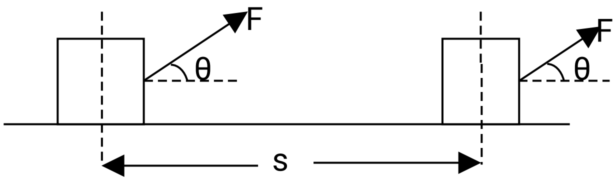
13. WORK, ENERGY AND POWER

13.1 Work done by a constant force

If a constant force F moves through a distance s, in the direction of the force then the work done W = F×s.



Consider the force acting at an angle θ to the direction of s.



Work done = (Fcos θ)×s

$$W = F \times s\cos\theta$$
$$W = \mathbf{F} \bullet \mathbf{s}$$

Hence work done is a dot product of force and displacement. A force acting normal to the direction of motion does no work.

Example 1

A force of magnitude 20 N acting in the direction of 3**i**−4**j**, moves a particle from A(-1,5) to B(7,10). Calculate the work done by the force.

Solution

$$\vec{AB} = (7\mathbf{i} + 10\mathbf{j}) - (-\mathbf{i} + 5\mathbf{j}) = 8\mathbf{i} + 5\mathbf{j}$$

$$\mathbf{F} = |\mathbf{F}|\hat{F} = 20\times\frac{1}{5}(3\mathbf{i} - 4\mathbf{j}) = 12\mathbf{i} - 16\mathbf{j}$$

$$W = \mathbf{F}\bullet\vec{AB}$$

$$= (12\mathbf{i} - 16\mathbf{j})\bullet(8\mathbf{i} + 5\mathbf{j})$$

$$= 12\times8 + -16\times5 = 16 \text{ J}$$

13.1.1 Work done against a force

If a force acts in a direction opposite to that in which displacement is covered, then work is done against such a force.

Work done against a force = magnitude of force × distance moved in opposite direction.

13.1.2 Work done against gravity

When a body of mass m kg ascends a vertical distance h, the work done against gravity, W = mgh.

13.1.3 Work done against friction

If an object moves along a rough surface it does work against friction.

Work done against friction = F×s.

F– magnitude of friction force.
s– distance moved in opposite direction.

13.2 Energy

This is the measure of a body's capacity to do work.

13.2.1 Kinetic energy (K. E)

This is the energy possessed by a body by virtue of its motion. Consider a constant force F acting on a body of mass m, initially at rest on a smooth horizontal surface and after moving a distance s along the surface it has a speed v.

Work done on body = F×s

From $v^2 = u^2 + 2as,\ u = 0$

$$a = \frac{v^2}{2s}$$

$$F = ma = \frac{mv^2}{2s}$$

Work done on the body $= \frac{mv^2}{2s}\times s = \frac{1}{2}mv^2$

This is the kinetic energy of a body of mass m moving with velocity v.

13.2.2 Potential Energy (P. E)

This is the energy possessed by a body by virtue of its position. When a body of mass m kg is raised through a vertical distance h, the work done on the body is mgh. This is stored as the potential energy of the body. Hence potential energy = mgh.

Principle of conservation of energy:
If a body moves without doing work against friction and gravity is the only external force affecting motion of the body, the sum of the body's kinetic energy and potential energy is constant. This is called the mechanical energy of the body.

Mechanical energy = K.E + P. E

Example 2

A particle of mass 2 kg is projected vertically with a speed of $10\ \text{m s}^{-1}$. Using the principle of conservation of energy, find its speed after it has moved a vertical distance of 4 metres downwards.

Solution

Loss in P.E = gain in K.E

$$mgh = \frac{1}{2}m(v^2-u^2)$$

$$v = \sqrt{u^2+2gh}$$

$$v = \sqrt{10^2+2\times9{\cdot}8\times4} = 13{\cdot}36\ \text{m s}^{-1}$$

Example 3

A mass is suspended from a fixed point O by an elastic string of natural length a, and when the mass is hanging freely, the length of the string is $\frac{5}{3}a$.

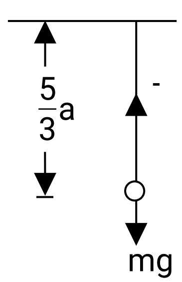
(a) Show by the principle of conservation of energy, that if the mass is allowed to fall from rest at O the maximum length of the string in the subsequent motion is 3a.

(b) Find the speed with which the mass is moving when it is a distance 2a from O.

(c) Determine the acceleration of the mass when the string is at its maximum length.

Solution

(a)



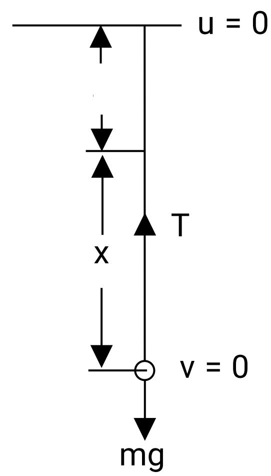
In equilibrium: T = mg

From Hooke's law

$$T = \frac{\lambda x}{l_0},\ x = \frac{5}{3}a - a = \frac{2}{3}a,\ l_0 = a.$$

$$T = \frac{\lambda\left(\frac{2}{3}a\right)}{a} = \frac{2}{3}\lambda$$

$$\therefore\ \frac{2}{3}\lambda = mg \Rightarrow \lambda = \frac{3}{2}mg$$



By conservation of energy:
Loss in potential energy = Elastic P.E stored

$$mg(a + x) = \frac{\lambda x^2}{2a}$$

$$mg(a + x) = \frac{3}{2}mg\times\frac{x^2}{2a}$$

$$4a(x + a) = 3x^2$$

$$3x^2 - 4ax = 4a^2 \Rightarrow x^2 - \frac{4}{3}ax = \frac{4}{3}a^2$$

$$\left(x-\frac{2}{3}a\right)^2 = \left(\frac{2}{3}a\right)^2 + \frac{4}{3}a^2 \Rightarrow x = \frac{2}{3}a \pm \frac{4}{3}a$$

Either $\quad x = \frac{2}{3}a - \frac{4}{3}a = -\frac{2}{3}a$

$$x = \frac{2}{3}a + \frac{4}{3}a = 2a$$

$\therefore\ x = 2a$

Hence $l_{max} = a + 2a = 3a$

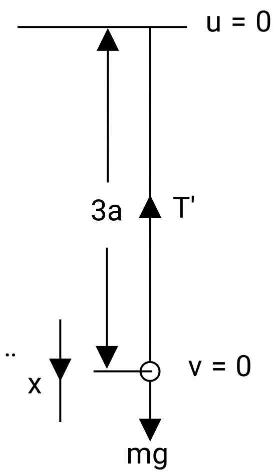
(b) Loss in potential energy and kinetic energy = Elastic potential energy stored

$$mg(2a-a) + \frac{1}{2}m(u^2-v^2) = \frac{\lambda a^2}{2a}$$

$$mga + \frac{1}{2}m\left(2ag-v^2\right) = \frac{3}{2}mg\times\frac{a}{2}$$

$$v = \frac{1}{2}\sqrt{10ag}$$

(c)



$$mg - T' = m\ddot{x}$$

$$mg - \frac{\lambda x}{a} = m\ddot{x}$$

$$mg - \frac{3}{2}mg\times\frac{(3a-a)}{a} = m\ddot{x}$$

$$\ddot{x} = -2g = -2\times9\cdot8 = -19\cdot6\ \text{m s}^{-2}$$

$$\ddot{x} = 19\cdot6\ \text{m s}^{-2}\ \text{upwards.}$$

Example 4

Point A is at the bottom of a rough plane which is inclined at an angle θ to the horizontal. A body of mass m is projected from A, along and up along a line of greatest slope. The coefficient of friction between the body and the plane is μ. The body comes to rest at a point B, a distance x from A. Show that the:

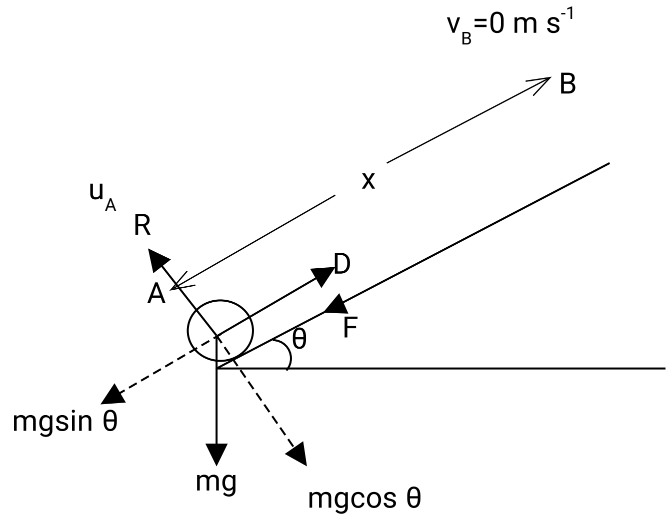
(a) work done against friction when the body moves from A to B and back to A is given by 2μmgxcos θ.

(b) initial speed of the body is $\sqrt{2gx(\sin\theta+\mu\cos\theta)}$.

(c) body returns to A with speed $\sqrt{2gx(\sin\theta-\mu\cos\theta)}$.

(d) Are there any circumstances under which the body will not return to A?

Solution



(a) Resolving normal to plane

$$R = mg\cos\theta$$

From $F = \mu R \Rightarrow F = \mu mg\cos\theta$

Work done against friction is moving from A to B and back is:

$$W = (\mu mg\cos\theta)\times2x$$

$$W = 2\mu mgx\cos\theta$$

(b) Let u_A be the initial speed of the body.

Loss in Kinetic energy = Work done against friction and gravity from A to B.

$$\frac{1}{2}m(u_A^2-0) = \mu mg\cos\theta\times x + mg\sin\theta\times x$$

$$u_A = \sqrt{2gx\left(\sin\theta+\mu\cos\theta\right)}$$

(c) Let v_A be the speed with which the body returns to A.

Loss in Kinetic energy from A to B and back to A = Total work done against friction.

$$\frac{1}{2}m\left(u_A^2-v_A^2\right) = 2\mu mgx\cos\theta$$

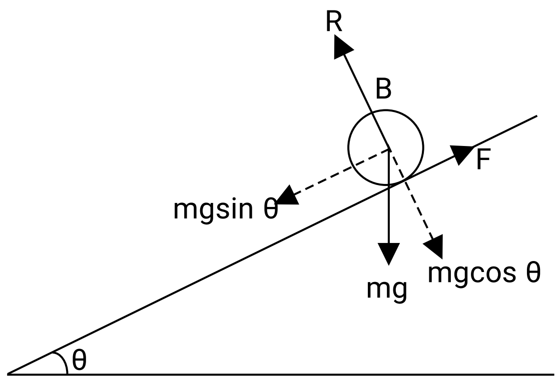
$$u_A^2 - v_A^2 = 4\mu gx\cos\theta$$

$$v_A^2 = 2gx(\sin\theta + \mu\cos\theta) - 4\mu gx\cos\theta$$

$$v_A = \sqrt{2gx\left(\sin\theta-\mu\cos\theta\right)}$$

(d) At the instant when the body reach B,

$v_B = 0$



Resolving normal to plane: $R = mg\cos\theta$

The body remains at B if $F_{max} \geq mg\sin\theta$,

$F_{max} = \mu R$

$\mu mg\cos\theta \geq mg\sin\theta \Rightarrow \mu \geq \tan\theta$

Hence the body will not return to A if $\mu \geq \tan\theta$, the body remains at rest at B.

13.3 Work-Energy theorem

It states that the work done by the resultant force acting on a body is equal to the change in kinetic energy of the body.

Proof:

Consider a body of mass m acted on by a resultant force F, initially moving with velocity u and attains a final velocity v through a distance s. Work done by resultant force, $W = F \times s$.

But $F = ma$

$W = m \times as$

From $v^2 = u^2 + 2as$

$$a = \frac{v^2 - u^2}{2s}$$

$$W = ms \times \frac{v^2 - u^2}{2s}$$

$$W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2 = \text{Change in kinetic energy}$$

Example 5

When t = 0 a body of mass 5 kg is at rest at a point A, position vector $(3\mathbf{i}+7\mathbf{j}+2\mathbf{k})$ m. The body is then subjected to a constant force $\mathbf{F} = (10\mathbf{i}+15\mathbf{j}-5\mathbf{k})$ N causing it to accelerate and 2 seconds later, the body passes through the point B. Find:

(a) The acceleration of the body in vector form.

(b) The velocity of the body as it passes through B.

(c) The kinetic energy of the body when t = 2 seconds.

(d) $\vec{AB}$

(e) The position vector of point B.

(f) $\mathbf{F} \bullet \vec{AB}$, the work done by the force in these first 2 seconds and verify that this equals the change in kinetic energy of the body in travelling from A to B.

<u>Solution:</u>

(a) $m = 5$ kg, $\mathbf{r}(0) = (3\mathbf{i}+7\mathbf{j}+2\mathbf{k})$ m, $\mathbf{F} = (10\mathbf{i}+15\mathbf{j}-5\mathbf{k})$ N

From $\mathbf{F} = m\mathbf{a}$

$5\mathbf{a} = 10\mathbf{i} + 15\mathbf{j} - 5\mathbf{k}$

$\mathbf{a} = (2\mathbf{i}+3\mathbf{j}-\mathbf{k})$ m s^{-2}

(b) $\mathbf{v}_B = \mathbf{v}_A + \mathbf{a}t$, $\mathbf{v}_A = 0$

$\mathbf{v}_B = (2\mathbf{i}+3\mathbf{j}-\mathbf{k}) \times 2 = (4\mathbf{i}+6\mathbf{j}-2\mathbf{k})$ m s^{-1}

(c) Kinetic energy $= \frac{1}{2}mv^2 = \frac{1}{2}m\mathbf{v}\bullet\mathbf{v}$

Kinetic energy at B $= \frac{1}{2}m\mathbf{v}_B\bullet\mathbf{v}_B$

$$= \frac{1}{2}\times 5\times(4\mathbf{i} + 6\mathbf{j} - 2\mathbf{k})\bullet(4\mathbf{i} + 6\mathbf{j} - 2\mathbf{k})$$

$$= \frac{5}{2}(16+36+4) = 140 \text{ J}$$

(d) From $\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$, $\mathbf{u} = \mathbf{0}$

$$\vec{AB} = \frac{1}{2}\times(2\mathbf{i}+3\mathbf{j}-\mathbf{k})\times 2^2$$

$$\Rightarrow \vec{AB} = (4\mathbf{i}+6\mathbf{j}-2\mathbf{k}) \text{ m}$$

(e) $\mathbf{r}_B = \mathbf{r}_A + \vec{AB}$

$= (3\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}) + (4\mathbf{i} + 6\mathbf{j} - 2\mathbf{k})$

$= (7i+13j)$ m

(f) $\vec{F}\cdot\vec{AB} = (10i + 15j - 5k)\cdot(4i + 6j - 2k)$

$= 40 + 90 + 10 = 140$ J

= Change in kinetic energy from A to B.

13.4 POWER

Power is the rate at which work is done.

$$\text{Power} = \frac{\text{Work done}}{\text{Time taken}}$$

13.4.1 Pump raising water and ejecting water

When a pump is used in raising and also ejecting water at a certain speed, the total work done is the potential energy and kinetic energy given to the water each second.
The density of water, $\rho = 1000$ kg m^{-3}.

(i) **Pump raising water:**

When a pump raises water of mass m kg through a vertical distance h, in a time t, the power developed by the pump is $P = \frac{mgh}{t}$.

(ii) **Pump ejecting water:**

If the pump is used in ejecting m kg of water from an horizontal pipe at a velocity u in time t, the power developed by pump is given by $P = \frac{\frac{1}{2}mu^2}{t}$.

(iii) **Pump raising and ejecting water:**

If the pump is used in raising m kg of water through a vertical height h in time t and ejects it at a speed u, the power developed by pump is given by

$$P = \frac{mgh}{t} + \frac{\frac{1}{2}mu^2}{t}.$$

Example 6

In each minute a pump draws 2·4 m^3 of water from a well 5 m below ground and issues it at ground level through a pipe of cross-sectional area 50 cm^2. Find the:

(a) speed at which water leaves the pipe.

(b) rate at which the pump is working. If in fact the pump is only 75% efficient, find the rate at which it must work.

Solution

$V = 2{\cdot}4$ m^3

$m = 2\cdot4\times1000 = 2400$ kg

$h = 5$ m

$A = 50$ cm^2 $= 5\times10^{-3}$ m^2

(a) $\frac{V}{t} = A\times\frac{l}{t} = Au$, where u is the Speed

$$\frac{2{\cdot}4}{60} = 5\times10^{-3}u$$

$u = 8$ m s^{-1}

(b) $$P = \frac{\frac{1}{2}mu^2}{t} + \frac{mgh}{t}$$

$$= \frac{1}{2}\times\frac{2400}{60}\times8^2 + \frac{2400\times9{\cdot}8\times5}{60}$$

$= 3240$ W

$P = 3\cdot24$ kW

Let P' be the rate at which the pump must work

75% of P' = P

$$\frac{75}{100}\times P' = 3\cdot24$$

$P' = 4\cdot32$ kW

13.4.2 Vehicles in motion

Consider a vehicle being driven along a road by a forward force F applied by the engine. Suppose the engine is working at a constant rate of P watts, then;
P = Work done per second

$$\text{Power} = \frac{\text{Work done}}{\text{Time taken}} = \text{Force} \times \frac{\text{Distance}}{\text{Time taken}}$$

This implies that $P = F\times v$

Hence $F = \frac{P}{v}$ gives the tractive force offered by

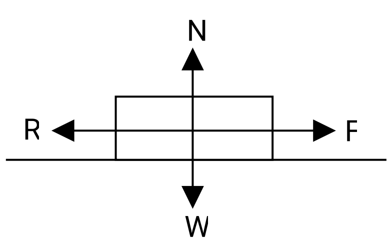
the engine. If the velocity changes with time, the tractive force also changes if the engine is assumed to work at a constant rate.
The power of a moving vehicle is supplied by its engine. The tractive force of an engine is the pushing force it exerts.

To solve problems involving moving vehicles:

1. Draw a clear force diagram.
   Note: 'Non-gravitational resistance' means 'frictional force'.
2. Resolve perpendicular to the direction of motion.
3. If the velocity is;
   (a) constant (vehicle moving with steady speed), then resolve forces parallel to the direction of motion (Newton's first law.)
   (b) not constant (vehicle accelerating), then find the resultant force acting and write down the equation of motion in the direction of motion.
4. Use, Power = Tractive force × Speed
   Note: At maximum speed, the acceleration is zero.

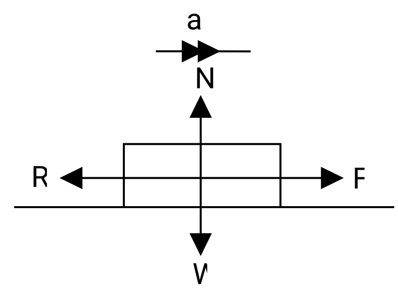
Common situations:
The following are some common cases which arise in problems;

1. **Vehicles on the level:**
   (a) Moving with steady speed v



Resolve (↑): N = W
Resolve (→): F = R
Power: P = Fv

(b) Moving with acceleration a and instantaneous speed v

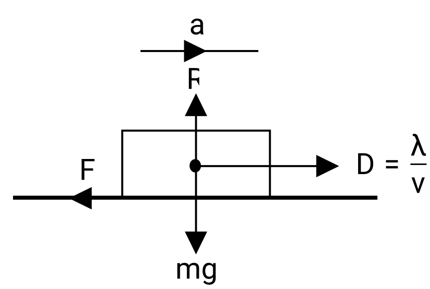


Resolve (↑): N = W
Resolve (→): F−R = ma
Power: P = Fv

Note that if the vehicle is retarding R > F, then a will be in the opposite direction.

Example 7
A car of mass m develops a constant power λ while moving along a level road against a resistance which is directly proportional to the car's velocity at any time. Given that the maximum velocity is α prove that its acceleration at a lower speed β is $\frac{\lambda(\alpha+\beta)(\alpha-\beta)}{m\beta\alpha^2}$.

Solution



$F \propto v \Rightarrow F = kv$

$\frac{\lambda}{v} - kv = ma$

At $v_{max} = \alpha$, $a = 0$

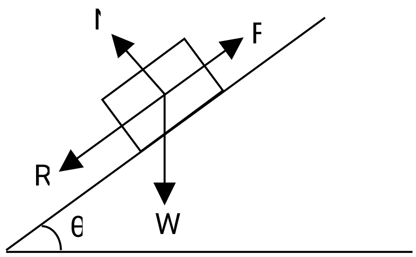
$\frac{\lambda}{\alpha} - k\alpha = 0 \Rightarrow k = \frac{\lambda}{\alpha^2}$

At a lower speed β:

$\frac{\lambda}{\beta} - \frac{\lambda\beta}{\alpha^2} = ma \Rightarrow ma = \frac{\lambda(\alpha^2-\beta^2)}{\beta\alpha^2}$

$a = \frac{\lambda(\alpha+\beta)(\alpha-\beta)}{m\beta\alpha^2}$

2. **Vehicle on a slope of inclination θ:**
   (a) Moving with steady speed v
      (i) Up the slope



Resolving normal to plane:

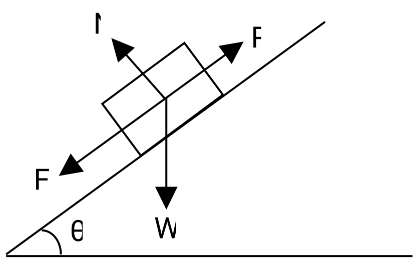
$N = W\cos\theta$

Resolving along plane:

$F = R + W\sin\theta$

Power: $P = Fv$

(ii) Down the slope



Resolving normal to plane:

$N = W\cos\theta$

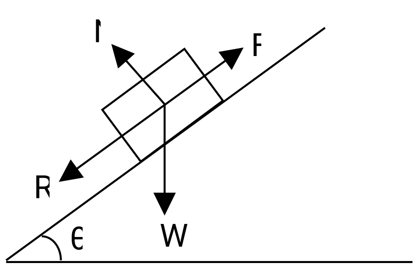
Resolving along plane:

$R = F + W\sin\theta$

Power: $P = Fv$

(b) Moving with acceleration a and instantaneous speed v

(i) Up the slope



Resolving normal to plane:

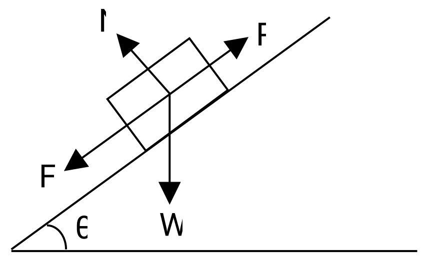
$N = W\cos\theta$

Equation of motion along the plane:

$F-(R + W\sin\theta) = ma$

Power: $P = Fv$

(ii) Down the slope



Resolving normal to plane:

$N = W\cos\theta$

Equation of motion along the plane:

$(F + W\sin\theta) - R = ma$

Power: $P = Fv$

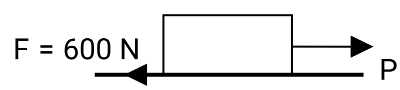
Example 8

A car of mass 1·2 tonnes is travelling along a straight, horizontal road at a constant speed of 120 km h^{-1} against a resistance of 600 N. Calculate in kW, the effective power being exerted. Given that the resistance is proportional to the square of the velocity. Calculate also the:

(i) power required to go down a hill of 1 in 30 (along the slope) at a steady speed of 120 km h^{-1}.

(ii) acceleration of the car up this hill with the engine working at 20 kW at the instant when the speed is 80 km h^{-1}.

Solution



$$v = 120 \text{ km h}^{-1} = \left(120\times\frac{1000}{60\times60}\right) \text{ m s}^{-1}$$

$$= \frac{100}{3} \text{ m s}^{-1}$$

$$\frac{P}{v} = 600 \Rightarrow P = 600v$$

$$P = 600\times\frac{100}{3} = 20\,000 \text{ W}$$

$$P = 20 \text{ kW}$$

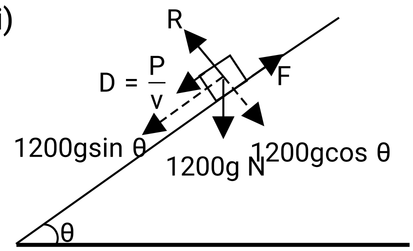
If $F \propto v^2 \Rightarrow F = kv^2$, F− resistance, k− constant

When $F = 600$ N, $v = \frac{100}{3}$

$$\text{m s}^{-1} \Rightarrow 600 = k\times\left(\frac{100}{3}\right)^2 \Rightarrow k = \frac{27}{50}$$

$$\therefore\ F = \frac{27}{50}v^2$$

(i)



$$v = \frac{100}{3}\ \text{m s}^{-1}$$

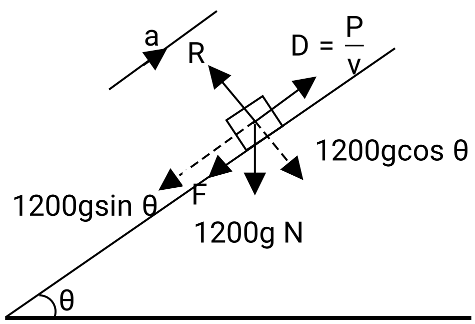
$$\theta = \sin^{-1}\left(\frac{1}{30}\right)$$

$$(D + 1200g\sin\theta) = F$$

$$\frac{P}{\left(\frac{100}{3}\right)} + 1200\times9\cdot8\times\frac{1}{30} = \frac{27}{50}\times\left(\frac{100}{3}\right)^2$$

$$P = 6933\frac{1}{3}\ \text{W}$$

(ii)



$$v = \frac{200}{9}\ \text{m s}^{-1}$$

$$\theta = \sin^{-1}\left(\frac{1}{30}\right)$$

$$\frac{P}{v} - (1200g\sin\theta + F) = 1200a$$

$$\frac{20\times10^3}{\left(\frac{200}{9}\right)} - 1200\times9\cdot8\times\frac{1}{30} - \frac{27}{50}\times\left(\frac{200}{9}\right)^2 = 1200a$$

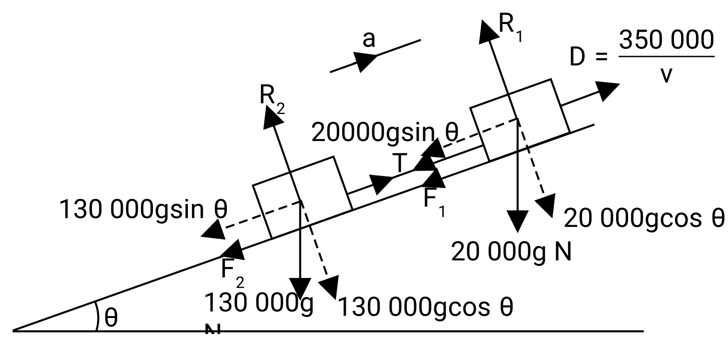
$$a = 0\cdot201\ \text{m s}^{-2}$$

Example 9

A locomotive of mass 20 000 kg is connected to carriages of total mass 130 000 kg by means of coupling. The train climbs a straight truck inclined at $\sin^{-1}\left(\frac{1}{200}\right)$ to the horizontal with the engine of the locomotive working at 350 kW. The non-gravitational resistances opposing this motion are constant and total 2000 N for the locomotive and 8000 N for the carriages. Given that at a particular instant the train is moving at 15 m s^{-1}, calculate the:

(a) driving force produced by the engine of the locomotive.
(b) acceleration of the train.
(c) tension in the coupling between the locomotive and the carriages.
(d) Show that the greatest steady speed that the train can achieve up this incline under the given conditions is 20 m s^{-1}. If the train sustains this speed for 2 km, measured along the track, calculate the work done by the engine of the locomotive in covering this distance. (Use g = 10 m s^{-2})

Solution



$$\theta = \sin^{-1}\left(\frac{1}{200}\right);\ F_1 = 2000\ \text{N};\ F_2 = 8000\ \text{N}$$

(a) $$D = \frac{350\ 000}{15} = \frac{70\ 000}{3}\ \text{N}$$

$$= 23\ 333\frac{1}{3}\ \text{N}$$

(b) For locomotive:

$$\frac{70\ 000}{3} - (T + 20\ 000g\sin\theta + 2000) = 20\ 000a$$

$$\frac{64\ 000}{3} - 20\ 000\times10\times\frac{1}{200} - T = 20\ 000a$$

$$\frac{61\ 000}{3} - T = 20\ 000a \ \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

For Carriages:

$$T - (130\ 000g\sin\theta + 8000) = 130\ 000a$$

$$T - 130\ 000\times10\times\frac{1}{200} - 8000 = 130\ 000a$$

$$T - 14\ 500 = 130\ 000a \ \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Adding equation (i) and equation (ii):

$$\frac{17\,500}{3} = 150\,000a$$

$$\Rightarrow a = \frac{7}{180} \text{ m s}^{-2}$$

(b) From equation (ii):

$$T = 14\,500 + 130\,000 \times \frac{7}{180}$$

$$T = 19\,555{\cdot}56 \text{ N}$$

(c) For locomotive:

$$\frac{350\,000}{v} - \left(T + 20\,000 \times 10 \times \frac{1}{200} + 2000\right) = 20\,000a \ldots\ldots\ldots\ldots\text{(iii)}$$

For carriages:

$$T - \left(130\,000 \times 10 \times \frac{1}{200} + 8000\right) = 130\,000a \ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}$$

Adding equation (iii) and equation (iv):

$$\frac{350\,000}{v} - 17\,500 = 150\,000a$$

At v_{max} ; $a = 0 \Rightarrow \frac{350\,000}{v_{max}} - 17\,500 = 0 \Rightarrow v_{max} = 20 \text{ m s}^{-1}$

At $v_{max} = 20 \text{ m s}^{-1}$; $F = \frac{350\,000}{20}$

$$\Rightarrow F = 17\,500 \text{ N}$$

$$W = F \times s \Rightarrow W = 17\,500 \times 2000$$

$$\Rightarrow W = 3{\cdot}5 \times 10^{7} \text{ J}$$

Example 10

A car of mass M kg has an engine which works at a constant rate of 2H kW. The car has a constant speed v m s^{-1} along a horizontal road.

(a) Find in terms of M, H, v, g and θ the acceleration of the car when travelling:

  (i) up a road of inclination θ with a speed of $\frac{3}{4}$v m s^{-1}.

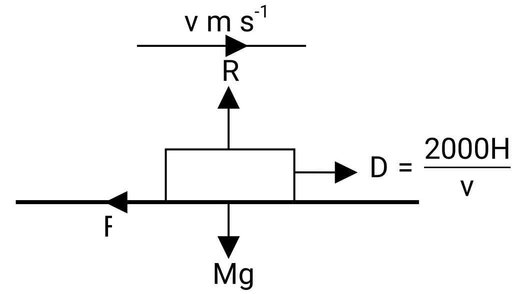
  (ii) down the same road with a speed of $\frac{3}{5}$v m s^{-1}, the resistance to motion of the car apart from the gravitational force being constant.

(b) If the acceleration in a(ii) above is three times that of a(i) above, find the angle of inclination θ of the road.

(c) If the car continues directly up the road, in case a(i) above, show that its maximum speed is $\frac{12}{13}$v m s^{-1}.

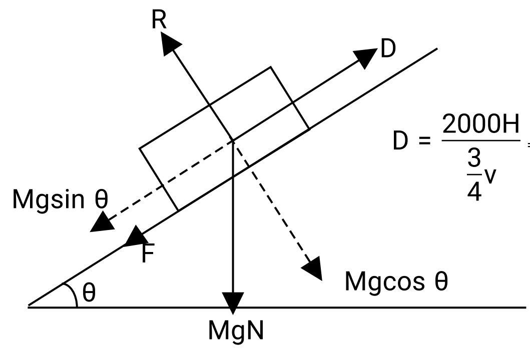
Solution:

On horizontal road



$$D = F \Rightarrow F = \frac{2000H}{v}$$

(a) (i) ascending with acceleration a_1



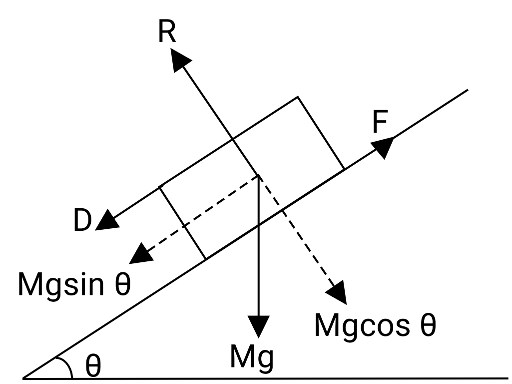
$$D = \frac{2000H}{\frac{3}{4}v} \Rightarrow D = \frac{8000H}{3v}$$

$$\frac{8000H}{3v} - (Mg\sin\theta + F) = Ma_1$$

$$\frac{8000H}{3v} - Mg\sin\theta - \frac{2000H}{v} = Ma_1$$

$$a_1 = \frac{2000H}{3Mv} - g\sin\theta$$

(ii) descending with acceleration a_2



$$D = \frac{2000H}{\left(\frac{3}{5}v\right)} = \frac{10\,000H}{3v}$$

$$(D + Mg\sin\theta) - F = Ma_2$$

$$\frac{10\,000H}{3v} + Mg\sin\theta - \frac{2000H}{v} = Ma_2$$

$$a_2 = \frac{4000H}{3Mv} + g\sin\theta$$

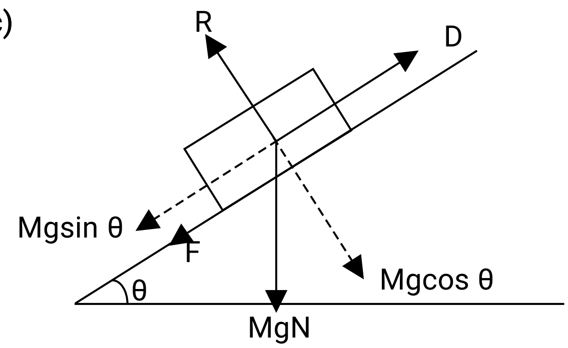
(b) From: $a_2 = 3a_1$

$$\frac{4000H}{3Mv} + g\sin\theta = 3\left(\frac{2000H}{3Mv} - g\sin\theta\right)$$

$$4g\sin\theta = \frac{2000H}{3Mv} \Rightarrow \sin\theta = \frac{500H}{3Mvg}$$

$$\theta = \sin^{-1}\left(\frac{500H}{3Mvg}\right)$$

(c)



At v_{max}; $a = 0 \Rightarrow D = \frac{2000H}{v_{max}}$

$$\frac{2000H}{v_{max}} - (F + Mg\sin\theta) = 0$$

$$\frac{2000H}{v_{max}} - \frac{2000H}{v} - Mg\times\frac{500H}{3Mvg} = 0$$

$$v_{max} = \frac{12}{13}v \text{ m s}^{-1}$$

Example 11

(a) A pump raises 2 m^3 of water through a vertical distance of 10 m in 90 seconds, discharging it at a speed of $2{\cdot}5\text{ m s}^{-1}$. Show that the power it develops is approximately $2\cdot25$ kW.

(b) A car of mass 1000 kg has a maximum speed of 150 km h^{-1} on a level rough road with the engine working at 60 kW.

(i) Calculate the coefficient of friction between the car and the road if all the resistance is due to friction.

(ii) Given that the tractive force remains constant and the non-gravitational resistance in both cases varies as the square of the speed, find the greatest slope on which a speed of 120 km h^{-1} could be maintained.

Solution

(a) Volume, $V=2\text{ m}^3$

Density of water, $\rho=1000\text{ kg m}^{-3}$

$h = 10$ m

$m = \rho\times V = 1000\times2 = 2000$ kg

$t = 90$ s

velocity, $u=2{\cdot}5\text{ m s}^{-1}$

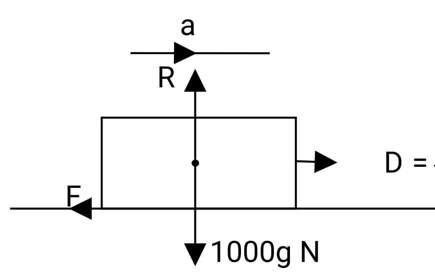
Power = P.E per second + K.E per second

$$P = \frac{mgh}{t} + \frac{\frac{1}{2}mu^2}{t}$$

$$P = \frac{2000\times9{\cdot}8\times10}{90} + \frac{\frac{1}{2}\times2000\times2{\cdot}5^2}{90}$$

$$P = 2247\cdot22W \simeq 2\cdot25\text{ kW}$$

(b)



$$D = \frac{\text{Power}}{\text{Speed}}$$

$$150\text{ km h}^{-1} = 150\times\frac{1000}{60\times60} = \frac{125}{3}\text{ m s}^{-1}$$

Equation of motion:

$$D - F = ma$$

$$\left(\frac{60\,000}{v}\right) - F = 1000a$$

At maximum speed, $a = 0$

$$\frac{60\,000}{\left(\frac{125}{3}\right)} - F = 0 \Rightarrow F = 1440\text{ N}$$

Resolving vertically:

$$R = 1000g$$

$$R = 1000 \times 9 \cdot 8 = 9800 \text{ N}$$

(i) From: $F = \mu R$

$1440 = \mu \times 9800$

$\mu = 0 \cdot 15$

(ii) If $F \propto v$

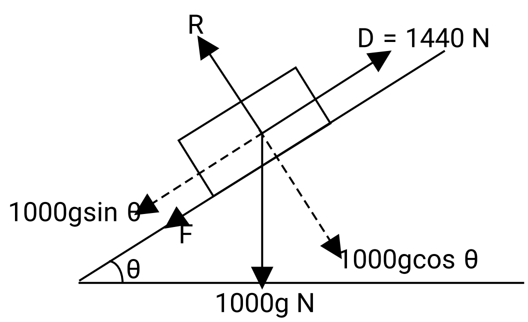
$F = kv^2$, when $F = 1440$ N ; $v = \frac{125}{3}$ m s^{-1}

$$1440 = \left(\frac{125}{3}\right)^2 k \Rightarrow k = \frac{2592}{3125}$$

$$\Rightarrow F = \frac{2592}{3125}v^2$$

When $v = 120$ km h^{-1} $= \frac{100}{3}$ m s^{-1}

$$F = \frac{2592}{3125} \times \left(\frac{100}{3}\right)^2 = 921 \cdot 6 \text{ N}$$



$D = F + 1000g\sin\theta$

$1440 = 921 \cdot 6 + 1000 \times 9 \cdot 8 \sin\theta$

$\theta = 3 \cdot 0°$

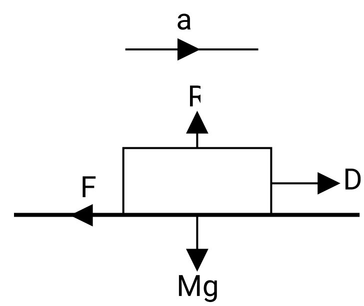
Example 12

A train of mass M starts from rest at A and travels with uniform acceleration for a time t_1. Steam is then shut off and the train comes to rest at B (without breaks being applied). The distance from A to B is x and the total time taken is t. The resistance due to friction is β times the weight of the train.

(i) Prove that $t - t_1 = \frac{2x}{\beta g t}$.

(ii) The greatest rate of working of the engine during the journey is $\frac{2xM\beta^2 g^2 t}{(\beta g t^2 - 2x)}$.

Solution

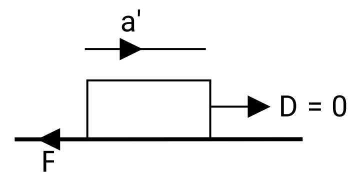


$$D - F = Ma; \text{ but } F = \beta R = \beta Mg$$

$$D - \beta Mg = Ma \Rightarrow a = \left(\frac{D}{M} - \beta g\right)$$

From $v = u + at$

$$v = \left(\frac{D}{M} - \beta g\right)t_1$$



$$0 - \beta Mg = Ma' \Rightarrow a' = -\beta g$$

From $v = u + at$

$$0 = \left(\frac{D}{M} - \beta g\right)t_1 - \beta g(t - t_1)$$

$$D = \frac{\beta Mgt}{t_1} \quad \text{.............................. (i)}$$

From $s = ut + \frac{1}{2}at^2$

$$s_1 = \frac{1}{2}\left(\frac{D}{M} - \beta g\right)t_1^2 = \frac{1}{2}\left(\frac{\beta gt}{t_1} - \beta g\right)t_1^2$$

$$s_1 = \frac{1}{2}\beta g t_1 (t - t_1) \text{......................(ii)}$$

$$s_2 = \left(\frac{D}{M} - \beta g\right)t_1(t - t_1) - \frac{1}{2}\beta g(t - t_1)^2$$

$$s_2 = \left(\frac{\beta gt}{t_1} - \beta g\right)t_1(t - t_1) - \frac{1}{2}\beta g(t - t_1)^2$$

$$= \frac{1}{2}\beta g(t - t_1)^2$$

(i) But $s_1 + s_2 = x$

$$\frac{1}{2}\beta g t_1(t - t_1) + \frac{1}{2}\beta g(t - t_1)^2 = x$$

$$\beta g t(t - t_1) = 2x$$

$$\Rightarrow t - t_1 = \frac{2x}{\beta g t} \quad \text{.......................... (iii)}$$

(ii) From equation (i): $D = \frac{\beta Mgt}{t_1}$

But from equation (iii): $t_1 = \frac{\beta g t^2 - 2x}{\beta g t}$

Hence: $D = \frac{\beta Mgt \times \beta gt}{\beta gt^2 - 2x} = \frac{M\beta^2 g^2 t^2}{\beta gt^2 - 2x}$

$P_{max} = D \times v_{max}$

$v_{max} = \left(\frac{D}{M} - \beta g\right)t_1 = \left(\frac{\beta gt}{t_1} - \beta g\right)t_1$

$$= \beta g\left(t - t_1\right) = \beta g \times \frac{2x}{\beta gt} = \frac{2x}{t}$$

$P_{max} = \frac{M\beta^2 g^2 t^2}{\beta gt^2 - 2x} \times \frac{2x}{t}$

$P_{max} = \frac{2xM\beta^2 g^2 t}{\left(\beta gt^2 - 2x\right)}$

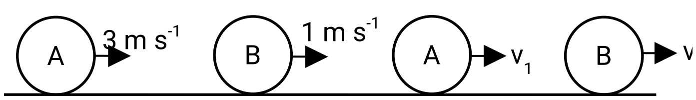
Example 13

(a) A particle A of mass 1 kg is moving with a velocity of 3 m s^{-1} when it collides with particle B of mass 2 kg which is moving with a velocity of 1 m s^{-1} in the same direction as A. After collision the particles separate with a relative velocity of 1 m s^{-1}. Find the:
   (i) velocities of the particles after collision.
   (ii) loss in kinetic energy.

(b) The force opposing motion of a car is $\left(\alpha + \beta v^2\right)$ N, where v is the speed of the car in m s^{-1}. The power required to maintain a steady speed of 20 m s^{-1} on the level road is 6•3 kW, and at 30 m s^{-1} it is 15•3 kW. Find the values of α and β, and hence calculate the power required for a steady speed of 40 m s^{-1}

Solution

(a) (i) By conservation of momentum



$1 \times 3 + 2 \times 1 = 1 \times v_1 + 2 \times v_2$

$$v_1 + 2v_2 = 5 \quad \text{.................................. (i)}$$

Also: $v_2 - v_1 = 1$ .... (ii)

Adding equation (i) and equation (ii)

$$3v_2 = 6 \Rightarrow v_2 = 2 \text{ m s}^{-1}$$

From equation (i): $2 - v_1 = 1 \Rightarrow v_1 = 1$ m s^{-1}

(ii) The loss in kinetic energy:

Initial kinetic energy

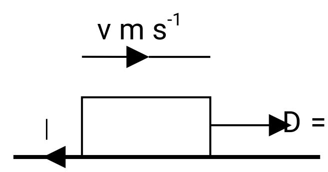
$$= \frac{1}{2} \times 1 \times 3^2 + \frac{1}{2} \times 2 \times 2^2 = 5 \cdot 5 \text{ J}$$

Kinetic energy after impact

$$= \frac{1}{2} \times 1 \times 1^2 + \frac{1}{2} \times 2 \times 2^2 = 4 \cdot 5 \text{ J}$$

Loss in kinetic energy = 5•5 - 4•5 = 1 J

(b)



$D = \frac{P}{v}$; $F = \alpha + \beta v^2$

At steady speed: D = F

When v=2 m s^{-1} and P = 6•3 kW

$$\frac{6 \cdot 3 \times 10^3}{20} = \alpha + \beta \times 20^2$$

$$\alpha + 400\beta = 315 \quad \text{..................... (i)}$$

When v = 30 m s^{-1} and P = 15•3 kW

$$\frac{15 \cdot 3 \times 10^3}{30} = \alpha + \beta \times 30^2$$

$$\alpha + 900\beta = 510 \quad \text{..................... (ii)}$$

Subtracting equation (i) from equation (ii)

$$500\beta = 195 \Rightarrow \beta = 0.39$$

From equation (i): $\alpha = 510 - 900 \times 0 \cdot 39 = 159$

Hence $\frac{P}{v} = 159 + 0 \cdot 39v^2$

When v=40 m s^{-1}

$$\frac{P}{40} = 159 + 0 \cdot 39 \times 40^2$$

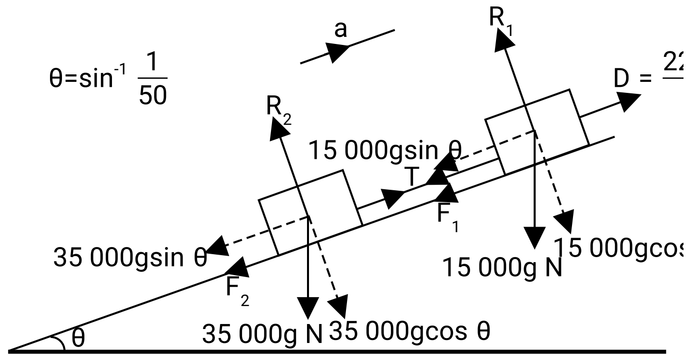
$$\Rightarrow P = 31\,320\text{W} = 31 \cdot 32 \text{ kW}$$

Example 14

A locomotive of mass 15 tonnes, working at a rate of 220 kW, pulls a train of mass 35 tonnes up a track of inclination 1 in 50. When the speed is 10 m s^{-1}, the acceleration is 0•23 m s^{-2}. Find the frictional resistance at this speed. Given that the resistance is proportional to the speed of the train, find the greatest speed of the train up the slope if the rate of working is unchanged. (Take

$g = 10\ m\ s^{-2}$)

Solution



For locomotive:

$$\frac{220\ 000}{v} - 15\ 000g\sin\theta - T - F_1 = 15\ 000a$$

$$\frac{220\ 000}{10} - 15\ 000\times10\times\frac{1}{50} - T - F_1$$

$$= 15\ 000\times0{\cdot}23$$

$$T + F_1 = 15\ 550 \text{ .... (i)}$$

For train:

$$T - 35\ 000g\sin\theta - F_2 = 35\ 000a$$

$$T - F_2 = 35\ 000\times10\times\frac{1}{50} + 35\ 000\times0.23$$

$$T - F_2 = 15\ 050 \text{ .... (ii)}$$

Subtracting equation (ii) from equation (i);

$$F_1 + F_2 = 500\ N$$

$F = 500$ N, where $F = F_1 + F_2$

Since $F \propto v \Rightarrow F = kv$

When $F = 500$ N, $v = 10$

$$500 = 10k \Rightarrow k = 50 \quad \therefore \quad F = 50v$$

For locomotive:

$$\frac{220\ 000}{v_{max}} - 15\ 000\times10\times\frac{1}{50} - T - F_1 = 15000\times0$$

$$\frac{220\ 000}{v_{max}} = 3000 + T + F_1 \text{ .... (iii)}$$

For train:

$$T - 35\ 000\times10\times\frac{1}{50} - F_2 = 3500\times0$$

$$T = 7000 + F_2 \text{ .... (iv)}$$

From equation (iii) and equation (iv)

$$\frac{22\times10^4}{v_{max}} = (F_1+F_2) + 10\ 000$$

But $(F_1+F_2) = F = kv \Rightarrow F = 50v_{max}$

$$\frac{220\ 000}{v_{max}} = 50v_{max} + 10\ 000$$

Let $x = v_{max}$

$$\frac{4400}{x} = x + 200$$

$$x^2 + 200x = 440$$

$$(x+100)^2 = 4400 + 100^2$$

$$\Rightarrow x = -100 \pm 120$$

Either $x = 20$ or $x = -220$

Hence $x = 20$

Implying that $v_{max} = 20\ m\ s^{-1}$

Exercises

Exercise: 13A

1. A block of mass 6·5 kg is projected with a velocity of $4\ m\ s^{-1}$ up a line of greatest slope of a rough plane. Calculate the initial kinetic energy of the block. The coefficient of friction between the block and the plane is $\frac{2}{3}$ and the plane makes an angle θ with the horizontal where $\sin\theta = \frac{5}{13}$. The block travels a distance of d m up the plane before coming instantaneously to rest. Express in terms of d the:
   (i) potential energy gained by the block in coming to rest.
   (ii) work done against friction by the block in coming to rest. Hence calculate d. (Use $g = 10\ m\ s^{-2}$)
2. A car of mass 1600 kg is free wheeling down a hill of slope 1 in 25. When the car descends 200 m along the plane, its speed increases from $3\ m\ s^{-1}$ to $10\ m\ s^{-1}$. Find the:
   (i) change in kinetic energy of the car.
   (ii) average frictional resistance to motion.
3. A bullet of mass 20 g is fired with a velocity of

400 m s^{-1} into a wooden block of mass 2 kg on a smooth horizontal surface. If the bullet gets embedded in the block, find the loss in kinetic energy of the system.

4. A box of mass 16 kg is pulled from rest a distance of 5 m across a smooth horizontal floor by a cable inclined at 60° to the horizontal. Find the work done by the tension in the cable given its magnitude is 25 N.

5. A particle of mass 6 kg sliding across a rough horizontal plane comes to rest in a distance of 8 m. Given it had an initial speed of 10 m s^{-1}, find the work done against friction. Find also the coefficient of friction between the particle and the plane.

6. Find the kinetic energy gained by a body of mass 2 kg falling freely from rest through a distance of 10 m. If a vertical force P N brings the body to rest in a further distance of 8 m, find the value of P.

7. A particle of mass 5 kg slides a distance of 9 m down a rough plane inclined at an angle α to the horizontal, where $\sin \alpha = \frac{3}{5}$. The coefficient of friction is 0·5. Find the work done by gravity and work done against friction. Find also the velocity attained by the particle.

8. A ball is thrown vertically downwards with a speed of 3·5 m s^{-1}. Find its speed when it has travelled a distance of 5 m.

9. A particle is projected vertically upwards from a point A with a speed of 21 m s^{-1}. Find its position relative to A when its speed is:
   (a) 4·2 m s^{-1}
   (b) 35 m s^{-1}

10. A particle is moving at a speed of 7 m s^{-1} when it begins to ascend a slope inclined at an angle α to the horizontal where $\sin \alpha = \frac{5}{13}$. The coefficient of friction between the particle and the slope is $\frac{5}{8}$. Find the:
    (a) speed of the particle when it has travelled 1·95 m up the slope.
    (b) total distance the particle travels up the slope before coming to rest.

11. A block of mass 5 kg is projected with velocity 5·6 m s^{-1} up a line of greatest slope of a rough plane inclined at an angle α to the horizontal, where $\sin \alpha = \frac{3}{5}$. If the block travels 2 m up the slope before coming to rest, find the work done against friction. Find also the coefficient of friction between the block and the plane.

12. A particle of mass 4 kg moves under the action of a force **F** in the interval of time from t = 0 to t = 2. When t = 0, the velocity of the particle is 2**i**−3**j**. Given that **F** = 6t**i** + 2**j**, find the:
    (a) impulse of **F**.
    (b) velocity of the particle when t = 2.
    (c) gain in kinetic energy.

13. A bullet of mass 0·04 kg travelling at 300 m s^{-1} hits a fixed wooden block and penetrates a distance of 4 cm. Find the average resistance of the wood.

14. A particle released from rest on a smooth plane inclined at an angle α to the horizontal reaches a speed of 6 m s^{-1} after travelling 3 m down the plane. Find sin α.

15. A body of mass 10 kg moving under the action of a constant force accelerates from 2 m s^{-1} to 5 m s^{-1}. Find the work done by the force. Given that the magnitude of the force is 7 N, find the distance moved by the body.

16. A brick of mass 3 kg slides in a straight line on a rough horizontal floor with an initial speed of 8 m s^{-1}. If the brick is brought to rest after moving 12 m by the constant frictional force, calculate the:
    (a) kinetic energy lost by the brick in coming to rest.
    (b) coefficient of friction between the brick and the floor.

17. A car of mass 650 kg is travelling on a road which is inclined at 5°. At a certain point P on the road, the car's speed is 15 m s^{-1}. The point Q is 0·4 km down the hill from P and at Q the car's speed is 35 m s^{-1}. If the driving force on the car is constant as it moves down the hill from P to Q and that any resistance is negligible, find the magnitude of the driving force of the car's engine.

18. A block of mass 5 kg is sliding down a

rough plane inclined at 35° to the horizontal. If the acceleration of the block is $3{\cdot}2\ m\ s^{-2}$ and the distance moved by the block is $0{\cdot}5$ m, find the coefficient of friction and work done against friction.

19. Find the work done against gravity when a person of mass 80 kg climbs a vertical distance of 25 m.

20. A rough surface is inclined at an angle $\tan^{-1}\left(\frac{5}{12}\right)$ to the horizontal. A body of mass 130 kg is pulled at a uniform speed a distance of 50 m up the surface by a force acting along a line of greatest slope. The coefficient of friction between the body and the surface is $\frac{2}{7}$. Find the:
    (a) frictional force acting.
    (b) work done against friction.
    (c) work done against gravity.

Exercise: 13B

1. A particle is acted on by a force $(7i+2j-4k)$ N and displaces it from a point with position vector $(i+2j+3k)$ m to a point with position vector $(5i+4j+k)$ m. Find the work done by the force.

2. A car of mass 1000 kg travelling along a straight horizontal road accelerates uniformly from $15\ m\ s^{-1}$ to $25\ m\ s^{-1}$ in a distance of 320 m. If the resistance to motion is 145 N, find the driving force of the engine.

3. A particle of mass 5 kg at rest at a point (1,-4,4) is acted upon by forces $F_1 = 3\mathbf{i} + 3\mathbf{j}$ N, $F_2 = 2\mathbf{j} + 4\mathbf{k}$ N and $F_3 = 2\mathbf{i} + 6\mathbf{k}$ N.
    (i) Find the position and momentum of the particle after 4 seconds.
    (ii) Work done by the forces in 4 seconds.

4. A train of mass 300 tonnes is travelling up a hill of slope 1 in 200. The resistance due to friction is $0{\cdot}0688$ N per kg mass of the train. If the tractive force of the engine is 39 200 N, find how far the train travels and the time taken for it to attain a speed of $18\ km\ h^{-1}$ from rest.

5. 5. If the effective power developed by a crane is 15 kW, find the time taken to lift a load of 50 kN through a vertical distance of 12 m.

6. If the engine of a train travelling at a steady speed of $25\ m\ s^{-1}$ is working at a rate of 1800 kW, find the magnitude of the resistance to motion.

7. A constant force acting on a particle of mass 15 kg moves it along a straight line with a speed of $10\ m\ s^{-1}$, the rate at which work is done by the force is 50 W. If the particle starts from rest, determine the time it takes to move a distance of 100 m.

8. A block is pulled up an incline of $\sin^{-1}\frac{1}{20}$ to the horizontal at a steady speed of $6\ m\ s^{-1}$. If the work done against gravity in one second is 400 J, find the mass of the block.

Exercise: 13C

1. A pump raises 40 kg of water per second through a vertical distance of 10 m and delivers it in a jet with speed $12\ m\ s^{-1}$. Find the effective power developed by the pump.

2. A pump raises water at a rate of 250 litres per minute through a vertical distance of 8 m and projects it into a lake at $2\ m\ s^{-1}$. Find the power developed by the pump.

3. A pump raises water through a vertical height of 15 m and delivers it at a speed of $10\ m\ s^{-1}$ through a circular pipe of internal diameter 12 cm. Find the mass of water raised per second and effective power of the pump.

4. A pump raises water at a rate of 500 kg per minute through a vertical distance of 3 m. If the water is delivered at $2{\cdot}5\ m\ s^{-1}$, find the power developed.

5. In every minute a pump draws $6\ m^3$ of water from a well and issues it at a speed of $5\ m\ s^{-1}$ from a nozzle situated 4 m above the level from which the water was drawn. Find the average rate at which the pump is working.

6. In every minute a pump draws $5\ m^3$ of water from a well and issues it at a speed of $6\ m\ s^{-1}$ from a nozzle situated 6 m above the level from which the water was drawn. Find the average rate at which the pump is working.

7. A pump draws water from a tank and issues it at a speed of 8 m s^{-1} from the end of a pipe of cross-sectional area 0•01 m^2, situated 10 m above the level from which water is drawn. Find the:
   (a) mass of water issued from the pipe in each second
   (b) rate at which the pump is working.

8. A pump draws water from a tank and issues it at a speed of 10 m s^{-1} from the end of a hose of cross-sectional area 5 cm^2, situated 4 m above the level from which the water is drawn. Find the rate at which the pump is working.

Exercise: 13D

1. (a) A rough slope of length 10 m is inclined at $\tan^{-1}\left(\frac{3}{4}\right)$ to the horizontal. A block of mass 50 kg is released from rest at the top of the slope and travels down the slope reaching the bottom with a speed of 8 m s^{-1}. Find the work done against friction and the coefficient of friction between the block and the surface.
   (b) The maximum power developed by the engine of a car of mass 200 kg is 44 kW. When the car is travelling at 20 km h^{-1} up an incline of 1 in 8 it accelerates at 2 m s^{-2}. At what rate will it accelerate when travelling down an incline of 1 in 16 at 20 km h^{-1} when the non-gravitational resistance to motion is the same.

2. A car travelling at 80 km h^{-1} up a rough inclined plane of inclination 1 in 80 has an acceleration of 1•5 m s^{-2} at this instant. If the car has a mass of 420 kg and the engine is working at a constant rate of 56 kW.
   (i) Find the friction force.
   (ii) What would be the maximum speed the car attains on a level road with the same resistance.

3. (a) A truck of mass 18 tonnes travels up a slope of inclination $\sin^{-1}\frac{1}{50}$ against a resistance of 0•1 N per kilogram. Find the tractive force required to produce an acceleration of 0.05 ms^{-2} and power which is developed given the speed is 10 m s^{-1} at this instant.
   (b) A second truck of mass 25 tonnes experiencing the same resistance and a maximum power of 120 kW follows the first truck up the slope. If the first truck maintains the same power while on the slope, show that the distance between them will decrease when both are travelling at maximum speed. (Use g = 10 m s^{-2})

4. A car of mass 900 kg is travelling at 20 m s^{-1} along a horizontal road against constant resistance of 600 N. Find the power developed by the engine if the car is:
   (a) moving with constant speed.
   (b) accelerating at 0•15 m s^{-2}.
   (c) decelerating at 0•5 m s^{-2}.

5. A car of mass 1000 kg is travelling down an incline of 1 in 28 against a constant resistance of 750 N. Find the maximum speed that can be reached by the car when the engine is working at the rate of 12 kW.

6. A lorry of mass 2400 kg is moving at a steady speed of 60 km h^{-1} against a constant frictional resistance of 2 kN. Find the power developed by the engine if the lorry is travelling:
   (a) along a horizontal road.
   (b) up a hill of inclination $\sin^{-1}\left(\frac{1}{21}\right)$.
   (c) down the same hill.

7. A train of mass 50 tonnes is ascending an incline of 1 in 60 with the engine working at a rate of 200 kW. When the train is travelling at 10 m s^{-1}, its acceleration is 0•1 m s^{-2}. Find the frictional resistance to motion.
   Assuming that the resistance and the power of the engine remain constant, find the maximum speed the train can attain up this slope.

8. The constant non-gravitational resistance to the motion of a car of mass 1500 kg is 750 N. The engine of the car works at a constant rate of 20 kW.
   (a) Find in km h^{-1} the maximum speed:
      (i) on level, and
      (ii) directly up a road inclined at an angle arcsin $\left(\frac{1}{12}\right)$ to the horizontal.
   (b) Find also in m s^{-2}, the acceleration of the car when it is travelling at 72 km h^{-1} on a level road. (Use g = 10 m s^{-2}).

9. A car has a maximum speed of 108 km h^{-1} when moving along a horizontal road with the engine working at 36 kW. Calculate the total resistance to motion of the car.

The car which is of mass 800 kg can move down a road inclined at an angle α to the horizontal at a maximum speed of 108 km h^{-1} with the engine working at 30 kW against the same total resistance. Calculate sin α.

Given that the total resistance varies as the square of the speed, find the rate at which the engine is working at the instant when the car is moving along a horizontal road with an acceleration of 0•5 m s^{-2} at a speed of 54 km h^{-1}.

(Take g = 10 m s^{-2})

10. A car of mass 2 tonnes moves from rest down a road of inclination $\sin^{-1}\left(\frac{1}{20}\right)$ to the horizontal. Given that the engine develops a power of 64 • 8 kW and the resistance to motion is 500 N, find the acceleration of the car when its velocity is 10 m s^{-1}.

11. A car of mass 750 kg is travelling along a horizontal road. If the resistance to motion totals 240 N and the car's engine is working at a constant rate of 12 kW. Find the:
   (a) maximum velocity of the car.
   (b) car's acceleration when its velocity is 30 m s^{-1}.

12. A car of mass 1500 kg is capable of working at a maximum rate of 20 kW. It is being driven up a hill whose slope is 1 in 12 at 8 m s^{-1}. If the non-gravitational resistance to motion is 150 N, find the greatest acceleration it can have at that speed.

13. The resistance to motion of a lorry of mass 10 tonnes is proportional to the square of its speed. The lorry maintains a steady speed of 54 km h^{-1} when travelling up a hill of inclination 1 in 120 with its engine working at 60 kW. Determine the acceleration of the lorry when it is travelling down the same incline with its engine working at 35 kW at the instant when the speed is 27 km h^{-1}.

14. A car of mass 900 kg travels up a hill inclined at 10° to the horizontal, against a constant resistance force of 250 N. If the maximum speed is 45 km h^{-1}, determine the power output of the engine.

15. A bus of mass 5 tonnes free wheels down a slope of inclination 1 in 40 at a constant speed. Assuming the non-gravitational resistance remains the same, find the rate at which the engine must work in order to drive the bus up the same incline at a steady speed of 12 km h^{-1}. If the power is suddenly increased to 10 kW, find in m s^{-2}, the immediate acceleration of the bus.

16. A brick of mass 0 • 8 kg slides 6 m down an incline of $\sin^{-1}\left(\frac{3}{5}\right)$ to the horizontal with an initial speed of 0•4 m s^{-1}. If at the bottom it has a speed of 5•4 m s^{-1}, calculate the work done against resistive forces.

17. Determine the work done when the force (3i+4j) N moves its point of application along the curve (1 + 3cos θ)i + (2 + 3sin θ)j from the point where θ = 0 to the point where $\theta = \frac{2}{3}\pi$. (The unit of distance being a metre)

18. A car of mass 2 tonnes developing a constant power of 20 kW travels with a constant velocity against a constant resistance of 2500 N. Find the:
   (i) constant velocity.
   (ii) acceleration of the car while travelling at half of the velocity in (i) above.

19. A car of mass 2000 kg developing a constant power of 12 kW attains a maximum speed of 108 km h^{-1} while travelling along a level part of a road that offers a constant non-gravitational resistance.
   (a) Find the non-gravitational resistance.
   (b) While travelling at maximum speed the car starts to descend a slope of 1 in 20 along the road, calculate the acceleration of the car when its speed is 120 km h^{-1}. (Assuming power developed remains constant at 12 kW)

20. A car of mass 500 kg tows another of mass 100 kg up a hill inclined at $\sin^{-1}\frac{1}{10}$ to the horizontal. The resistance to motion of each car is 0 • 5 N per kg. Find the tension in

the tow bar at the instant when their speed is 10 m s^{-1} and the power output of the towing car is 150 kW.

Exercise: 13E

1. (a) A particle of mass m kg is projected with a velocity of 10 m s^{-1} up a rough plane of inclination 30° to the horizontal. If the coefficient of friction between the particle and the plane is $\frac{1}{4}$ calculate how far up the plane the particle travels.

   (b) A car is working at 5 kW and is travelling at a constant speed of 75 km h^{-1}. Find the resistance to motion.

2. A train is running at $12{\cdot}5\text{ m s}^{-1}$, when it is a distance of 500 m from a station. Power is then shut off and the train runs against a uniform resistance equal to $\frac{1}{100}$ of the weight of the train. If the uniform breaking force that can be exerted on the train provides an additional resistance equal to $\frac{1}{10}$ of the weight of the train, find how far from the station the brakes must be applied so that the train may stop there.

3. A rough surface is inclined at an angle θ to the horizontal. A body of mass m is pulled at a uniform speed a distance x up the surface by a force acting along a line of greatest slope. The coefficient of friction between the body and the plane is μ. If the only resistances to motion are those due to gravity and friction, show that the total work done on the body is $mgx(\sin\theta + \mu\cos\theta)$.

4. Point A is situated at the bottom of a rough plane which is inclined at an angle $\tan^{-1}\left(\frac{3}{4}\right)$ to the horizontal. A body is projected from A with a speed of 14 m s^{-1} along and up a line of greatest slope. The coefficient of friction between the body and the plane is $0{\cdot}25$. The body first comes to rest at a point B. Find the distance AB.

5. A cyclist and his bike have a combined mass of 75 kg and the maximum rate at which the cyclist can work is 392 W. If the greatest speed with which the cyclist can ride along a level road is 8 m s^{-1}, find the magnitude of the constant resistance to motion. With the resistance unchanged, find the greatest speed at which the cyclist can ascend a hill of inclination $\sin^{-1}\left(\frac{1}{15}\right)$.

6. When a car is moving on any road with speed $v\text{ m s}^{-1}$ the resistance to its motion is $(a+bv^2)$ N, where a and b are positive constants. When the car moves on a level road with the engine working at a steady rate of 53 kW, it moves a steady speed of 40 m s^{-1}. When the engine is working at a steady rate of 24 kW the car can travel on level road at a steady speed of 30 m s^{-1}. Find a and b and hence deduce that when the car is moving with speed 34 m s^{-1}, the resistance to its motion is 992 N. Given that the car has a mass of 1200 kg, find its acceleration on a level road at the instant when the car engine is working at a rate of 51 kW and the car is moving with speed 34 m s^{-1} so that the resistance to motion is 992 N. The car can ascend a hill at a steady speed of 34 m s^{-1} with engine working at a steady rate of 68 kW. Find the angle of inclination of the hill to the horizontal.

7. A car of mass 4 tonnes develops a constant power of $29{\cdot}4$ kW while ascending a hill of 1 in 20 against a constant non-gravitational resistance of 2000 N.

   (a) Calculate the:

      (i) maximum speed of the car up the hill.

      (ii) speed of the car while accelerating at $0{\cdot}5\text{ m s}^{-2}$.

   (b) Determine the acceleration of the car down the same hill if it develops the same power, while travelling at the same speed as in a(i) above.

8. With its engine working at a constant rate of $9{\cdot}8$ kW, a car of mass 800 kg can descend a slope of 1 in 56 at twice the steady speed that it can ascend the same slope, the resistances to motion remaining the same throughout. Find the magnitude of the resistance and the speed of descent.

Answers to exercises

Exercise: 13A

1. 52 J (i) 25d (ii) 40d ; d = 0•8 m 2. (i) 72 800 J (ii) 263•2 N
3. 1584•16 J 4. 62•5 J 5. 300 J ; $\frac{125}{196}$ 6. 196 J ; 44•1
7. 264•6 J ; 176•4 J ; 5•94 m s^{-1} 8. 10•5 m s^{-1} 9. (a) 21•6 m above A (b) 40 m below A 10. (a) 3•5 m s^{-1} (b) 2•6 m 11. 19•6 J ; $\frac{1}{4}$
12. (a) $(12i+4j)$ N s (b) $(5i-2j)$ m s^{-1} (c) 32 J 13. 45 kN
14. $\frac{30}{49}$ 15. 105 J ; 15 m 16. (a) 96 J (b) 0•272 17. 257•318 N
18. 0•3016 ; 6•0526 J 19. 19 600 J 20. (a) 336 N (b) 16 800 J (c) 24 500 J

Exercise: 13B

1. 40 J 2. 770 N 3. (i) $(9i+4j+20k)$ m ; $(20i+20j+40k)$ kg m s^{-1} (ii) 240 J 4. 971•5026 m ; 388•6 s 5. 40 s 6. 72 kN 7. $10\sqrt{6}$ s
8. 136•054 kg

Exercise: 13C

1. 6•8 kW 2. 335 W 3. 113 kg ; 22•3 kW
4. 271•042 W 5. 5•17 kW
6. 6•4 kW 7. (a) 80 kg (b) 10•4 kW 8. 446 W

Exercise: 13D

1. (a) 1340 J ; 0•342 (b) 3•8375 m s^{-2} 2. (i) 1838•55 N (ii) 30•46 m s^{-1}
3. (a) 6•3 kN ; 63 kW (b) $v_{max} = 11\frac{2}{3}$ m s^{-1} ; v'_{max} = 16 m s^{-1} since $v_{max} < v'_{max}$ distance decreases 4. (a) 12 kW (b) 14•7 kW (c) 3 kW
5. 30 m s^{-1} 6. (a) $33\frac{1}{3}$ kW (b) 52 kW (c) $14\frac{2}{3}$ kW
7. $6\frac{5}{6}$ kN ; $13\frac{1}{3}$ m s^{-1} 8. (a) (i) 96 km h^{-1} (ii) 36 km h^{-1} (b) $\frac{1}{6}$ m s^{-2}
9. 1200 N ; $\frac{1}{40}$; 10•5 kW 10. 3•48 m s^{-2} 11. (a) 50 m s^{-1} (b) $\frac{16}{75}$ m s^{-2}
12. $\frac{3}{4}$ m s^{-2} 13. $\frac{15}{32}$ m s^{-2} 14. 22•3 kW 15. 8•17 kW ; 0•11 m s^{-2}
16. 16•624 J 17. -3•108 J
18. (i) 8 m s^{-1} (ii) 1•25 m s^{-2} 19. (a) 400 N (b) 0•47 m s^{-2}
20. 2500 N

Exercise: 13E

1. (a) 7.12 m (b) 240 N 2. 29•72 m 3. 4. 12•5 m
5. 49 N ; 4 m s^{-1} 6. a = 125 ; b = $\frac{3}{4}$; 0•42 m s^{-2} ; 4•9°
7. (a) (i) 7•424 m s^{-1} (ii) 4•933 m s^{-1} (b) 0•98 m s^{-2}
8. 420 N ; 35 m s^{-1}

14. ELASTICITY AND HOOKE'S LAW

We shall consider the behavior of springs and elastic strings. The un-stretched length of an elastic string is called its natural length.

14.1 Hooke's law

Hooke's law for an elastic string or spring is $T = \frac{\lambda x}{l}$, where T is the tension in the string or spring, λ is the modulus of elasticity, x is the extension and l is its natural (un-stretched) length. A negative extension of a spring is a compression. Work done in stretching the string or spring is given by $\frac{\lambda x^2}{2l}$.

14.2 Conservation of energy of an elastic string

If there is no work done against friction and the only external force which does work is gravity, then;

Total energy = P.E due to gravity + P.E in spring+ Kinetic energy = Constant

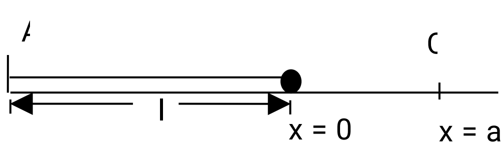
When solving problems it is useful to write the principle of conservation of energy in the form:

Initial total energy = Final total energy

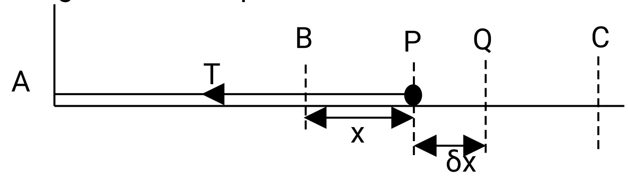
Potential energy stored in an elastic string:

The work done in stretching an elastic string is stored as its elastic potential energy, when the stretching force is removed the string uses this energy stored to regain its natural length. This energy is also a form of potential energy.

Consider an elastic string of modulus λ and natural length l, having one end attached to point A and the other to a particle resting on a smooth horizontal surface.



If x is the extension of the string and the particle is initially at a point B, where x = 0 and is pulled to point C, where x = a, the work done in stretching the string from point P to point Q through a small displacement δx is $\delta W = F\delta x$.



Since δx is very small the force in the string is almost constant

$$F = T = \frac{\lambda x}{l}.$$

$$\delta W \simeq \frac{\lambda x}{l}\delta x$$

The total work done in stretching the string from B to C is

$$W \simeq \lim_{\delta x \to 0} \sum_{x=0}^{x=a} \frac{\lambda x}{l}\delta x$$

$$\Rightarrow W = \int_0^a \frac{\lambda x}{l} dx$$

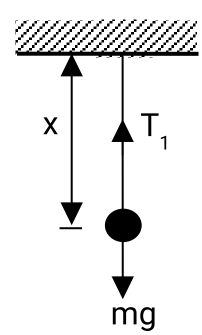
$$W = \frac{\lambda a^2}{2l}$$

Hence the elastic potential energy in a string stretched by distance a from its natural length l is $\frac{\lambda a^2}{2l}$.

Example 1

An elastic string fixed at one end has length x, when supporting a mass m kg and its length increased to x' when an additional mass M kg is supported. Prove that the natural length of the string is $x - \alpha(x' - x)$, where $\alpha = \frac{m}{M}$.

Solution



Let l_0 be the natural length of the string and λ its modulus of elasticity.

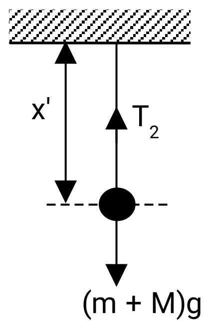
Resolving vertically:

$$T_1 = mg$$

From Hooke's law:

$$T_1 = \frac{\lambda(x-l_o)}{l_o} \Rightarrow \frac{\lambda(x-l_o)}{l_o} = mg$$

$$\lambda = \frac{mgl_o}{x-l_o} \quad \text{.................................. (i)}$$



Resolving vertically:

$$T_2 = (m + M)g$$

From Hooke's law:

$$T_2 = \frac{\lambda(x'-l_o)}{l_o}$$

$$\Rightarrow \frac{\lambda(x'-l_o)}{l_o} = (m + M)g \quad \text{......... (ii)}$$

From equation (i) and equation (ii):

$$\frac{mgl_o}{x-l_o} \times \frac{(x'-l_o)}{l_o} = (m + M)g$$

$$m(x'-l_0) = (m + M)(x-l_0)$$

$$\Rightarrow \alpha(x'-l_0) = (\alpha + 1)(x-l_0)$$

$$l_0 = x - \alpha(x'-x) \text{ as required.}$$

Example 2

An elastic string of natural length 1•2 m and modulus of elasticity 8 N is stretched until the extending force is 6 N. Find the extension and work done.

Solution:

$$l_0 = 1 \cdot 2 \text{ m}, \lambda = 8 \text{ N}$$

$$T = \frac{\lambda x}{l_0}$$

$$6 = \frac{8x}{1\cdot 2} \Rightarrow x = 0 \cdot 9 \text{ m}$$

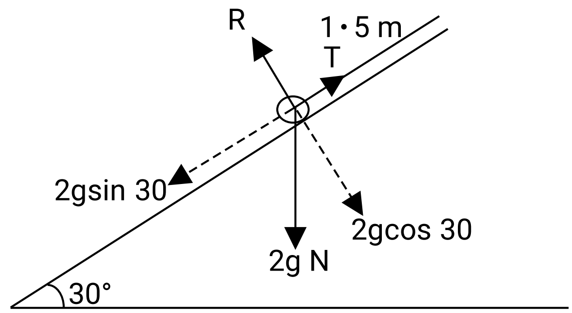
$$W = \frac{\lambda x^2}{2l_0}$$

$$= \frac{8 \times 0\cdot 9^2}{2 \times 1\cdot 2} = 2 \cdot 7 \text{ J}$$

Example 3

A smooth surface is inclined at 30° to the horizontal. A body A of mass 2 kg is held at rest on the surface by a light elastic string of modulus 2g N. If the other end of the string is at a point on the surface which is 1•5 m away from A up a line of greatest slope, find the natural length of the string.

Solution



Resolving along the plane:

$$T = 2g\sin 30$$

From Hooke's law, $T = \frac{\lambda x}{l_0}$

$$T = \frac{2g(1\cdot 5 - l_0)}{l_0}$$

$$\therefore 2g\sin 30 = \frac{2g(1\cdot 5 - l_0)}{l_0}$$

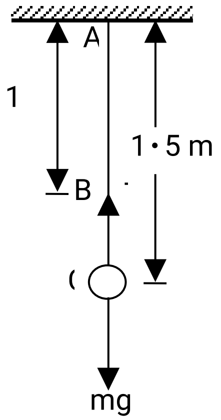
$$\frac{1}{2}l_0 = 1 \cdot 5 - l_0$$

$$l_0 = 1 \text{ m}$$

Example 4

A particle of mass 2 kg is attached to one end of a string of natural length 1 m. The other end is attached to a fixed point A. Initially the particle is held at A and is then released, it falls vertically downwards and comes to rest at a point 1•5 m below A. Using the principle of conservation of energy or otherwise, find the modulus of elasticity of the string.

Solution



From A to a point B, 1 m below A:
Loss in potential energy = Gain in kinetic energy

Gain in kinetic energy = 2g × 1

$$= 2\times9\cdot8$$

$$= 19\cdot6 \text{ J}$$

From B to a point C, 1·5 m below A:

Loss in potential energy and kinetic energy = Elastic potential energy stored

$$19\cdot6 + 2gx = \frac{\lambda x^2}{2l_0}$$

$$19\cdot6 + 2\times9\cdot8\times(1\cdot5\text{-}1) = \frac{\lambda(1\cdot5\text{-}1)^2}{2\times1}$$

$$\lambda = 235\cdot2 \text{ N}$$

Example 5

Find the work done in stretching a spring of natural length 1·5 m and modulus 9 N from a length of 2 m to a length of 2·5 m.

Solution

$$W = \frac{\lambda x^2}{2l_0}$$

$$W_1 = \frac{9\times(2\text{-}1\cdot5)^2}{2\times1\cdot5}$$

$$= 0\cdot75 \text{ J}$$

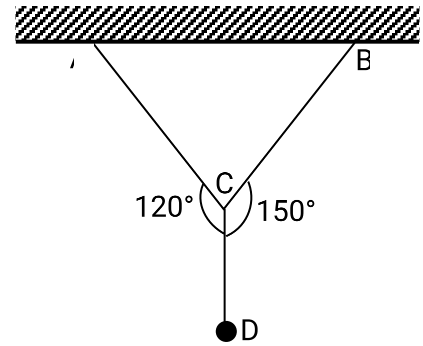
$$W_2 = \frac{9\times(2\cdot5\text{-}1\cdot5)^2}{2\times1\cdot5} = 3 \text{ J}$$

$$\Delta W = W_2 - W_1$$

$$= 3-0\cdot75 = 2\cdot25 \text{ J}$$

Example 6

AC, BC and CD are identical strings of different lengths.

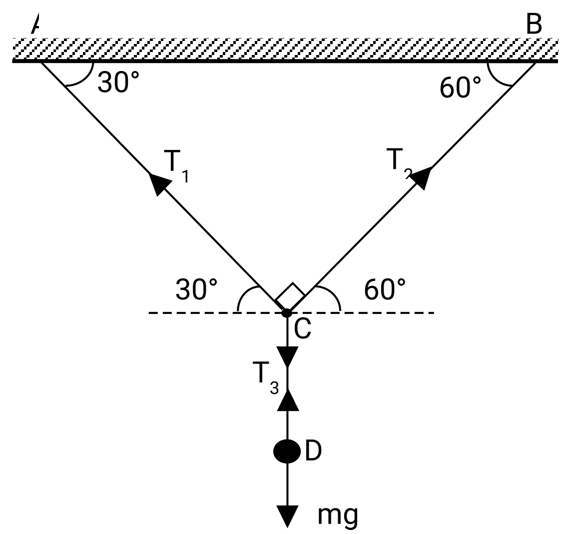


AC and CD have natural lengths l_0 and l respectively and modulus of elasticity of the strings is 3mg. A particle of mass m rests at D.

(a) Show that the equilibrium length of AC is $\frac{7}{6}l_0$ and D rests at a distance $\frac{1}{12}(7l_0 + 16l)$ below the support.

(b) Find the natural length BC.

Solution



(a) In equilibrium:

For D: $T_3 = mg$

Equilibrium of forces at point C:

Resolving horizontally:

$$T_1\cos 30 = T_2\cos 60$$

$$T_1\times\frac{\sqrt{3}}{2} = T_2\times\frac{1}{2}$$

$$T_2 = T_1\sqrt{3} \quad \text{.......................... (i)}$$

Resolving vertically:

$$T_1\sin 30 + T_2\sin 60 = T_3$$

$$\frac{1}{2}T_1 + \frac{\sqrt{3}}{2}T_2 = mg$$

$$T_1 + \sqrt{3}T_2 = 2mg \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

From equation (i) and equation (ii)

$$T_1 + \sqrt{3}\left(\sqrt{3}T_1\right) = 2mg$$

$$T_1 = \frac{1}{2}mg$$

$$T_2 = \frac{\sqrt{3}}{2}mg$$

For string AC; let l_1 be the length of AC in equilibrium;

From Hooke's law:

$$T_1 = \frac{\lambda x}{l_0}$$

$$\therefore \frac{1}{2}mg = \frac{3mg\left(l_1 - l_0\right)}{l_0} \Rightarrow \frac{1}{6}l_0 = l_1 - l_0$$

$$l_1 = \frac{7}{6}l_0$$

For CD: Let l' be the length of CD in equilibrium;

From Hooke's Law:

$$T_3 = \frac{\lambda x}{l}$$

$$\therefore mg = \frac{3mg\left(l' - l\right)}{l}$$

$$\frac{1}{3}l = l' - l$$

$$l' = \frac{4}{3}l$$

$$\text{Distance below AB} = \frac{7}{6}l_0 \sin 30 + \frac{4}{3}l$$

$$= \frac{7}{12}l_0 + \frac{4}{3}l$$

$$= \frac{1}{12}\left(7l_0 + 16l\right)$$

(b) For BC:

Let l_2 be the natural length of BC and l_2' the length in equilibrium position

$$\tan 60 = \frac{AC}{BC}$$

$$BC = \frac{AC}{\tan 60}$$

$$l_2' = \frac{7}{6}l_0 \div \tan 60$$

$$l_2' = \frac{7\sqrt{3}}{18}l_0$$

From Hooke's law

$$T_2 = \frac{\lambda x}{l_2}$$

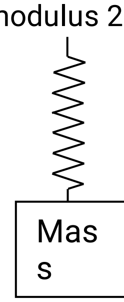
$$\frac{\sqrt{3}}{2}mg = \frac{3mg\left(l_2' - l_2\right)}{l_2} \Rightarrow \frac{\sqrt{3}}{6}l_2 = l_2' - l_2$$

$$l_2\left(\sqrt{3} + 6\right) = \frac{7\sqrt{3}}{3}l_0$$

$$l_2 = \frac{7\sqrt{3}l_0}{3\left(\sqrt{3} + 6\right)} \Rightarrow l_2 = \frac{7l_0}{3\left(1 + 2\sqrt{3}\right)}$$

Example 7

The diagram shows a body of mass 10 kg freely suspended from a light elastic spring of natural length 0・5 m and modulus 25g N.



Find the extension in the spring when the:

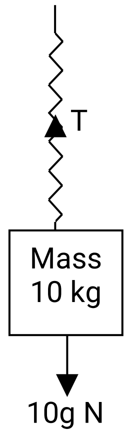
(a) body is at rest.

(b) top of the spring is moved upwards with constant velocity.

(c) top of the spring is moved upwards with a constant acceleration of 4•9 m s^{-2}.

(d) top of the spring is moved downwards with a constant acceleration of 4•9 m s^{-2}.

Solution



$$T = \frac{\lambda x}{l_0}$$

(a) $T = 10g \Rightarrow T = 10\times9\cdot8 \Rightarrow T = 98\text{ N}$

$$\therefore 98 = \frac{25\times9\cdot8\times x}{0\cdot5}$$

$x = 0\cdot2\text{ m}$

(b) $T-10g = 10\times0 \Rightarrow T = 10\times9\cdot8$

$\Rightarrow T = 98\text{ N}$

$$\therefore 98 = \frac{25\times9\cdot8\times x}{0\cdot5}$$

$x = 0\cdot2\text{ m}$

(c) $T-10g = 10\times4\cdot9$

$T = 10(9\cdot8+4\cdot9) = 147\text{ N}$

$$\therefore 147 = \frac{25\times9\cdot8\times x}{0\cdot5}$$

$x = 0\cdot3\text{ m}$

(d) $10g-T = 10\times4\cdot9$

$T = 10(9\cdot8-4\cdot9) = 49\text{ N}$

$$\therefore 49 = \frac{25\times9\cdot8\times x}{0\cdot5}$$

$x = 0\cdot1\text{ m}$

Exercises

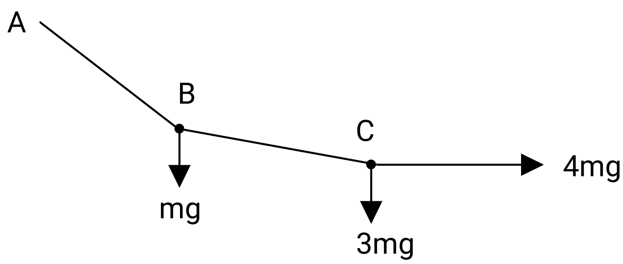
Exercise: 14A

1. An elastic string of natural length 2 m is stretched to $2\cdot5$ m. If its modulus of elasticity is 20 N, find the tension in the string.
2. A spring of natural length 60 cm is stretched to 1 m. If the tension in the spring is 6 N, find the modulus of elasticity.
3. A spring of natural length 10 cm is compressed to a length of 8 cm. If its modulus of elasticity is 12 N, find the thrust in the spring.
4. When the tension in an elastic string is 8 N, its length is $1\cdot6$ m. If the modulus of elasticity of the string is 24 N, find its natural length.
5. An elastic string of natural length 3a is fixed at one end and a particle of weight W is attached to the other end. When the particle hangs freely in equilibrium, the length of the string is 5a. If the string is held at an angle θ to the vertical by a horizontal force of magnitude $\frac{1}{2}W$, find the value of θ and prove that the new length of the string is $a\left(3+\sqrt{5}\right)$.
6. A particle of mass 3 kg is suspended from an elastic string of natural length $0\cdot5$ m and modulus 48 N. If the particle is pulled vertically downwards and then released when the length of the string is 1 m, find its acceleration at the instant that it is released.
7. An elastic string AB of natural length $1\cdot2$ m and modulus 10 N lies along a line of greatest slope of a smooth plane inclined at 30° to the horizontal. The end A is fixed and to the end B is attached a particle of weight 10 N. Find the length of the string when the particle at B rests in equilibrium on the plane.
8. The ends of an elastic string of natural length 4a are fixed to points A and B on the same horizontal level, where AB = 3a. A particle P of weight W is attached to the midpoint of the string and hangs in equilibrium at a depth of 2a below the level AB. Find the modulus of elasticity of

the string in terms of W.

9. A spring PQ of natural length 1·5 m and modulus λ N is fixed at P. The other end is joined to a second spring QR of natural length 1 m and modulus 2λ N. A particle of weight 15 N is then attached to the end R of the second spring. When the system is hanging freely in equilibrium the distance PR is 4 m. Find the value of λ.

10. A particle of mass 3m is tied to the end C and a particle of mass m is tied at the midpoint B of a light un-stretched elastic string ABC. The end A of the string is fixed and a horizontal force of magnitude 4mg is applied to the particle at C so that the system hangs in equilibrium as shown below.



Calculate the:

(i) tensions in AB and BC.

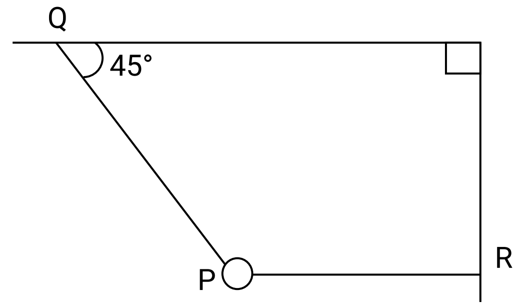
(ii) inclinations of AB and BC to the vertical.

(iii) Given that the modulus of elasticity of the string is 6mg, show that for this position of equilibrium AB:BC = $\left(6+4\sqrt{2}\right)$:11.

11. Find the work done in stretching a spring of natural length 1 m and modulus 10 N to a length of:

(a) 1·2 m (b) 2 m

12.

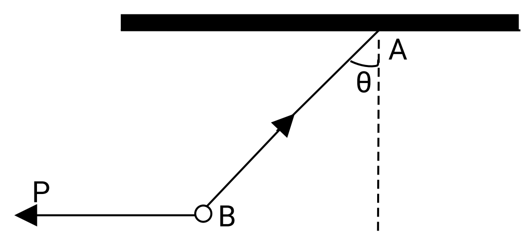


The diagram above shows a particle P of mass 1·6 kg which is attached to a light inextensible string PQ and to a string PR of natural length 12 cm and modulus of elasticity 40 N. The string PR is horizontal and the string PQ makes an angle of 45° with the horizontal. Find the:

(i) tension in the string PQ.

(ii) extension of the string PR.

13. In the diagram below AB is an elastic string of natural length 1 m and modulus 147 N, P is the horizontal force applied onto body B of mass $5\sqrt{3}$ kg. At equilibrium θ = 30°.



Find the:

(i) magnitude of P.

(ii) extension in the string AB.

Exercise: 14B

1. Find the energy stored in an elastic string of natural length 4 m and modulus 2 N when its length is:

(a) 5 m (b) 10 m

2. The work done in stretching an elastic string of natural length 2·5 m from 3 m to 4 m in length is 6 J. Find the modulus of the string.

3. When a mass of 5 kg is freely suspended from one end of a light elastic string the other end of which is fixed, the string extends to twice its natural length. Find the modulus of

the string.

4. The work done in extending a spring from its natural length 6a to a length 7a is ka. Find the modulus of the spring in terms of k and the work done in extending the length of the spring from 7a to 8a.

5. A particle of mass 8 kg is suspended from a fixed point by a spring of natural length 0•5 m and modulus 140 N. If the particle is released from rest with the spring vertical and un-stretched, find the distance it falls before coming to rest instantaneously.

6. A particle is suspended from a fixed point A by a light elastic string of natural length 2 m. When the particle hangs in equilibrium the length of the string is 2•5 m. Given that the particle is now released from rest at A, find the distance it has fallen when it first comes to rest.

7. An elastic string of natural length 0•6 m is stretched by 8 cm by a mass of 1 kg hanging on it. Determine the work done in stretching it from 0•65 m to 0•7 m.

8. A smooth surface inclined at 30° to the horizontal has a body of mass 6 kg held at rest on the surface by a light elastic string attached to a fixed point 2•5 m from the particle on the line of greatest slope, the particle being below the fixed point. If the modulus of elasticity of the string is 3g N, find:
   (i) its natural length.
   (ii) the extension in this position.

9. A body of mass m lies on a smooth horizontal surface and is connected to a point O on the surface by a light elastic string of natural length l and modulus λ. When the body moves with constant speed v around a horizontal path, centre at O, the extension in the string is $\frac{1}{4}l$, show that $\lambda = \frac{16mv^2}{5l}$.

10. An elastic string is of natural length 4 m and modulus 24 N. Find the work that must be done to stretch the string from a length of 5 m to a length of 6 m.

11. A light elastic string is of natural length 50 cm and modulus 147 N. One end of the string is attached to a fixed point and a body of mass 3 kg is freely suspended from the other end. Find the:
   (a) extension of the string in the equilibrium position.
   (b) energy stored in the string.

12. A spring is of natural length 50 cm and modulus 60 N. How much energy is released when the length of the spring is reduced from 1•5 m to 1 m?

Answers to exercises

Exercise: 14A

1. 5 N 2. 9 N 3. 2•4 N 4. 1•2 m 5. 26•6° 6. 6•2 m s^{-2}(upwards) 7. 1•8 m

8. $\frac{5}{2}$W 9. 20 10. (i) $4\sqrt{2}$mg; 5mg (ii) 45° ; 53•1° (iii)

11. (a) 0•2 J (b) 5 J 12. (i) 22•175 N (ii) 0•047 m

13. (i) 49 (ii) $\frac{2}{3}$ m

Exercise: 14B

1. (a) 0•25 J (b) 9 J 2. 15 N 3. 49 N **4.** 12k ; 3ka **5.** 0•56 m **6.** 4 m **7.** 0•46 J 8. (i) 1•25 m (ii) 1•25 m

9. 10. 9 J 11. (a) 0•1 m (b) 1•47 J 12. 45 J

15. USE OF CALCULUS

For uniformly accelerated motion the following equations are used:

$$v = u + at$$

$$s = ut + \frac{1}{2}at^2$$

$$v^2 = u^2 + 2as$$

However when the acceleration is variable, differentiation and integration are employed.
Velocity = Rate of change of displacement

$$v = \frac{ds}{dt} = \dot{s}$$

From $v = \frac{ds}{dt}$

$$s = \int v\,dt + \text{ constant}$$

Acceleration = Rate of change of velocity

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2} = \ddot{s}$$

Note: From $a = \frac{dv}{dt} \Rightarrow v = \int a\,dt + \text{ constant}$

15.1 Motion in i–j plane

For motion with non-uniform acceleration in **i**–**j** plane we employ calculus to each direction separately.

If **s** = x**i** + y**j** or $\mathbf{s} = \begin{pmatrix} x \\ y \end{pmatrix}$, where x and y are functions of time.

Then $\mathbf{v} = \dot{x}\,\mathbf{i} + \dot{y}\,\mathbf{j}$ or $\mathbf{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$

and $\mathbf{a} = \ddot{x}\,\mathbf{i} + \ddot{y}\,\mathbf{j}$ or $\mathbf{a} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix}$

15.2 Motion in i–j–k space

If **s** = x**i** + y**j** + z**k** or $\mathbf{s} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ where x, y and z are functions of time then; $\mathbf{v} = \dot{x}\,\mathbf{i} + \dot{y}\,\mathbf{j} + \dot{z}\,\mathbf{k}$ or $\mathbf{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$ and $\mathbf{a} = \ddot{x}\,\mathbf{i} + \ddot{y}\,\mathbf{j} + \ddot{z}\,\mathbf{k}$ or $\mathbf{a} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix}$.

15.3 Velocity as a function of displacement

From $a = \frac{dv}{dt} = \frac{dv}{ds} \times \frac{ds}{dt} = v\frac{dv}{ds}$ since $v = \frac{ds}{dt}$

The above two equations are employed when velocity is a function of displacement.

15.3.1 Average velocity of a particle moving along a curved path

If a particle passes through a point with position vector **r**(t_1) at time t_1 and through **r**(t_2) at time t_2, then average velocity $\mathbf{v} = \frac{\Delta r}{\Delta t}$.

Hence average velocity is given by:

$$\mathbf{v} = \frac{r(t_2) - r(t_1)}{t_2 - t_1}.$$

Example 1

The position vector of a particle of mass 2 kg is given as $\mathbf{r} = t^3\mathbf{i} + (t^2+5t)\mathbf{j}$ m. Determine:

(i) its average velocity in the time interval t = 1 to t = 2 s.

(ii) its speed at t = 2 s.

(iii) the power developed at t = 2 s.

Solution

(i) Average velocity $= \frac{\Delta r}{\Delta t} = \frac{r_2 - r_1}{t_2 - t_1}$

When t = 1 s ; r_1 = **i** + 6**j** ; when t = 2 s ; r_2 = 8**i** + 14**j**

Average velocity $= \frac{(8i+14j)-(i+6j)}{2-1}$

$= (7i+8j)\text{ m s}^{-1}$

(ii) $\mathbf{v} = \frac{dr}{dt} \Rightarrow \mathbf{v} = 3t^2\mathbf{i} + (2t+5)\mathbf{j}$; at t = 2 s ;

$\mathbf{v} = 12\mathbf{i} + 9\mathbf{j}$

$|\mathbf{v}| = \sqrt{12^2+9^2} \Rightarrow |\mathbf{v}| = 15 \text{ m s}^{-1}$

(iii) $\mathbf{a} = \frac{dv}{dt} = \frac{d}{dt}(3t^2\mathbf{i}+(2t+5)\mathbf{j})$

$\mathbf{a} = 6t\mathbf{i} + 2\mathbf{j}$

When t = 2 s; $\mathbf{a} = (6\times2)\mathbf{i} + 2\mathbf{j}$

$\Rightarrow \mathbf{a} = (12\mathbf{i}+2\mathbf{j}) \text{ m s}^{-2}$

$\mathbf{F} = m\mathbf{a} \Rightarrow \mathbf{F} = 2(12\mathbf{i}+2\mathbf{j}) = (24\mathbf{i}+4\mathbf{j}) \text{ N}$

When t = 2 s ; $\mathbf{v} = (12\mathbf{i}+9\mathbf{j}) \text{ m s}^{-1}$

$P = \mathbf{F}\bullet\mathbf{v} \Rightarrow P = (24\mathbf{i}+4\mathbf{j})\bullet(12\mathbf{i}+9\mathbf{j})$

$P = 24\times12 + 4\times9 = 324 \text{ W}$

15.3.2 Average acceleration of a particle along a curved path

If a particle has a velocity $\mathbf{v}(t_1)$ at time t_1 and a velocity $\mathbf{v}(t_2)$ at time t_2 then its average acceleration in the time interval t_1 to t_2 is given by:

average acceleration = $\frac{\Delta v}{\Delta t}$.

Hence average acceleration is given by: $\mathbf{a} = \frac{v(t_2)-v(t_1)}{t_2-t_1}$.

Example 2

The position vector of a particle at any time t s, is $\mathbf{r} = 4t^2\mathbf{i} + \frac{3}{\pi}\sin \pi t\mathbf{j}$ m. Find the magnitude of the average acceleration of the particle in the time interval t = 1 to t = 2 s.

Solution

$\mathbf{v} = \frac{dr}{dt} = \frac{d}{dt}\left(4t^2\mathbf{i}+\frac{3}{\pi}\sin \pi t\mathbf{j}\right)$

$\Rightarrow v = 8t\mathbf{i} + 3\cos \pi t\mathbf{j}$

When t = 1 s ; $v_1 = 8\mathbf{i} + 3\cos \pi\mathbf{j} \Rightarrow v_1 = 8\mathbf{i} - 3\mathbf{j}$

When t = 2 s ; $v_2 = 16\mathbf{i} + 3\cos 2\pi\mathbf{j}$

$\Rightarrow v_2 = 16\mathbf{i} + 3\mathbf{j}$

Average acceleration, $\mathbf{a} = \frac{\Delta v}{\Delta t} = \frac{v_2-v_1}{t_2-t_1}$

$\mathbf{a} = \frac{(16i+3j)-(8i-3j)}{2-1}$

$\mathbf{a} = (8i+6j) \text{ m s}^{-2}$

$|a| = \sqrt{8^2+6^2} \Rightarrow |a| = 10 \text{ m s}^{-2}$

15.4 Variable force

A variable force acting on a body in a constant direction produces a variable acceleration.

From $\mathbf{F} = m\mathbf{a}$

The acceleration $\mathbf{a}$ is

$\mathbf{a} = \frac{dv}{dt} = \frac{dv}{ds}\times\frac{ds}{dt} = v\frac{dv}{ds}$

Depending on the given information we can employ calculus to find the solution to the given problem.

Example 3

A particle of mass 2 kg initially at rest at (0,0,0) is acted upon by the force $\begin{pmatrix}2t\\t\\3t\end{pmatrix}$ N. Find the:

(i) acceleration at any time t.
(ii) velocity after 3 seconds.
(iii) distance travelled after 3 seconds.

Solution

(i) $2\mathbf{a} = \begin{pmatrix}2t\\t\\3t\end{pmatrix} \Rightarrow \mathbf{a} = \begin{pmatrix}t\\\frac{1}{2}t\\\frac{3}{2}t\end{pmatrix} \text{ m s}^{-2}$

(ii) $\mathbf{v} = \int\begin{pmatrix}t\\\frac{1}{2}t\\\frac{3}{2}t\end{pmatrix}dt \Rightarrow \mathbf{v} = \begin{pmatrix}\frac{1}{2}t^2\\\frac{1}{4}t^2\\\frac{3}{4}t^2\end{pmatrix} + c_1$

At t = 0 ; $\mathbf{v} = \mathbf{0} \Rightarrow c_1 = \mathbf{0}$

Hence $\mathbf{v} = \begin{pmatrix}\frac{1}{2}t^2\\\frac{1}{4}t^2\\\frac{3}{4}t^2\end{pmatrix}$

At t = 3 s ; $\mathbf{v} = \begin{pmatrix}\frac{1}{2}\times3^2\\\frac{1}{4}\times3^2\\\frac{3}{4}\times3^2\end{pmatrix} \Rightarrow \mathbf{v} = \begin{pmatrix}4\cdot5\\2\cdot25\\6\cdot75\end{pmatrix}$

(iii) $\mathbf{r} = \int \begin{pmatrix} \frac{1}{2}t^2 \\ \frac{1}{4}t^2 \\ \frac{3}{4}t^2 \end{pmatrix} dt \Rightarrow \mathbf{r} = \begin{pmatrix} \frac{1}{6}t^3 \\ \frac{1}{12}t^3 \\ \frac{1}{4}t^3 \end{pmatrix} + \mathbf{c}_2$

When $t = 0$; $\mathbf{r} = \mathbf{0} \Rightarrow \mathbf{c}_2 = \mathbf{0}$

$$\text{Hence } \mathbf{r} = \begin{pmatrix} \frac{1}{6}t^3 \\ \frac{1}{12}t^3 \\ \frac{1}{4}t^3 \end{pmatrix}$$

$$\text{When } t = 3 \text{ s} ; \mathbf{r}(3) = \begin{pmatrix} \frac{1}{6}\times 3^3 \\ \frac{1}{12}\times 3^3 \\ \frac{1}{4}\times 3^3 \end{pmatrix}$$

$$\Rightarrow \mathbf{r}(3) = \begin{pmatrix} 4 \cdot 5 \\ 2 \cdot 25 \\ 6 \cdot 75 \end{pmatrix}$$

$$|\mathbf{r}(3)| = \sqrt{4 \cdot 5^2 + 2 \cdot 25^2 + 6 \cdot 75^2}$$

$$\Rightarrow |\mathbf{r}(3)| = 8 \cdot 4187 \text{ m}$$

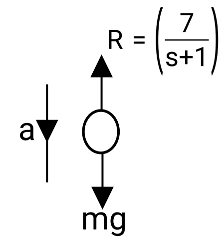
15.5 Motion in a resistive medium

When motion takes place in a resistive medium, we apply Newton's second law to obtain the equation of motion for the body.

Example 4

A body of mass 1 kg is released from rest and falls under gravity against a resistance of $\frac{7}{s+1}$ N, where s is the distance (in metres) that the body has fallen since release. Find the speed of the body when it has fallen a distance of $6 \cdot 4$ m. (Take $g = 10 \text{ m s}^{-2}$)

Solution



$$mg - \frac{7}{s+1} = ma \text{ ; } m = 1 \text{ kg ; } g = 10 \text{ m s}^{-2}$$

$$a = 10 - \frac{7}{s+1}$$

$$\text{But } a = \frac{dv}{dt} = \frac{dv}{ds}\times\frac{ds}{dt} = v\frac{dv}{ds}$$

$$v\frac{dv}{ds} = 10 - \frac{7}{s+1} \Rightarrow \int v\,dv = \int\left(10 - \frac{7}{s+1}\right)ds$$

$$\frac{1}{2}v^2 = 10s - 7\ln(s+1) + c$$

At $t = 0, v = 0, s = 0$

$$\Rightarrow 0 = 10\times 0 - 7\ln(1) + c \Rightarrow c = 0$$

$$\frac{1}{2}v^2 = 10s - 7\ln(s+1)$$

$$\Rightarrow v = \sqrt{2[10s - 7\ln(s+1)]}$$

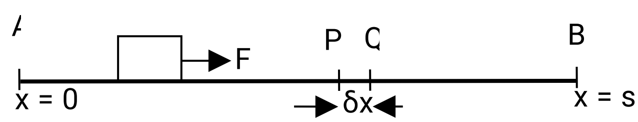
When $s = 6 \cdot 4$ m ;

$$v = \sqrt{2[10\times 6 \cdot 4 - 7\ln(6 \cdot 4 + 1)]} \Rightarrow v = 10 \text{ m s}^{-1}$$

15.6 Work done by a variable force

15.6.1 Variable force acting in a straight line (constant direction)

Consider a variable force **F** which acts on a body and moves it along a straight line from point A to point B through a displacement s.



The work done δW in moving the body from P to Q through a small displacement δx is $\delta W \simeq \mathbf{F}\delta x$.

The total work done by **F** in moving the body from A to B is $W = \int_{x=0}^{x=s} F\,dx$.

Example 5

A particle of mass 2 kg is acted upon by a force $24t^2\mathbf{i} + (36t-6)\mathbf{j} - 12t\mathbf{k}$. Initially the particle is at a point (3,-4,4) and moving with velocity $16\mathbf{i} + 15\mathbf{j} - 8\mathbf{k}$. Find the:

(a) speed of the particle after 2 seconds.

(b) distance covered by the particle in the first 2 seconds.

(c) rate of doing work when t = 2 seconds.

Solution

(a) $2\mathbf{a} = 24t^2\mathbf{i} + (36t-6)\mathbf{j} - 12t\mathbf{k}$

$\Rightarrow \mathbf{a} = 12t^2\mathbf{i} + (18t-3)\mathbf{j} - 6t\mathbf{k}$

$\mathbf{v} = \int[12t^2\mathbf{i}+(18t-3)\mathbf{j}-6t\mathbf{k}]dt \Rightarrow \mathbf{v}=4t^3\mathbf{i}+(9t^2-3t)\mathbf{j}-3t^2\mathbf{k}+c_1$

At t = 0

$\mathbf{v} = 16\mathbf{i} + 15\mathbf{j} - 8\mathbf{k} \Rightarrow c_1 = 16\mathbf{i} + 15\mathbf{j} - 8\mathbf{k}$

$\therefore \mathbf{v} = (4t^3+16)\mathbf{i} + (9t^2-3t+15)\mathbf{j} - (3t^2+8)\mathbf{k}$

When t = 2 s

$\mathbf{v} = (4\times2^3+16)\mathbf{i} + (9\times2^2-3\times2+15)\mathbf{j} - (3\times2^2+8)\mathbf{k}$

$\mathbf{v} = 48\mathbf{i} + 45\mathbf{j} - 20\mathbf{k}$

$|\mathbf{v}| = \sqrt{48^2+45^2+(-20)^2} \Rightarrow |\mathbf{v}| = 68\cdot7677$ m s^{-1}

(b) $\mathbf{r} = \int[(4t^3+16)\mathbf{i}+(9t^2-3t+15)\mathbf{j}-(3t^2+8)\mathbf{k}]dt$

$\mathbf{r} = (t^4+16t)\mathbf{i} + \left(3t^3-\frac{3}{2}t^2+15t\right)\mathbf{j} - (t^3+8t)\mathbf{k} + c_2$

At t = 0 ; $\mathbf{r} = 3\mathbf{i} - 4\mathbf{j} + 4\mathbf{k} \Rightarrow c_2 = 3\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$

$\mathbf{r}(t) = (t^4+16t+3)\mathbf{i} + \left(3t^3-\frac{3}{2}t^2+15t-4\right)\mathbf{j} + (4-8t-t^3)\mathbf{k}$

When t = 2 s;

$\mathbf{r}(2) = (2^4+16\times2+3)\mathbf{i} + \left(3\times2^3-\frac{3}{2}\times2^2+15\times2-4\right)\mathbf{j} + (4-8\times2-2^3)\mathbf{k}$

$\mathbf{r}(2) = 51\mathbf{i} + 14\mathbf{j} - 20\mathbf{k}$

$\mathbf{s} = \mathbf{r}(2) - \mathbf{r}(0) = (51\mathbf{i}+14\mathbf{j}-20\mathbf{k}) - (3\mathbf{i}-4\mathbf{j}+4\mathbf{k})$

$\mathbf{s} = 48\mathbf{i} + 18\mathbf{j} - 24\mathbf{k}$

$|\mathbf{s}| = \sqrt{48^2+18^2+(-24)^2} \Rightarrow |\mathbf{s}| = 56\cdot604$ m

(c) When t = 2

$\mathbf{F} = 24\times2^2\mathbf{i} + (36\times2-6)\mathbf{j} - 12\times2\mathbf{k}$

Hence $\mathbf{F} = 96\mathbf{i} - 66\mathbf{j} - 24\mathbf{k}$

$\mathbf{v} = 48\mathbf{i} + 45\mathbf{j} - 20\mathbf{k}$

Rate of doing work = Power

$P = \mathbf{F}\bullet\mathbf{v}$

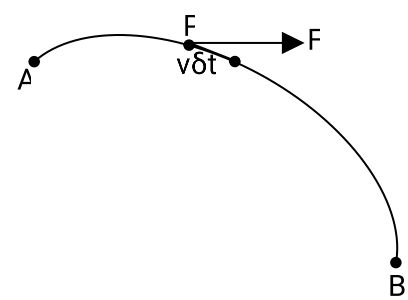
$\Rightarrow P = (96\mathbf{i}-66\mathbf{j}-24\mathbf{k})\bullet(48\mathbf{i}+45\mathbf{j}-20\mathbf{k})$

$= 96\times48 - 66\times45 - 24\times-20$

= 2118 W

15.6.2 Variable force acting along a curved path

Consider a variable **F** acting on a particle P and moves it from A to B.



If the velocity at P is **v** then $\vec{PQ} \simeq \mathbf{v}\delta t$, work done by **F** to move the particle from P to Q is $\delta W \simeq \mathbf{F}\bullet\vec{PQ}$.

$\delta W \simeq \mathbf{F}\bullet(\mathbf{v}\delta t) = (\mathbf{F}\bullet\mathbf{v})\delta t$ ………………… (i)

The rate at which the force is working when the particle is at P is: $\lim_{\delta t\to0}\frac{(\mathbf{F}\bullet\mathbf{v})\delta t}{\delta t} = \mathbf{F}\bullet\mathbf{v}$.

Hence for a variable for **F**(t) acting on a particle and causing it to move with velocity **v**(t), the power developed by the force at time, t is P = **F•v**.

From equation (i), the total work done over AB is:

$W = \lim_{\delta t\to0}\sum_{t_A}^{t_B}(\mathbf{F}\bullet\mathbf{v})\delta t$,

where t_A and t_B are times, when the particle is at A and B respectively.

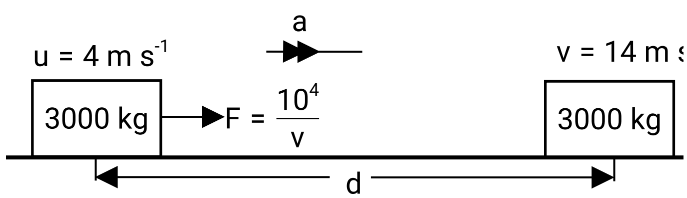
By using calculus we obtain $W = \int_{t_A}^{t_B}(\mathbf{F}\bullet\mathbf{v})dt$

Hence for a variable force **F**(t) acting on a particle and causing it to move with velocity **v**(t) the total work done by the force in the time interval from t_A to t_B is given by: $W = \int_{t_A}^{t_B}(\mathbf{F}\bullet\mathbf{v})dt$.

Example 6

A car of mass 3000 kg developing a constant power of 10 kW increases its speed from 4 m s^{-1} to 14 m s^{-1} covering a distance d, taking T seconds in the process. Find T and show that d = 268 m.

Solution



$$\frac{10^4}{v} = 3000a, \text{ but } a = \frac{dv}{dt}$$

$$\frac{10^4}{v} = 3000\frac{dv}{dt}$$

$$\int_0^T \frac{10}{3}dt = \int_4^{14} v\,dv$$

$$\left[\frac{10}{3}t\right]_0^T = \left[\frac{1}{2}v^2\right]_4^{14}$$

$$\frac{10T}{3} = \frac{1}{2}(14^2 - 4^2)$$

$$T = 27 \text{ s}$$

$$\text{Also } \frac{10^4}{v} = 3000a, \text{ but } a = \frac{dv}{dt} = v\frac{dv}{dx}$$

$$\frac{10^4}{v} = 3000v\frac{dv}{dx}$$

$$\int_0^d \frac{10}{3}dx = \int_4^{14} v^2 dv$$

$$\left[\frac{10}{3}x\right]_0^d = \left[\frac{1}{3}v^3\right]_4^{14}$$

$$\frac{10d}{3} = \frac{1}{3}(14^3 - 4^3)$$

$$d = 268 \text{ m}$$

Example 7

A particle moving with an acceleration given by $\mathbf{a} = 4e^{-3t}\mathbf{i} + 12\sin t\,\mathbf{j} - 7\cos t\,\mathbf{k}$ is located at the point (5,-6,2) and has velocity $\mathbf{v} = 11\mathbf{i} - 8\mathbf{j} + 3\mathbf{k}$ at time t = 0. Find the:

(i) magnitude of acceleration when t = 0.

(ii) velocity at any time t.

(iii) displacement at any time t.

Solution

(i) At t = 0; $\mathbf{a} = 4e^{-3\times0}\mathbf{i} + 12\sin 0\mathbf{j} - 7\cos 0\mathbf{k}$

$$\mathbf{a} = 4\mathbf{i} - 7\mathbf{k} \Rightarrow |a| = \sqrt{4^2+(-7)^2}$$

$$\Rightarrow |a| = \sqrt{65} = 8\cdot062 \text{ m s}^{-2}$$

(ii) $\mathbf{v} = \int[4e^{-3t}\mathbf{i}+12\sin t\mathbf{j}-7\cos t\mathbf{k}]dt$

$$\mathbf{v} = -\frac{4}{3}e^{-3t}\mathbf{i} - 12\cos t\mathbf{j} - 7\sin t\mathbf{k} + c_1$$

At t = 0; $11\mathbf{i} - 8\mathbf{j} + 3\mathbf{k} = -\frac{4}{3}\mathbf{i} - 12\mathbf{j} + c_1$

$$\Rightarrow c_1 = \frac{37}{3}\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$$

Hence

$$\mathbf{v} = \frac{1}{3}(37-4e^{-3t})\mathbf{i} + (4-12\cos t)\mathbf{j} + (3-7\sin t)\mathbf{k}$$

(iii) $\mathbf{r} = \int\left[\frac{1}{3}(37-4e^{-3t})\mathbf{i}+(4-12\cos t)\mathbf{j}+(3-7\sin t)\mathbf{k}\right]dt$

$$\mathbf{r} = \left[\frac{1}{3}\left(37t+\frac{4}{3}e^{-3t}\right)\mathbf{i}+(4t-12\sin t)\mathbf{j}+(3t+7\cos t)\mathbf{k}\right] + c_2$$

At t = 0 ; $5\mathbf{i} - 6\mathbf{j} + 2\mathbf{k} = \frac{4}{9}\mathbf{i} + 7\mathbf{k} + c_2$

$$\Rightarrow c_2 = \frac{41}{9}\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}$$

$\therefore$

$$\mathbf{r}(t) = \left[\frac{1}{3}\left(\frac{41}{3}+37t+\frac{4}{3}e^{-3t}\right)\mathbf{i}+(4t-6-12\sin t)\mathbf{j}+(3t-5+7\cos t)\mathbf{k}\right]$$

Displacement at any time, t : $\mathbf{s} = \mathbf{r}(t) - \mathbf{r}(0)$

$$\mathbf{s} = \left[\frac{1}{3}\left(\frac{41}{3}+37t+\frac{4}{3}e^{-3t}\right)\mathbf{i}+(4t-6-12\sin t)\mathbf{j}+(3t-5+7\cos t)\mathbf{k}\right] - [5\mathbf{i}-6\mathbf{j}+2\mathbf{k}]$$

$$\mathbf{s} = \left[\frac{1}{3}\left(37t+\frac{4}{3}(e^{-3t}-1)\right)\mathbf{i}+4(t-3\sin t)\mathbf{j}+(3t-7(1-\cos t))\mathbf{k}\right]$$

Exercises

Exercise: 15A

1. A particle of mass m moves so that its

position vector at time t is **r**. Find its velocity **v** and force **F** acting on it given that;

(a) **r** = $4t^2$**i** + $(3t^3-2t)$**j**

(b) **r** = 5**i**$-2t^3$**j** + (t^2-1)**k**

(c) **r** = $(2t-t^2)$**i** + (3sin 2t)**j**

(d) **r** = te^{-t}**i**$-2t$**j**

2. 2. A particle of mass 10 kg moves such that its position vector after time t seconds is **r** = (cos 2t)**i** + (4sin 2t + 3)**j**. Find the:

(a) speed of the particle when $t = \frac{\pi}{3}$.

(b) force acting on the particle when $t = \frac{\pi}{2}$.

3. 3. A particle of mass 4 kg starts from rest at a point with position vector 2**i**−3**j** and moves under the action of a constant force, **F** = 8**i** + 20**j** N. Find the velocity and position vector of the particle after 5 seconds.

4. At time t the force acting on a particle of mass 3 kg is **F** = 6**i** - $3t^2$**j** + 54t**k**. At the time t = 0 the particle is at a point with position vector **i**−5**j**−**k** and its velocity is 3**i** + 3**j**. Find the position vector of the particle at time t = 1.

5. 5. A particle of mass 2 kg is acted upon at time t by a force **F** = 8**i**−4cos t**j** + 2t**k**. When t = 0, the velocity of the particle is 6**i**. Find an expression for the velocity of the particle at time t.

6. A particle moves such that its position vector at any time t is given **r** = $(3t^2-1)$**i** + $(4t^3 + t-1)$**j**. Find the:

(a) speed,

(b) magnitude of acceleration, at t = 2.

7. An object of mass 5 kg is initially at rest at a point with position vector -2**i** + **j**. If it is acted on by a force **F** = 2**i** + 3**j**−4**k**, find its:

(i) acceleration.

(ii) velocity after 3 s.

(iii) distance from the origin after 3 s.

8. The position vector **r** of a particle in motion is given by **r** = 5**i** + t^3**j** + t^2**k**, where t is the time in seconds. Determine the:

(i) velocity and acceleration of the particle at any time.

(ii) speed at time t = 4 seconds.

9. A particle of mass 3 kg is acted upon at time t by a force **F** = 6**i**−$36t^2$**j** + 54t**k**. At time t = 0, the particle is at a point with position vector **i**−5**j**−**k** and its velocity is 3(**i** + **j**). Find the position vector of the particle any time t.

10. A body of mass 3 kg moves so that its position vector at time t is given by **r** = t^2**i** + sin t**j** + cos t**k**.

(i) Find the magnitude of the resultant force.

(ii) Show that the velocity is initially perpendicular to the direction of motion of the body.

Exercise: 15B

1. The position vector of a particle is given by **r** = sin $\frac{\pi}{6}$t**i** + cos $\frac{\pi}{6}$t**j** + t^2**k**. Find the speed and magnitude of acceleration of the particle after 3 s.

2. The velocity of a particle at any time t is v(t) = -asin ωt + bωcos ωt. Find an expression for the displacement x at any time, given that x = 0 when time t = 0.

3. A force **F** = t^2**i** + 3t**j** + 4**k** acts on a body of mass 2 kg. Initially the body is at rest at a point 2**i** - **j** + **k**. Find the:

(i) speed of the body after 5 s.

(ii) distance of the body from the origin after 2 s.

(iii) work done by **F** in the interval from t = 0 to t = 4 s.

4. The force acting on a 4 kg particle is F = (5+4t) N where t is the time in seconds. If it is initially moving at a speed of 5 m s^{-1}, find its speed after 3 seconds.

5. A particle starting with an initial velocity 3**i**−6**j**−3**k** m s^{-1} has an acceleration of 6t**i** + 6**j**−2**k** at any time t. Find the velocity and speed of the particle at t = 2 s.

6. A resultant force **F** = [(4t−1)**i** + 4**j**)] N acts on a particle of mass 0•5 kg initially at rest at an origin O. Calculate the speed of the particle after 2 seconds.

7. A body starts from the origin with a velocity of 3 m s^{-1} and an acceleration given by a = 6t−4. Find the velocity and displacement at time t.

8. If v = (3t−2)(t−4) and s = 8 m when t = 1 s, find the:

(a) initial speed of the body.

(b) values of t when the body is at rest.

(c) acceleration of the body when t = 3 s.

(d) distance the body is from O when t = 2 s, where O is the location of the body at t = 0.

Exercise: 15C

1. A particle of mass 4 kg moves such that $\mathbf{r} = \begin{pmatrix} t^3-t^2-4t+3 \\ t^3-2t^2+3t-7 \end{pmatrix}$.

(a) Calculate the times when the particle crosses the line y = x.

(b) Find the velocity of the particle at t = 4 s.

(c) Find an expression for the acceleration **a** in terms of t and hence calculate the force acting on the particle at $t = \frac{2}{3}$ s.

2. If $\dot{\mathbf{r}} = 4t^3\mathbf{i} + 6t\mathbf{j} - 3t^2\mathbf{k}$ and, when t = 1 s, $\mathbf{r} = (14\mathbf{i}+6\mathbf{j}-3\mathbf{k})$ m, find:

(a) $\ddot{\mathbf{r}}$ when t = 3 s.

(b) **r** when t = 0 s.

3. A particle starts from rest at (2,0,0) and moves such that $\ddot{\mathbf{r}} = \begin{pmatrix} 16\cos 4t \\ 8\sin 2t \\ \sin t-2\sin 2t \end{pmatrix}$. Find the:

(a) acceleration when t = π s.

(b) velocity when $t = \frac{\pi}{2}$ s.

(c) displacement when $t = \frac{\pi}{4}$ s.

4. If v = 2r + 3, find t when r = 3 m given that r = 0 when t = 0.

5. If $\ddot{r} = 4r + 2$ and initially r = 0 and $\dot{r} = 1$ m s^{-1}, find the:

(a) value of $\dot{r}$ when r = 3 m.

(b) value of t when r = 3 m.

6. If $\ddot{r} = r + 2$ and initially r = 0 and $\dot{r} = 2$ m s^{-1}, find an expression for:

(a) $\dot{r}$ as a function of r.

(b) t as a function of r.

7. A body moves along a straight line with its acceleration at time t given by a = (5−2v) m s^{-2}, where v is the velocity of the body at time t. When t = 0, the body is at rest. Show that $t = \frac{1}{2}\ln\left(\frac{5}{5-2v}\right)$ s, and hence obtain an expression for v as a function of t. Show that as the motion continues, the velocity of the body approaches a maximum value and find this maximum value.

8. A body of mass 8 kg is projected vertically upwards with an initial speed of 10 m s^{-1}. The body experiences a resisting force of $\frac{v}{5}$ N, v being the speed of the body. Find the height above its point of projection when it instantaneously comes to rest.

9. A body of mass 10 kg is projected vertically upwards through a viscous liquid with an initial speed of 12 m s^{-1}. The body experiences a resisting force of $\frac{1}{2}v^2$ N, where v is the speed of the body. Find the distance above its point of projection at which it instantaneously comes to rest.

Answers to exercises

Exercise: 15A

1. (a) $\mathbf{v} = 8t\mathbf{i} + (9t^2-2)\mathbf{j}$; $2m(4\mathbf{i}+9t\mathbf{j})$ (b) $\mathbf{F} = -6t^2\mathbf{j} + 2t\mathbf{k}$; $2m(-6t\mathbf{j}+\mathbf{k})$

(c) $\mathbf{v} = (2-2t)\mathbf{i} + 6\cos 2t\mathbf{j}$;

$\mathbf{F} = -2m(\mathbf{i}+6\sin 2t\mathbf{j})$

(d) $\mathbf{v} = e^{-t}(1-t)\mathbf{i} - 2\mathbf{j}$; $\mathbf{F} = me^{-t}(t-2)\mathbf{i}$

2. (a) $\sqrt{19}$ m s^{-1} (b) 40**i** N

3. $(10\mathbf{i}+25\mathbf{j})$ m s^{-1} ; $(27\mathbf{i}+59{\cdot}5\mathbf{j})$ m

4. $5\mathbf{i} - \frac{25}{12}\mathbf{j} + 2\mathbf{k}$

5. $(4t+6)\mathbf{i} - 2\sin t\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

6. (a) 50•448 m s^{-1} (b) 48•3735 m s^{-2}

7. (i) $\frac{1}{5}(2\mathbf{i}+3\mathbf{j}-4\mathbf{k})$ (ii) $\frac{3}{5}(2\mathbf{i}+3\mathbf{j}-4\mathbf{k})$ (iii) 5•1662 m

8. (i) $(3t^2\mathbf{j}+2t\mathbf{k})$; $(6t\mathbf{j}+2\mathbf{k})$ (ii) 48•662 m s^{-1}

9. $(t^2+3t+1)\mathbf{i} + (3t-t^4-5)\mathbf{j} + (3t^3-1)\mathbf{k}$

10. (i) $3\sqrt{5}$ N (ii)

Exercise: 15B

1. $6{\cdot}02$ m s^{-1}; $2{\cdot}0187$ m s^{-2}

$r(t) = \frac{a}{\omega}(\cos \omega t-1) + b\sin \omega t$

3. (i) 29.76 m s^{-1} (ii) 5.754 m (iii) 321.78 J

4. $13{\cdot}25$ m s^{-1} 5. $(15i+6j-7k)$; $17{\cdot}61$ m s^{-1} 6. 20 m s^{-1}

7. $v = 3t^2-4t + 3$; $s = t^3-2t^2 + 3t$

8. (a) 8 m s^{-1} (b) $\frac{2}{3}$ s ; 4 s (c) 4 m s^{-2} (d) 2 m

Exercise: 15C

1. (a) 2 s ; 5 s (b) $\begin{pmatrix}36\\35\end{pmatrix}$ m s^{-1} (c) $\begin{pmatrix}6t-2\\6t-4\end{pmatrix}$; $\begin{pmatrix}8\\0\end{pmatrix}$ N

2. (a) $(108i+6j-18k)$ m s^{-2} (b) $(13i+3j-2k)$ m

3. (a) $\begin{pmatrix}16\\0\\0\end{pmatrix}$ m s^{-2} (b) $\begin{pmatrix}0\\8\\-1\end{pmatrix}$ m s^{-1} (c) $\begin{pmatrix}4\\\pi-2\\\frac{1}{2}-\frac{1}{\sqrt{2}}\end{pmatrix}$ m

4. 0.549 s 5.(a) 7 m s^{-1} (b) 0.973 s

6. (a) $\dot{r} = r + 2$ (b) $t = \ln\left(\frac{r+2}{2}\right)$

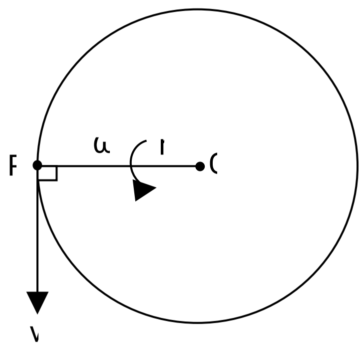
7. $v = \frac{5}{2}(1-e^{-2t})$; $2{\cdot}5$ m s^{-1} 8. $5{\cdot}02$ m 9. $5{\cdot}51$ m

16. CIRCULAR MOTION

There are different kinds of circular motion and in this chapter motion in horizontal and vertical circles is to be considered.

16.1 Motion in a horizontal circle

Consider a particle P of mass m, moving in a horizontal circle, centre O, radius r, with constant speed v.



The linear speed v of P is directed along the tangent to the circle at P. The constant angular velocity ω of P is $\omega = \frac{v}{r}$ measured in radians per second (rad s^{-1}). There is no acceleration along the tangent since the particle moves with constant speed around the circle. The acceleration of P is in the direction $\vec{PO}$, that is, towards the centre of the circle and is given by $a = \frac{v^2}{r}$.

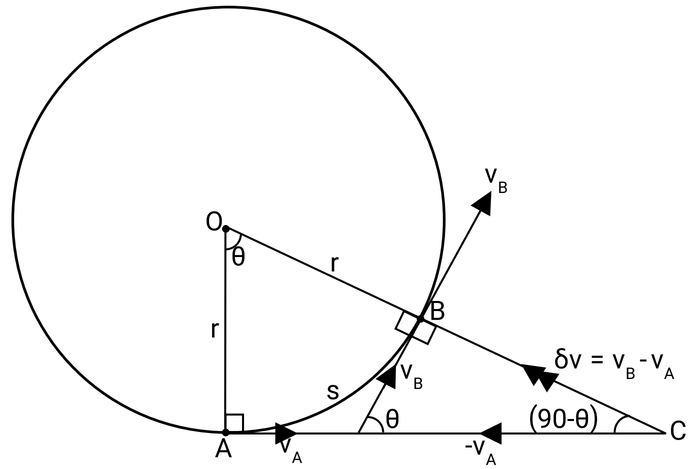
By Newton's 2nd law, this acceleration must be produced by a force which is also directed towards the centre of the circle.

From $F = ma \Rightarrow F = \frac{mv^2}{r}$.

This force may be tension in a string, a frictional force, etc…, and because it is always directed towards the centre, it is called the **centripetal force**.

16.1.2 Derivation

Consider a body describing a circle of radius r at a constant speed v as shown:



From $\frac{s}{2\pi r} = \frac{\theta}{2\pi} \Rightarrow s = r\theta$

The speed $v = \frac{s}{t} \Rightarrow v = \frac{r\theta}{t} = r\left(\frac{\theta}{t}\right)$

The quantity $\omega = \frac{\theta}{t}$ is called the angular velocity of the body.

Hence $v = \omega r$

The acceleration $a = \frac{v_B - v_A}{t} = \frac{\delta v}{t}$

But $t = \frac{s}{v} \Rightarrow t = \frac{r\theta}{v}$ and $\sin\theta = \frac{\delta v}{|v_A|} = \frac{\delta v}{v}$

$$\Rightarrow \delta v = v\sin\theta$$

But for small angles in radians as $\theta \to 0$,

$$\sin\theta \to \theta \Rightarrow \delta v = v\theta$$

Hence $a = \frac{v\theta}{\left(\frac{r\theta}{v}\right)} \Rightarrow a = \frac{v^2\theta}{r\theta} \Rightarrow a = \frac{v^2}{r}$.

Since δv is towards the centre, that is, along line CO, the acceleration is also directed towards the centre and is called **centripetal acceleration**.

From Newton's second law: $F = ma \Rightarrow F = \frac{mv^2}{r}$.

This force is towards the centre of the circular path and is thus called centripetal(centre seeking) force. This force enables a body to move in a circular path.

Problem solving:
When solving problems in which P describes a horizontal circle, centre O, with constant speed or constant angular velocity.

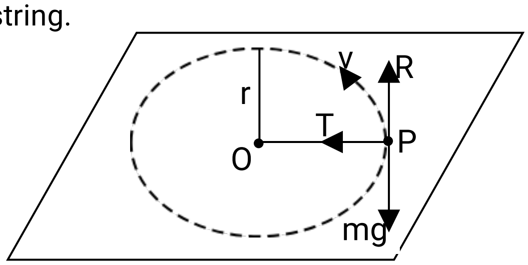
1. Draw a clear force diagram.
2. Resolve vertically. Since the particle does not

move up or down, forces must balance in this direction.
3. Write down an equation for motion along the radius.

Some common situations:

1. Particle on a string:

A particle attached to one end of a string, the other end being on a smooth horizontal surface. When the particle describes circular motion, the force towards the centre is provided by tension in the string.

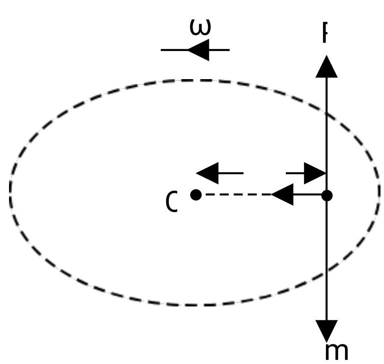


Resolving vertically: $R = mg$

Resolving horizontally: $T = \frac{mv^2}{r}$

2. A particle on a rotating disc:

If a particle rests on a rotating horizontal disc, the only horizontal force acting on the particle is the frictional force between the particle and the surface of the disc. For any particular surface, there will be a maximum value of F, that is $F_{max} = \mu R$ and then the particle will be at the point of slipping.

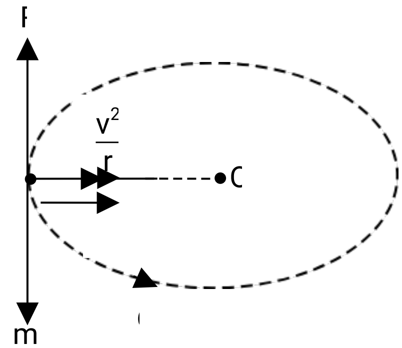


$(\uparrow)$:$R = mg$

$(\rightarrow)$:$F = \frac{mv^2}{r}$

3. Bead on a circular wire:

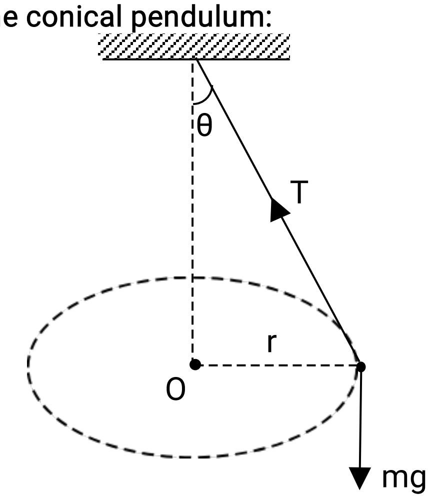
If a bead is threaded on a smooth horizontal circular wire and moves at speed v, the necessary force towards the centre of the circular wire is provided by the force between the bead and the wire. In addition the wire supports the weight of the bead, and the vertical reaction R equals mg.



$(\uparrow)$:$R = mg$

$(\rightarrow)$:$F = \frac{mv^2}{r}$

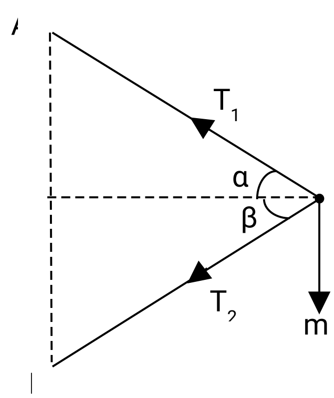
4. The conical pendulum:



$(\uparrow)$:$T\cos\theta = mg$

$(\rightarrow)$:$T\sin\theta = \frac{mv^2}{r}$

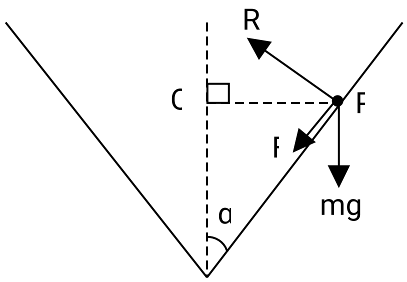
5. A particle fixed along a string whose other ends are fixed and is whirled in a horizontal circle:



$(\uparrow)$: $T_1 \sin \alpha - T_2 \sin \beta = mg$

$(\rightarrow)$: $T_1 \cos \alpha + T_2 \cos \beta = \dfrac{mv^2}{r}$

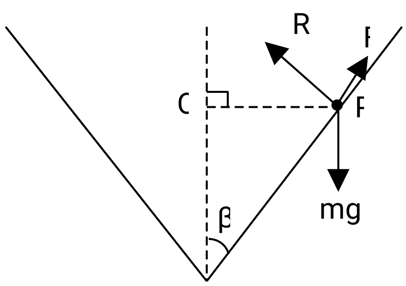
6. A particle P moving inside a hollow cone, with friction:
   (a) P about to move up the cone (v– a maximum)



   (b) $(\uparrow)$: $R \sin \alpha - F \cos \alpha = mg$

   (c) $(\rightarrow)$: $R \cos \alpha + F \sin \alpha = \dfrac{mv^2}{r}$

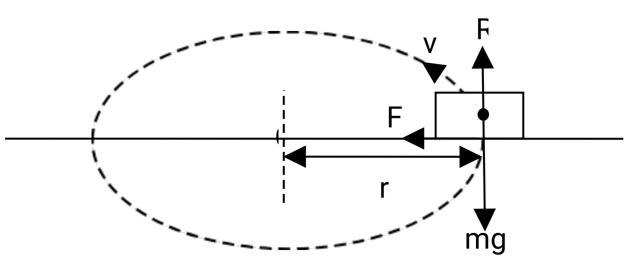
   (d) P about to move down the cone (v– a minimum)



$(\uparrow)$: $R \sin \beta + F \cos \beta = mg$

$(\rightarrow)$: $R \cos \beta - F \sin \beta = \dfrac{mv^2}{r}$

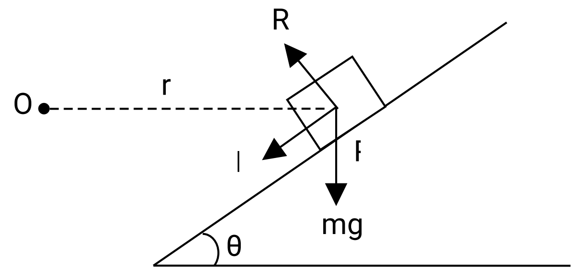
7. A car rounding a rough horizontal track:



$(\uparrow)$: $R = mg$

$(\rightarrow)$: $F = \dfrac{mv^2}{r}$

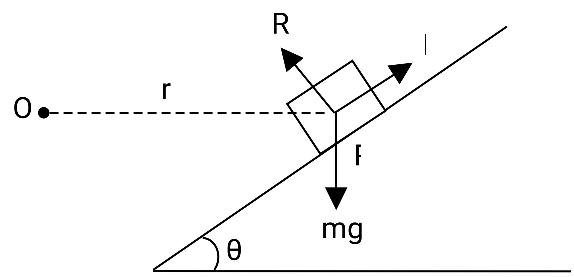
8. A car rounding a banked track:
   (a) Car cornering at maximum speed



$(\uparrow)$: $R \cos \theta - F \sin \theta = mg$

$(\rightarrow)$: $R \sin \theta + F \cos \theta = \dfrac{mv^2}{r}$

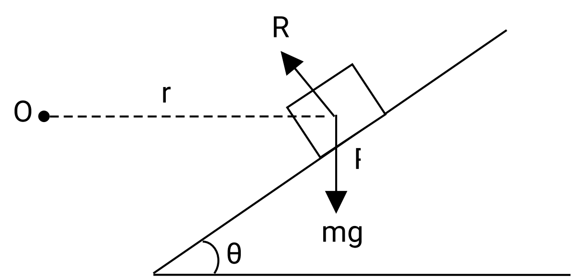
   (b) Car cornering at minimum speed



$(\uparrow)$: $R \cos \theta + F \sin \theta = mg$

$(\rightarrow)$: $R \sin \theta - F \cos \theta = \dfrac{mv^2}{r}$

   (c) Car cornering at the design speed



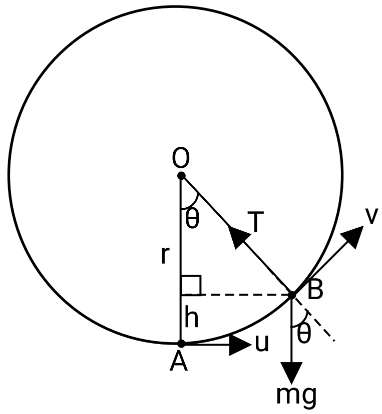
$(\uparrow)$: $R \cos \theta = mg$

$(\rightarrow)$: $R \sin \theta = \dfrac{mv^2}{r}$

16.2 Motion in a vertical circle

If a particle describes a vertical circle when whirled at the end of a string or by sliding on a smooth spherical surface, we apply the principle of conservation of energy to analyse its motion.

Consider a particle of mass m attached to the end of a light inextensible string describing a vertical circle of radius r.



If the particle is given an initial horizontal speed u from the lowest point A, and at B it has rotated through an angle θ and has speed v.
Taking A as the reference point;

At A: Potential energy = 0 ; Kinetic energy = $\frac{1}{2}mu^2$

Mechanical energy = $\frac{1}{2}mu^2$

At B: Potential energy = mgh = $mgr(1-\cos\theta)$;

Kinetic energy = $\frac{1}{2}mv^2$

Mechanical energy = $mgr(1-\cos\theta) + \frac{1}{2}mv^2$

From $v^2 = u^2 + 2as$

$v^2 = u^2 - 2gr(1-\cos\theta)$

Mechanical energy =

$mgr(1-\cos\theta) + \frac{1}{2}m[u^2-2gr(1-\cos\theta)] = \frac{1}{2}mu^2$

Hence mechanical energy is conserved.

Consider the motion of the particle.

At B:

Along the tangent;

$$0 - mg\sin\theta = ma \Rightarrow a = -g\sin\theta$$

Along the radius;

$$T - mg\cos\theta = \frac{mv^2}{r} \Rightarrow T = mg\cos\theta + \frac{mv^2}{r}$$

From $v^2 = u^2 - 2gr(1-\cos\theta)$

$$T = mg\cos\theta + \frac{m}{r}[u^2-2gr(1-\cos\theta)]$$

$$T = \frac{mu^2}{r} - mg(2-3\cos\theta)$$

This gives the tension in the string at any angle θ, motion is possible if the string does not become slack.

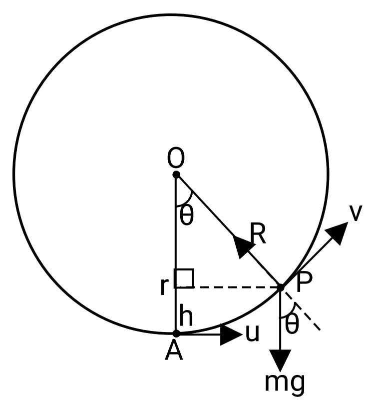
Hence $T \geq 0 \Rightarrow \frac{mu^2}{r} - mg(2-3\cos\theta) \geq 0$

$$\Rightarrow u \geq \sqrt{rg(2-3\cos\theta)}$$

At the instant when the string first slackens, T = 0.

16.2.1 Motion of a particle on a smooth spherical surface

Consider a particle, P of mass, m free to move on the inside of a smooth circular (spherical) surface of radius r.



Along the tangent:

$$0 - mg\sin\theta = ma \Rightarrow a = -g\sin\theta$$

Along the radius:

$$R - mg\cos\theta = \frac{mv^2}{r} \text{..........................(i)}$$

If the particle is projected from the lowest point A with horizontal speed u;

By conservation of mechanical energy:

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mgr(1-\cos\theta)$$

$$\Rightarrow v^2 = u^2 - 2gr(1-\cos\theta) \text{......(ii)}$$

From (i) and (ii)

$$R = mg\cos\theta + \frac{m}{r}[u^2-2gr(1-\cos\theta)]$$

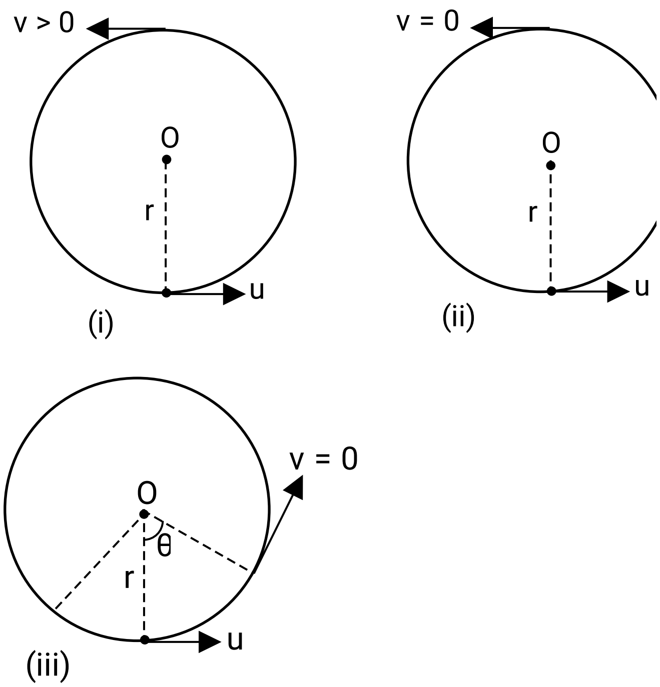
$$R = \frac{mu^2}{r} - mg(2\text{-}3\cos\theta)$$

Motion continues if $R \geq 0 \Rightarrow u \geq \sqrt{rg(2\text{-}3\cos\theta)}$

Note that if the particle is on the outside of the sphere, the reaction is along the radius but away from the centre of the sphere.

16.2.2 Analysis of two main types of motion in a vertical circle

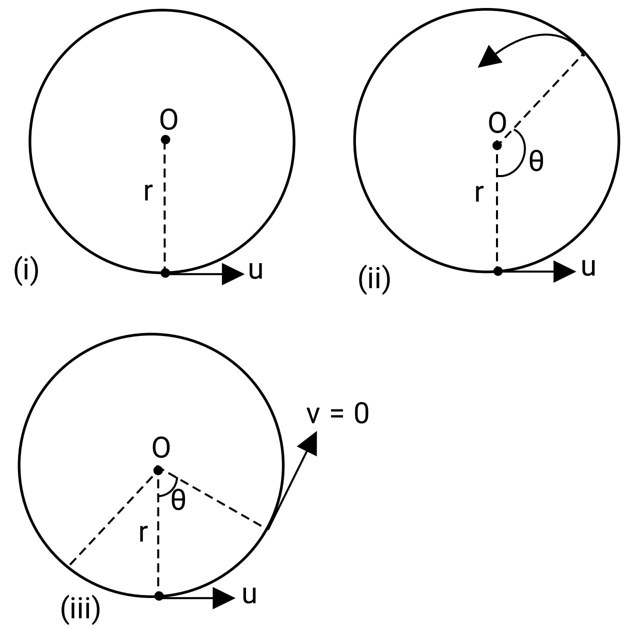
1. The particle cannot leave the circular path, for example a bead threaded on a vertical wire. The particle can do one of the three behaviors below.
   - (i) Complete the circle if $v > 0$ at the top and $u^2 > 4rg$.
   - (ii) Come to rest at the top if $v = 0$ at the top and $u^2 = 4rg$.
   - (iii) Oscillate if $v = 0$ for $0 < \theta < \pi$ and $u^2 < 4rg$.



2. The particle can leave the circular path and become a projectile, for example a particle attached to a string. The particle can do one of the three behaviors below.

(T is the tension in the string or a normal reaction)

- (i) Complete the circle if $T \geq 0$ for all values of θ and $u^2 \geq 5rg$.
- (ii) Become a projectile if $T = 0$ and $\frac{\pi}{2} < \theta \leq \pi$ and $2rg < u^2 < 5rg$.
- (iii) Oscillate if $v = 0$ for $\theta \leq \frac{\pi}{2}$ and $u^2 \leq rg$.

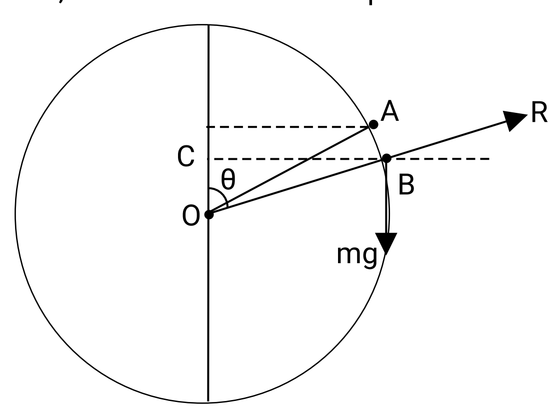


Example 1

A particle is released from rest at a height $\frac{3}{2}r$ above the lowest point of a sphere, it slides down the outside of the smooth sphere of radius r. Prove that it leaves the sphere at a height $\frac{1}{3}r$ above the centre. Show that when the particle is at a distance $r\sqrt{2}$ from the vertical diameter of the sphere, it is at a depth 4r below the centre of the sphere.

Solution

Since the particle slides on the outside of the sphere, the reaction R on the sphere is outwards.



In moving from A to B, the particle falls through a vertical height

$$h = \frac{1}{2}r - \frac{1}{3}r = \frac{1}{6}r$$

Since the particle starts from rest at A, its initial velocity is zero.

At B:

From $v^2 = u^2 + 2as$, $u = 0$, $s = h = \frac{1}{6}r$.

$$v^2 = 2g\times\frac{1}{6}r \Rightarrow v = \sqrt{\frac{1}{3}rg}$$

Inclination of OB to the vertical.

From $\cos\theta = \frac{\left(\frac{1}{3}r\right)}{r} = \frac{1}{3} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{3}\right)$

Along the radius:

$mg\cos\theta - R = \frac{mv^2}{r}$

$$R = mg\cos\theta - \frac{mv^2}{r} = mg\times\frac{1}{3} - \frac{m}{r}\times\frac{1}{3}rg$$

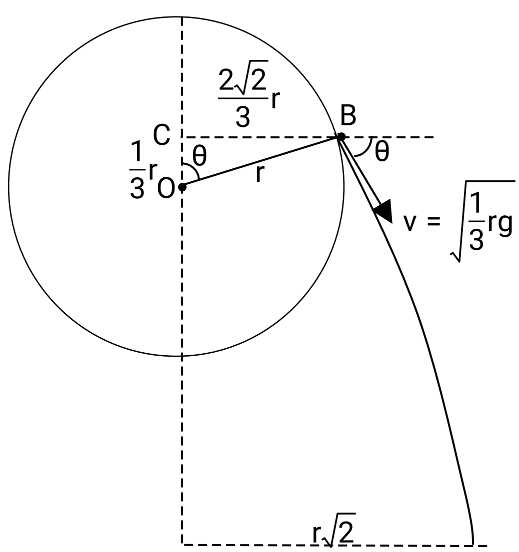
$$\Rightarrow R = 0.$$

Since the reaction is zero at B, the particle will leave the sphere when at a distance $\frac{1}{3}r$ above its centre.

The particle moves as a projectile

$CB^2 = r^2 - \left(\frac{1}{3}r\right)^2 \Rightarrow CB = \frac{2\sqrt{2}}{3}r$

The particle leaves at a distance $\frac{2\sqrt{2}}{3}r$ from the vertical diameter.



The particle leaves at a tangent to the radius.

$$x = (v\cos\theta)t$$

$$y = (v\sin\theta)t + \frac{1}{2}gt^2$$

When the particle is at a distance $r\sqrt{2}$ from the vertical diameter

$$x = r\sqrt{2} - \frac{2\sqrt{2}}{3}r = \frac{\sqrt{2}}{3}r$$

$$\therefore\ \frac{\sqrt{2}}{3}r = \frac{1}{3}\times\left(\sqrt{\frac{1}{3}rg}\right)t \Rightarrow t = \sqrt{\frac{6r}{g}}$$

$$y = \sqrt{\frac{1}{3}rg}\times\frac{2\sqrt{2}}{3}\times\sqrt{\frac{6r}{g}} + \frac{1}{2}g\times\frac{6r}{g}$$

$$y = \frac{4}{3}r + 3r = \frac{13}{3}r$$

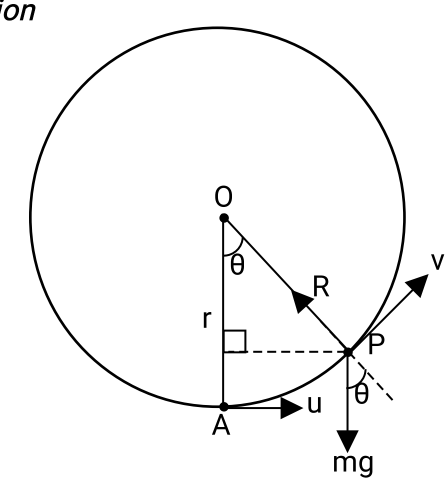
This is the distance of the particle below B.

Vertical distance below O $= \frac{13}{3}r - \frac{r}{3} = 4r$

Example 2

A particle is placed on the lowest point of the inside of a smooth spherical shell of internal radius 3a m and is given a horizontal velocity of $\sqrt{13ag}$ m s^{-1}. How high above the point of projection does the particle rise?

Solution



Along the radius:

$$R - mg\cos\theta = \frac{mv^2}{r}\text{.............................(i)}$$

If the particle is projected from the lowest point A with horizontal speed u;

By conservation of mechanical energy:

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mgr(1-\cos\theta)$$

$$\Rightarrow v^2 = u^2 - 2gr(1-\cos\theta) \ldots\ldots(ii)$$

From (i) and (ii)

$$R = mg\cos\theta + \frac{m}{r}[u^2 - 2gr(1-\cos\theta)]$$

$$R = \frac{mu^2}{r} - mg(2-3\cos\theta)$$

When the particle stops rising, R = 0.

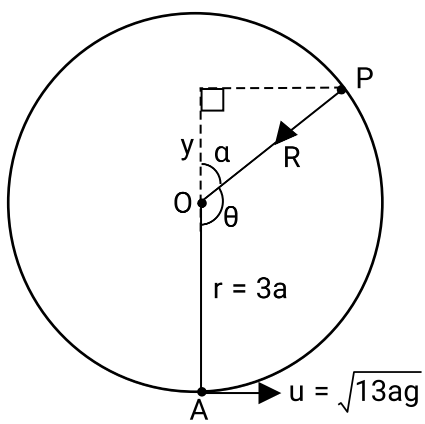
$$0 = \frac{mu^2}{r} - mg(2-3\cos\theta)$$

$$\Rightarrow u^2 = rg(2-3\cos\theta)$$

$$13ag = 3ag(2-3\cos\theta)$$

$$\cos\theta = \frac{-7}{9} \Rightarrow \theta = \pi - \cos^{-1}\frac{7}{9}$$

Let $\alpha = \cos^{-1}\frac{7}{9}$



$$\cos\alpha = \frac{y}{3a} \Rightarrow y = 3a\cos\alpha = 3a\times\frac{7}{9}$$

$$y = \frac{7}{3}a$$

$$h = 3a + \frac{7}{3}a = \frac{16}{3}a \text{ m}$$

Example 3

An inelastic string of length a meters is fixed at one end P and carries a particle of mass 3 kg at its other end Q. The particle is describing a horizontal circle of radius 0•8 m with an angular speed of 5 rad s^{-1}. Determine the:

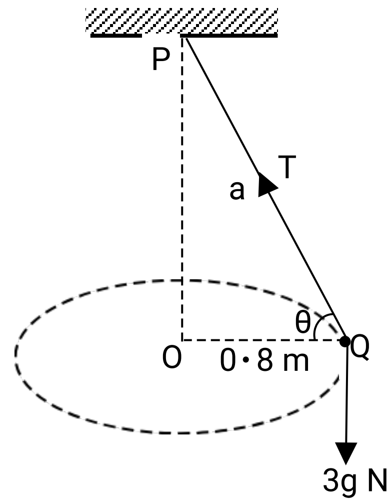
(a) (i) angle the string makes with the horizontal.
(ii) tension in the string.

(b) value of a.

(c) linear speed of the particle.

Solution

(a)



$(\rightarrow)$: $T\cos\theta = 3\times5^2\times0{\cdot}8$

$T\cos\theta = 60$ .... (i)

$(\uparrow)$: $T\sin\theta = 3g$

$T\sin\theta = 3\times9{\cdot}8$

$T\sin\theta = 29{\cdot}4$ .... (ii)

Dividing (ii) by (i):

(i) $\tan\theta = \frac{29{\cdot}4}{60}$

$\theta = 26{\cdot}1°$

(ii) From (i): $T\cos 26{\cdot}1 = 60$

$T = 66{\cdot}816$ N

(b) $\cos\theta = \frac{0{\cdot}8}{a}$

$a = \frac{0{\cdot}8}{\cos 26{\cdot}1} = 0{\cdot}8908$ m

(c) $v = \omega r$

$= 5\times0{\cdot}8$

$= 4$ m s^{-1}

Example 4

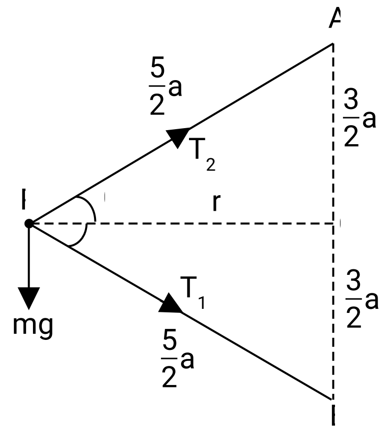
A light inextensible string of length 5a metres has one end attached to a point A and the other to a point B which is vertically below A and 3a metres from it. A particle P of mass m kg is fastened to the midpoint of the string and moves with speed u in a circle whose centre is the midpoint of AB. Show that the tensions in the upper and lower strings are

$$\left(\frac{15mu^2+40mga}{48a}\right) \quad \text{and} \quad \left(\frac{15mu^2-40mga}{48a}\right)$$

respectively.

Hence deduce that the motion is possible if $3u^2 \geq 8ga$.

Solution



By pythogoras theorem:

$$r^2 = \left(\frac{5}{2}a\right)^2 - \left(\frac{3}{2}a\right)^2 \Rightarrow r = 2a$$

Resolving horizontally:

$$\left(T_1+T_2\right)\cos\theta = \frac{mu^2}{2a}$$

$$\left(T_1+T_2\right)\times\frac{2a}{\left(\frac{5}{2}a\right)} = \frac{mu^2}{2a}$$

$$T_1 + T_2 = \frac{5mu^2}{8a} \text{..........................} \text{ (i)}$$

Resolving vertically:

$$\left(T_2-T_1\right)\sin\theta = mg$$

$$\left(T_2-T_1\right)\times\frac{\left(\frac{3}{2}a\right)}{\left(\frac{5}{2}a\right)} = mg$$

$$T_2-T_1 = \frac{5}{3}mg \text{ ....} \text{ (ii)}$$

Adding (i) and (ii):

$$2T_2 = \frac{5mu^2}{8a} + \frac{5}{3}mg$$

$$T_2 = \left(\frac{15mu^2+40mga}{48a}\right)$$

From (i) $T_1 = \frac{5mu^2}{8a} - \left(\frac{15mu^2+40mga}{48a}\right)$

$$T_1 = \left(\frac{15mu^2-40mga}{48a}\right)$$

Motion is possible if both strings are taut, that is,

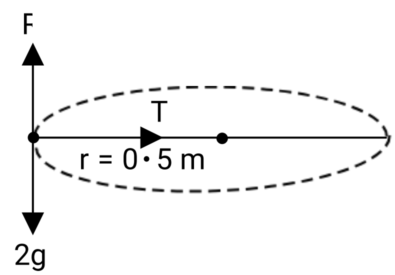
$$T_1 \geq 0 \Rightarrow \left(\frac{15mu^2-40mga}{48a}\right) \geq 0$$

Hence $3u^2 \geq 8ga$.

Example 5

One end of a light inextensible string of length $0 \cdot 5$ m is attached to a fixed point O on a smooth horizontal surface and a particle of mass 2 kg is attached to its other end. If the particle moves in a horizontal circle, centre O, with a speed of 5 m s^{-1}, find the tension in the string and the reaction of the particle on the surface.

Solution:



Horizontally: $T = \frac{mv^2}{r}$

$$T = \frac{2\times5^2}{0\cdot5} \Rightarrow T = 100 \text{ N}$$

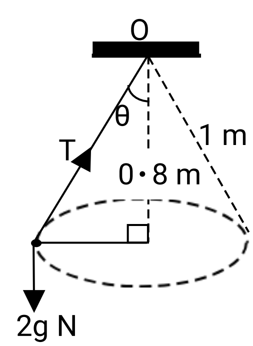
Vertically: $R = 2g$

$$R = 2\times9\cdot8 \Rightarrow R = 19\cdot6 \text{ N}$$

Example 6

A particle of mass 2 kg is attached by a light inextensible string of length 1 m to a fixed point O. The particle is made to move in a horizontal circle whose centre is 80 cm vertically below O. Find the tension in the string and the speed of the particle.

Solution:



$$r = \sqrt{1^2 - 0{\cdot}8^2} \Rightarrow r = 0{\cdot}6 \text{ m}$$

Resolving vertically:

$$T\cos\theta = 2g$$

$$T \times \left(\frac{0{\cdot}8}{1}\right) = 2 \times 9{\cdot}8$$

$$T = 24{\cdot}5 \text{ N}$$

Resolving horizontally:

$$T\sin\theta = \frac{mv^2}{r}$$

$$24{\cdot}5 \times \left(\frac{0{\cdot}6}{1}\right) = \frac{2v^2}{0{\cdot}6}$$

$$v = 2{\cdot}1 \text{ m s}^{-1}$$

Exercises

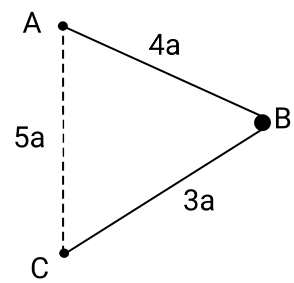
Exercise: 16A

1. A particle is suspended from a point A by a string of length 1 m. The particle is then made to describe a horizontal circle with angular velocity ω rad s^{-1} in a conical pendulum. Show that the angle made by the string to the vertical is $\theta = \cos^{-1}\left(\frac{g}{\omega^2}\right)$.
2. Two light inextensible strings AB and BC each of length l, are attached to a particle of mass m, at B, the other ends A and C are fixed to two points in a vertical line, such that A is a distance l above C. The particle describes a horizontal circle with constant angular velocity ω. Find the:
   (a) tensions in AB and BC.
   (b) least value of ω so that both strings shall be taut.
3. A pendulum of length l, is whirled in a horizontal circle such that its string makes an angle of 30° with the vertical. Show that the angular velocity of the bob is given by $\omega^2 = \frac{2g}{l\sqrt{3}}$.
4. A particle of mass 5 kg is whirled in a horizontal circle of radius 2 m with a constant speed of 3 m s^{-1}. Find the tension in the string.
5. One end of a light inextensible string of length $1{\cdot}6$ m is attached to a fixed point O on a smooth horizontal surface and a particle of mass 4 kg is attached to the other end. If the string will break when the tension in it exceeds 90 N, find the maximum speed at which the particle can move in a horizontal circle with centre O.
6. A particle of mass 5 kg is attached by a light inextensible string of length $3{\cdot}25$ m to a fixed point. The particle moves in a horizontal circle with a constant angular velocity of 2.8 rad s^{-1}. Find the tension in the string and the radius of the circle.
7. A particle of mass m is projected from the top of a smooth sphere of radius a. It slides down the outside surface of the sphere and leaves the surface of the sphere with speed $\sqrt{\frac{4}{5}ag}$.
   Find the:
   (a) vertical distance travelled by the particle while it is in contact with the sphere.
   (b) speed of projection.
   (c) speed of the particle when it is level with the horizontal diameter of the sphere.
8. A small bead P, of mass m, is threaded on a smooth thin wire of radius r and centre O which is fixed in a vertical plane. The bead is projected along the wire with speed u from the lowest point A. The bead comes to instantaneous rest at a point where $\angle POA = 120°$. Show that $u = \sqrt{3rg}$. Find in terms of r, the height of P above A at the instant when the reaction of the bead on the wire is zero.

Exercise: 16B

1. A particle of mass 2 kg is suspended from a fixed point by a light elastic string of natural length 1 m and modulus $19{\cdot}6$ N. The particle is moving in a horizontal circle with constant speed v m s^{-1}. Given that the length of the string is $2{\cdot}25$ m, find the value of v.
2. Two light strings AB and BC are each attached at B to a particle of mass m. The string AB is elastic of natural length 2a and modulus 3mg. The string BC is inextensible of length 3a. The ends A and C are fixed with C vertically below A and AC = 5a. The particle moves with constant speed in a horizontal circle with both strings taut and AB = 4a as

shown.

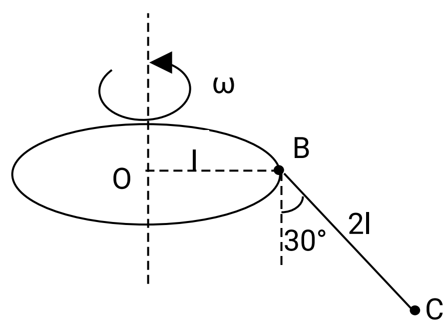


(a) Find the tension in string AB.

(b) Find the tension in string BC.

(c) Show that the speed of the particle is $\sqrt{\left(\frac{44ga}{5}\right)}$.

3.



A particle C of mass m is attached by means of a light inextensible string of length 2l to a particle B fixed at the edge of a horizontal rotating disc of radius l and centre O. The system rotates with constant angular speed ω about a vertical axis through O. The plane OBC remains vertical and the string makes an angle of 30° with the vertical as shown in the diagram above. Show that $\omega^2 = \frac{g}{2l\sqrt{3}}$.

4. A light inextensible string of length 0.72m is attached to points A and B where A is vertically above B and AB = 0•48 m. If a smooth ring P of mass 50 grammes is threaded on the string and is made to move in a horizontal circle about B, find the:

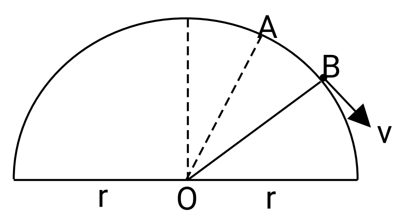
(a) tension in the string.

(b) angular speed of the ring.

5. A particle is suspended at the end of a light inextensible string of length r from a point O. When the string is vertical, the particle is projected horizontally with speed u. When the particle has turned through an angle θ in a vertical plane, prove that the tension in the string is given by $m\left(3g\cos\theta + \frac{u^2}{r} - 2g\right)$. Given that $u = \sqrt{\frac{7}{2}rg}$, show further that when the string first slackens, the particle is at a height $\frac{3}{2}r$ above the lowest point and that at that instant the string makes an angle of 60° with the upward vertical.

6. A particle is suspended from a fixed point by a string of length l. It is then projected horizontally so as to describe part of a circle in a vertical plane. Show that if the parabolic path of the particle after the string becomes slack passes through the original point of projection, the speed of projection is $\sqrt{\frac{7gl}{2}}$.

7. A particle is released from rest on the surface of a solid hemisphere at a point A such that OA makes an angle $\cos^{-1}\frac{7}{8}$ with the upward vertical as shown below.



The particle slides freely until it leaves the surface of the hemisphere at point B with speed v. Given that OB makes an angle θ with the upward vertical, prove that:

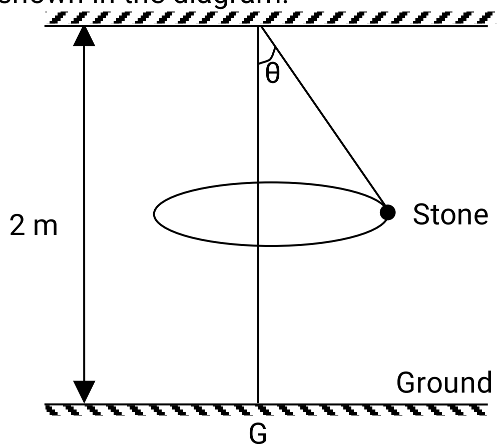
(a) $\cos\theta = \frac{7}{12}$

(b) $v^2 = \frac{7}{12}rg$

8. (a) (i) Show that the centripetal acceleration of a body moving in a circular path of radius r is given by $a = \frac{v^2}{r}$.

(ii) Derive the expression for the angle inclination to the horizontal necessary for a rider moving a round a circular track of radius r without skidding at a speed v, in terms of g, r and v.

(b) A stone of mass 0•5 kg is tied to one end of a string 1 m long. The point of suspension of the string is 2 m above the ground. The stone is whirled in a horizontal circle with increasing angular velocity as shown in the diagram.



The string will break when the tension in it is 12 • 5 N and semi-vertical angle θ_{max}.

(i) Calculate the value of θ_{max}.

(ii) How far from point G will the stone hit the ground?

(iii) What will be the velocity of the stone when it hits the ground?

Answers to Exercises

Exercise: 16A

1. 2. (a) $\frac{1}{2}m(\omega^2 l+2g)$; $\frac{1}{2}m(\omega^2 l-2g)$ (b) $\sqrt{\frac{2g}{l}}$ 3.

4. 22 • 5 N

5. 6 m s^{-1} 6. 127 • 4 N ; 3 m 7. (a) $\frac{a}{5}$

(b) $\sqrt{\frac{2}{5}ag}$ (c) $2\sqrt{\frac{3}{5}ag}$

8. $\frac{4}{3}r$

Exercise: 16B

1. 3•15 m s^{-1} 2. (a) 3mg (b) $\frac{7}{3}mg$ (c)

3. 4. (a) 0 • 5308 N (b) 8•5732 rad s^{-1} 5. 6. 7. (a) (b)

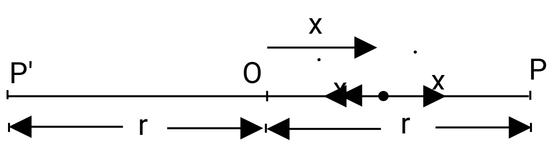
8. (a) (i) (ii) $\tan\theta = \frac{rg}{v^2}$ (b)(i) $\theta_{max} = 66.9°$ (ii) 2 • 8 m (iii) 7•26 m s^{-1} at 50 • 7° below the horizontal

17. SIMPLE HARMONIC MOTION(S.H.M)

Simple harmonic motion is a kind of periodic motion in which the acceleration is directed towards a fixed point called equilibrium position and it is directly proportional to the displacement from this point.

17.1 Equations for simple harmonic motion

Consider the particle shown below performing S.H.M about equilibrium position O with extreme points P and P'. The displacement of the particle from O at time t, is x, and $v = \dot{x}$ is the velocity at this time.



Where $\ddot{x}$ is the acceleration and r is the amplitude.

Velocity reduces as the particle moves away from the equilibrium position and the particle comes to instantaneous rest at P and P' and accelerates towards O. The amplitude OP or OP' = r is the maximum displacement of the particle from the equilibrium position.

During simple harmonic motion:

$$\ddot{x} \propto x$$

$$\ddot{x} = -\omega^2 x \text{ ....(i)}$$

Where ω is a positive constant called angular velocity.

$$\text{But } \ddot{x} = \frac{dv}{dx}\times\frac{dx}{dt} = v\frac{dv}{dx}$$

$$v\frac{dv}{dx} = -\omega^2 x$$

$$\int v dv = -\omega^2 \int x dx$$

$$\frac{1}{2}v^2 = -\frac{1}{2}\omega^2 x^2 + c$$

When $x = \pm r$, $v = 0$, where $v = \dot{x}$

$$0 = -\frac{1}{2}\omega^2 r^2 + c \Rightarrow c = \frac{1}{2}\omega^2 r^2$$

Hence $v^2 = \omega^2(r^2 - x^2)$

$$\Rightarrow v = \pm\omega\sqrt{r^2 - x^2} \text{ .... (ii)}$$

$$v_{max} = \omega r, \text{ when } x = 0$$

Taking the positive root in (ii)

$$\frac{dx}{dt} = \omega\sqrt{r^2 - x^2}$$

$$\int \frac{dx}{\sqrt{r^2 - x^2}} = \int \omega dt \Rightarrow \sin^{-1}\left(\frac{x}{r}\right) = \omega t + \varnothing$$

$$x = r\sin(\omega t + \varnothing) \text{ .... (iii)}$$

Alternatively taking the negative root in (ii)

$$\frac{dx}{dt} = -\omega\sqrt{r^2 - x^2}$$

$$\int \frac{dx}{-\sqrt{r^2 - x^2}} = \int \omega dt$$

$$\Rightarrow \cos^{-1}\left(\frac{x}{r}\right) = \omega t + E$$

$$x = r\cos(\omega t + E) \text{ .... (iv)}$$

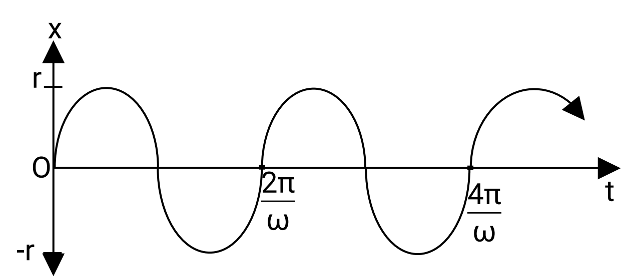
∅ and E are constants which can be obtained using the initial conditions.

Consider $x = r\sin(\omega t + \varnothing)$

If x = 0 at t = 0

$0 = r\sin\varnothing \Rightarrow \varnothing = 0$

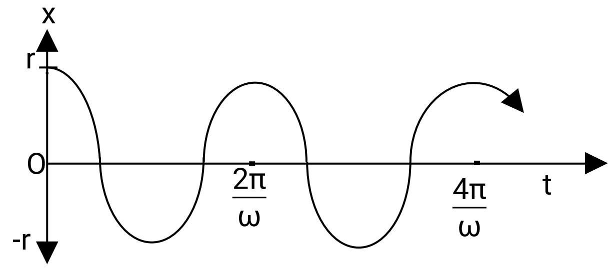
$\therefore\ x = r\sin\omega t$



If x = r at t = 0

$$r = r\sin\varnothing \Rightarrow \varnothing = \frac{\pi}{2}$$

$$\therefore\ x = r\sin\left(\omega t + \frac{\pi}{2}\right) \Rightarrow x = r\cos\omega t$$



Note that after time $\frac{2\pi}{\omega}$ the motion repeats itself.

$T = \frac{2\pi}{\omega}$ is called the periodic time of the motion.

This is the time taken to complete one cycle.

Summary of equations for a particle executing S.H.M:

1. $\ddot{x} = -\omega^2 x$
2. $\ddot{x}_{max} = \omega^2 r$
3. $v^2 = \omega^2(r^2 - x^2)$
4. $v_{max} = \omega r$
5. $x = r\sin \omega t$, if $x = 0$ at $t = 0$
6. $x = r\cos \omega t$, if $x = r$ at $t = 0$
7. $T = \frac{2\pi}{\omega}$

Example 1

A particle executing simple harmonic motion starts from rest and while at a point 3 m from the equilibrium position it is travelling at 8 m s^{-1} with an acceleration of 12 m s^{-2}. Find the:

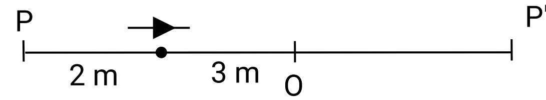
(i) angular velocity and amplitude of the motion.
(ii) shortest possible time the particle takes to come back to the given point.

Solution

When x = 3 m, v = 8 m s^{-1}, a = 12 m s^{-2}

(i)
$$a = -\omega^2 x$$
$$12 = -\omega^2(-3)$$
$$\omega = 2 \text{ rad s}^{-1}$$
$$v = \omega\sqrt{r^2 - x^2}$$
$$8 = 2\sqrt{r^2 - 3^2}$$
$$r = 5 \text{ m}$$

(ii)



$$x = r\cos \omega t$$
$$3 = 5\cos 2t$$
$$t = \frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$$
$$t_1 = \frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$$
$$t_1 = 0\cdot 464 \text{ s}$$
$$t_2 = \frac{1}{2}\left(2\pi - \cos^{-1}\frac{3}{5}\right)$$
$$t_2 = 2\cdot 678 \text{ s}$$
$$\text{Time taken} = t_2 - t_1$$
$$= 2\cdot 678 - 0\cdot 464$$
$$= 2\cdot 214 \text{ s}$$

Example 2

A particle moves in a straight line with simple harmonic motion of amplitude 2·5 m and period 2π seconds. Find the maximum speed and maximum acceleration of the particle.

Solution

$$r = 2\cdot 5 \text{ m}, T = 2\pi \text{ s}$$
$$v_{max} = \omega r, T = \frac{2\pi}{\omega} \Rightarrow 2\pi = \frac{2\pi}{\omega}$$
$$\Rightarrow \omega = 1 \text{ rad s}^{-1}$$
$$v_{max} = 1 \times 2\cdot 5$$
$$= 2\cdot 5 \text{ m s}^{-1}$$
$$a_{max} = \omega^2 r \Rightarrow a_{max} = 1^2 \times 2\cdot 5$$
$$= 2\cdot 5 \text{ m s}^{-2}$$

Example 3

A particle moving with simple harmonic motion in straight line has a maximum speed of 6 m s^{-1} and a maximum acceleration of 18 m s^{-2}. Find the:

(a) amplitude of the motion.
(b) period of the motion.
(c) speed of the particle when it is 1 m from the equilibrium position.

Solution:

$$v_{max} = 6 \text{ m s}^{-1}$$

$$\ddot{x}_{max} = 18 \text{ m s}^{-2}$$

$$v_{max} = \omega r$$

$$\omega r = 6 \text{ .... (i)}$$

$$\ddot{x}_{max} = \omega^2 r$$

$$\omega^2 r = 18 \text{............................ (ii)}$$

Dividing (ii) and (i)

$$\omega = 3 \text{ rad s}^{-1}$$

(a) From (i): $3r = 6 \Rightarrow r = 2$ m

(b) $$T = \frac{2\pi}{\omega}$$

$$= \frac{2\pi}{3} s$$

(c) When $x = 1$ m

$$v = \omega\sqrt{r^2 - x^2}$$

$$= 3\sqrt{2^2 - 1^2}$$

$$= 3\sqrt{3} \text{ m s}^{-1}$$

Example 4

A particle of mass 3 kg moves with simple harmonic motion in a straight line between points A and A' 15 m apart. Given that when the particle is 3 m from A its speed is 2 m s^{-1}, find the period of the motion. Find also the greatest force exerted on the particle during the motion.

Solution

From $v^2 = \omega^2(r^2 - x^2)$

$$2^2 = \omega^2\left(7{\cdot}5^2 - (7{\cdot}5 - 3)^2\right)$$

$$\omega = \frac{1}{3} \text{ rad s}^{-1}$$

$$T = \frac{2\pi}{\omega} \Rightarrow T = 2\pi \div \frac{1}{3}$$

$$T = 6\pi \text{ s}$$

$$F_{max} = m\omega^2 r$$

$$= 3 \times \left(\frac{1}{3}\right)^2 \times 7{\cdot}5$$

$$= 2{\cdot}5 \text{ N}$$

Example 5

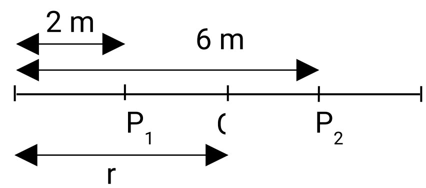
A particle performing simple harmonic motion starts from rest and passes through points P_1 and P_2 which are 2 m, 6 m respectively from the starting point with equal speeds. Given that 1 second after passing P_2 the particle next comes to rest, find the:

(i) period and amplitude of the motion.

(ii) time taken to move from P_1 directly to P_2.

Solution

(i)

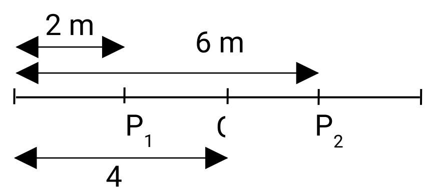


From $v^2 = \omega^2(r^2 - x^2)$

$$\omega^2[r^2 - (r-2)^2] = \omega^2[r^2 - (6-r)^2]$$

$$r^2 - (r-2)^2 = r^2 - (6-r)^2$$

$$r = 4 \text{ m}$$



Since the particle starts from rest, motion begins at the amplitude

From $x = r\cos\omega t$

$$x = 4\cos\omega t$$

At P_2; $x = -2$

$$-2 = 4\cos\omega t_2$$

$$t_2 = \frac{1}{\omega}\cos^{-1}\left(\frac{-1}{2}\right)$$

$$t_2 = \frac{1}{\omega}\left[\pi - \cos^{-1}\left(\frac{1}{2}\right)\right] = \frac{1}{\omega}\left(\pi - \frac{\pi}{3}\right)$$

$$t_2 = \frac{2\pi}{3\omega}$$

When particle comes to rest, $x = -4$, $t = t_3$

$$-4 = 4\cos\omega t_3$$

$$t_3 = \frac{1}{\omega}\left(\pi - \cos^{-1} 1\right)$$

$$t_3 = \frac{\pi}{\omega}$$

$$t_3 - t_2 = 1 \text{ s}$$

$$\frac{\pi}{\omega}-\frac{2\pi}{3\omega}=1$$

$$\frac{\pi}{3\omega}=1 \Rightarrow \omega=\frac{\pi}{3}\text{ rad s}^{-1}$$

$$T=2\pi \div \frac{\pi}{3}$$

$$T=6\text{ s}$$

(ii) At P_2, $t_2=\frac{2\pi}{3\omega}=\frac{2\pi}{3}\div\frac{\pi}{3}=2$ s

At P_1, $x = 2$

$$2=4\cos\left(\frac{\pi}{3}t_1\right)$$

$$t_1=\frac{3}{\pi}\cos^{-1}\left(\frac{1}{2}\right)=\frac{3}{\pi}\times\frac{\pi}{3}$$

$$t_1=1\text{ s}$$

Time taken to move from P_1 directly to P_2 is $t_2 - t_1 = 2-1 = 1$ s

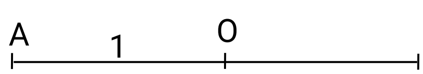
Example 6

A particle is released from rest at a point A, 1 m from a second point O. The particle accelerates towards O and moves with simple harmonic motion of time period 12 s and O as centre of oscillation. Find how far the particle is from O one second after release. How many seconds after release is the particle at the midpoint of OA for the:

(a) first time.

(b) second time.

Solution:



The amplitude, $r = 1$ m, $T = 12$ s $\Rightarrow 12=\frac{2\pi}{\omega}$

$$\Rightarrow \omega=\frac{\pi}{6}\text{ rad s}^{-1}$$

Since $x = r$, at $t = 0$

$$x=r\cos\omega t$$

$$x=1\cos\left(\frac{\pi}{6}t\right)$$

When $t = 1$ s

$$x=\cos\frac{\pi}{6}\Rightarrow x=\frac{\sqrt{3}}{2}\text{ m}$$

When $x=\frac{1}{2}$ m

$$\frac{1}{2}=\cos\left(\frac{\pi}{6}t\right)$$

$$t=\frac{6}{\pi}\cos^{-1}\left(\frac{1}{2}\right)$$

(a) $t_1=\frac{6}{\pi}\cos^{-1}\left(\frac{1}{2}\right)$

$t_1 = 2$ s

(b) $t_2=\frac{6}{\pi}\left[2\pi-\cos^{-1}\left(\frac{1}{2}\right)\right]$

$t_2=\frac{6}{\pi}\left[2\pi-\frac{\pi}{3}\right]$

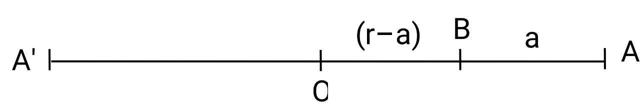
$t_2 = 10$ s

Example 7

A particle starts from rest and performs simple harmonic motion on a straight line. T seconds later, when its speed is half of its maximum speed, the distance covered is a metres. Prove that the:

(a) amplitude of the motion is $2a(2+\sqrt{3})$ m.
(b) period of the motion is 12T seconds.
(c) distance covered in the next T seconds is $a(\sqrt{3}+1)$ m.

Solution



r is the amplitude.

(a) $v_{max} = \omega r$

$v_B=\frac{1}{2}\omega r$ when $t = T$ and $x = r-a$

From $v^2=\omega^2(r^2-x^2)$

$$\left(\frac{\omega r}{2}\right)^2=\omega^2[r^2-(r-a)^2]$$

$$r^2-8ar+4a^2=0$$

$$r=\frac{8a\pm\sqrt{64a^2-4\times1\times4a^2}}{2}$$

$$r=2a(2\pm\sqrt{3})$$

Hence $r=2a(2+\sqrt{3})$ as required

(b) From $x = r\cos\omega t$

$(r-a) = r\cos\frac{2\pi T}{t_p}$, where t_P is the periodic time

$$2a(2+\sqrt{3})-a = 2a(2+\sqrt{3})\cos\frac{2\pi T}{t_p}$$

$$\frac{2\sqrt{3}+3}{2(\sqrt{3}+2)} = \cos\left(\frac{2\pi T}{t_p}\right)$$

$$\frac{\sqrt{3}}{2} = \cos\left(\frac{2\pi T}{t_p}\right)$$

$$\frac{\pi}{6} = \frac{2\pi T}{t_p}$$

$t_P = 12T$ as required

(c) From $x = r\cos\omega t$

$$x = 2a(\sqrt{3}+2)\cos\left(\frac{2\pi t}{t_p}\right)$$

When $t = 2T$

$$x = 2a(\sqrt{3}+2)\cos\left(\frac{2\pi\times 2T}{12T}\right)$$

$$= 2a(\sqrt{3}+2)\cos\left(\frac{\pi}{3}\right)$$

$$= a(\sqrt{3}+2)\ \text{m}$$

Distance covered in next T seconds:

$$= a(\sqrt{3}+2)-a$$

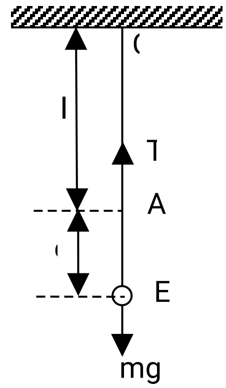
$$= a(\sqrt{3}+1)\ \text{m}$$

17.2 Forces producing simple harmonic motion

To show that a given set of forces acting on a particle produce simple harmonic motion, we must prove that the equation of motion of the particle can be expressed in the form $\ddot{x} = -\omega^2 x$.

1. Vertical string:

Consider a light elastic string of natural length l and modulus of elasticity λ, having one end fixed at a point O and a particle of mass m, attached to the other end.



If T is the tension in the string when the particle hangs at E in equilibrium;

From Hooke's law: $T = \frac{\lambda e}{l}$

In equilibrium: $T = mg$

Hence $\frac{\lambda e}{l} = mg$

If the particle is given a downward displacement x from E and released.

$$mg - T' = ma$$

$$mg - \frac{\lambda(x+e)}{l} = m\ddot{x}$$

$\ddot{x} = \frac{-\lambda}{ml}x$ which is of the form $\ddot{x} = -\omega^2 x$

hence S.H.M in which

$$\omega = \sqrt{\frac{\lambda}{ml}}$$

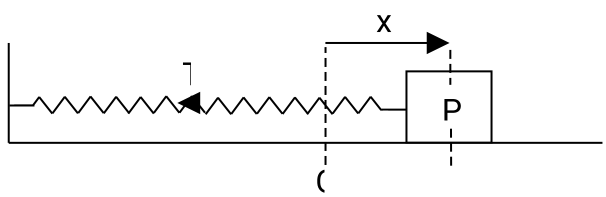
The period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{ml}{\lambda}}$

The amplitude r of the motion is the maximum displacement of the particle from E.

Note: For complete S.H.M, the string should remain taut throughout the motion. This is possible with $r < e$. For the case of a particle suspended from a spring, S.H.M is theoretically possible even when $r > e$ since a spring undergoes both compression and extension. The same derivation is used for the case of a spring.

2. Horizontal spring:

Consider a particle of mass m, at the end of a horizontal spring of modulus of elasticity λ and natural length l, resting on a smooth horizontal surface.



When the particle is given a small horizontal displacement x from O and released:

From Hooke's law $T = \frac{\lambda x}{l}$

Equation of motion;

$$O - T = m\ddot{x}$$

$$\frac{-\lambda x}{l} = m\ddot{x}$$

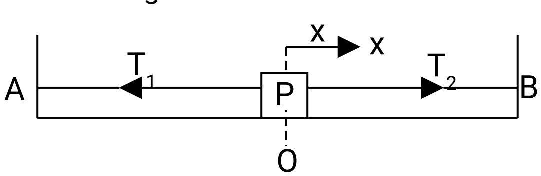
$$\ddot{x} = -\left(\frac{\lambda}{ml}\right)x$$

Hence S.H.M in which $\omega = \sqrt{\frac{\lambda}{ml}}$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{ml}{\lambda}}$$

3. Particle between two horizontal strings:

Consider a particle of mass m, between two horizontal strings as shown below.



In equilibrium $T_1 = T_2$ ………………….. (i)

This condition is used in locating the equilibrium position.

If the particle is given a displacement x towards B.

$T_2' - T_1' = m\ddot{x}$ ……………………….. (ii)

This equation is used to prove that the motion is simple harmonic.

Note: If string AP has natural length l_1 and modulus of elasticity λ_1 and string PB has natural length l_2 and modulus of elasticity λ_2 then:

$T_1 = \frac{\lambda_1 e_1}{l_1}$ and $T_2 = \frac{\lambda_2 e_2}{l_2}$ where e_1 and e_2 are respective extensions in AP and PB when the system is in equilibrium. Using equation (i) when the system is in equilibrium:

$$\frac{\lambda_1 e_1}{l_1} = \frac{\lambda_2 e_2}{l_2}$$

When P is given a small horizontal displacement x towards B:

$T_1' = \frac{\lambda_1(e_1+x)}{l_1}$ and $T_2' = \frac{\lambda_2(e_2-x)}{l_2}$

Using equation (ii) above:

$$\frac{\lambda_2(e_2-x)}{l_2} - \frac{\lambda_1(e_1+x)}{l_1} = m\ddot{x}$$

On simplifying:

$$m\ddot{x} = -\left(\frac{\lambda_1 l_2 + \lambda_2 l_1}{l_1 l_2}\right)x \Rightarrow \ddot{x} = -\left(\frac{\lambda_1 l_2 + \lambda_2 l_1}{m l_1 l_2}\right)x$$

Hence simple harmonic motion in which:

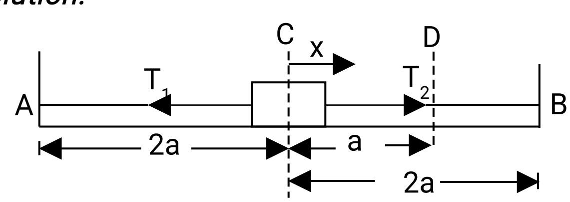
$\omega = \sqrt{\frac{\lambda_1 l_2 + \lambda_2 l_1}{m l_1 l_2}}$ and $T = 2\pi\sqrt{\frac{m l_1 l_2}{\lambda_1 l_2 + \lambda_2 l_1}}$.

Example 8

An elastic string of natural length 2a and modulus λ has its ends attached to two points A and B on a smooth horizontal table. The distance AB is 4a and C is the midpoint of AB. A particle of mass m, is attached to the midpoint of the string. The particle is then released from rest at D, the midpoint of CB. Denoting x the displacement of the particle from C, show that the equation of motion of the particle is $\ddot{x} + \frac{2\lambda}{ma}x = 0$.

Find the maximum speed of the particle and show that the time taken for the particle to move from D, directly to the midpoint of CD is $\frac{\pi}{3}\sqrt{\frac{ma}{2\lambda}}$.

Solution:



In equilibrium $T_1 = T_2$

$T_1 = \frac{\lambda e_1}{a}$ and $T_2 = \frac{\lambda e_2}{a}$ where e_1 and e_2 are respective extensions in AC and CB when the system is in equilibrium.

$\frac{\lambda e_1}{a} = \frac{\lambda e_2}{a} \Rightarrow e_1 = e_2$

Also $e_1 + e_2 = 2a$

Hence $e_1 = e_2 = a$

When the mass is given a displacement x from C towards B.

$$T_2' - T_1' = m\ddot{x}$$

$$\frac{\lambda(a-x)}{a} - \frac{\lambda(a+x)}{a} = m\ddot{x}$$

$$\frac{\lambda}{a}(a-x-a-x) = m\ddot{x}$$

$$m\ddot{x} = \frac{-2\lambda}{a}x$$

$$\ddot{x} + \frac{2\lambda}{ma}x = 0$$

The motion is simple harmonic with angular velocity $\omega = \sqrt{\frac{2\lambda}{am}}$

From $v_{max} = \omega r$, $r = a$

$$v_{max} = \left(\sqrt{\frac{2\lambda}{am}}\right)r = \left(\sqrt{\frac{2\lambda}{am}}\right)a$$

$$v_{max} = \sqrt{\frac{2a\lambda}{m}}$$

Since $x = r$ at $t = 0$; $x = r\cos\omega t$, $\omega = \sqrt{\frac{2\lambda}{am}}$

$$x = a\cos\left[\left(\sqrt{\frac{2\lambda}{am}}\right)t\right]$$

When $x = \frac{1}{2}a$ (at the midpoint of CD):

$$\frac{1}{2}a = a\cos\left[\left(\sqrt{\frac{2\lambda}{am}}\right)t\right]$$

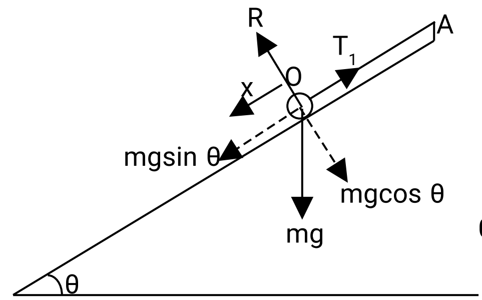
$$t = \sqrt{\frac{ma}{2\lambda}}\cos^{-1}\frac{1}{2}$$

$$t = \frac{\pi}{3}\sqrt{\frac{ma}{2\lambda}}$$

Example 9

A particle is attached to one end of a light elastic string, the other end of which is fastened to a fixed point A on a smooth plane inclined at an angle arc $\sin\frac{1}{4}$ to the horizontal. The particle rests in equilibrium at a point O on the plane with the string stretched along a line of greatest slope and extended by an amount c. If the particle is released from rest at a point P on AO produced, show that as long as the string remains taut the particle will oscillate with simple harmonic motion about O as a centre and state the periodic time.

Solution



$e = c$

$\theta = \sin^{-1}\frac{1}{4}$

Along plane;

In equilibrium: $T_1 = mg\sin\theta$

$$T_1 = mg\times\frac{1}{4} = \frac{1}{4}mg$$

From Hooke's law: $T_1 = \frac{\lambda c}{l_0}$

$$\text{Hence: } \frac{1}{4}mg = \frac{\lambda c}{l_0} \Rightarrow \lambda = \frac{mgl_0}{4c}$$

When the particle is given a displacement OP = x and released:

$$mg\sin\theta - T_2 = ma$$

$$ma = \frac{1}{4}mg - \frac{\lambda(c+x)}{l_0}$$

$$ma = \frac{1}{4}mg - \frac{\lambda c}{l_0} - \frac{\lambda x}{l_0}$$

$$a = -\frac{\lambda}{ml_0}x = -\frac{mgl_0}{4c}\times\frac{1}{ml_0}\times x$$

$$a = -\frac{g}{4c}x$$

Hence S.H.M in which $\omega = \sqrt{\frac{g}{4c}} = \frac{1}{2}\sqrt{\frac{g}{c}}$

The period, T of the motion is given by: $T = \frac{2\pi}{\omega}$

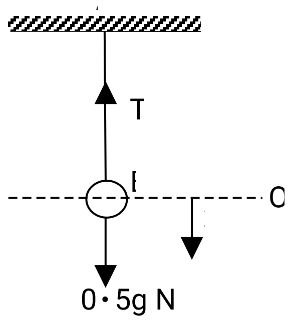
$$T = 4\pi\sqrt{\frac{c}{g}}.$$

Example 10

A body of mass 500 g is attached to end B of a light elastic string AB of natural length 50 cm. The system rests in equilibrium with the string vertical and end A fixed. The body is then pulled vertically downwards through a small displacement and released. If the ensuing motion is simple harmonic of period $\frac{\pi}{5}$ s, find the modulus of the string.

Solution

$$l_0 = 0 \cdot 5 \text{ m}$$



In equilibrium, $T = 0 \cdot 5g$

From Hooke's law; $T = \frac{\lambda e}{l_0} = \frac{\lambda e}{0 \cdot 5} = 2\lambda e$

$$\therefore 0 \cdot 5g = 2\lambda e \Rightarrow e = \frac{g}{4\lambda}$$

When the body is given a downward displacement x from equilibrium and released:

$$0 \cdot 5g - T' = m\ddot{x}$$

$$0 \cdot 5g - \frac{\lambda(x+e)}{l_0} = 0 \cdot 5\ddot{x}$$

$$0 \cdot 5g - \frac{\lambda x}{0 \cdot 5} - \lambda \times \frac{g}{4\lambda \times 0 \cdot 5} = 0 \cdot 5\ddot{x}$$

$$-2\lambda x = 0 \cdot 5\ddot{x}$$

$$\ddot{x} = -4\lambda x, \text{ Hence S.H.M}$$

$$\omega = \sqrt{4\lambda} = 2\sqrt{\lambda} \text{ rad s}^{-1}$$

$$T = \frac{2\pi}{2\sqrt{\lambda}} \Rightarrow \frac{\pi}{\sqrt{\lambda}} = \frac{\pi}{5}$$

$$\lambda = 25 \text{ N}$$

Exercises

Exercise: 17A

1. Find the periodic time of the simple harmonic motion governed by the following equations:
   (a) $\ddot{x} = -x$ (b) $\ddot{x} = -4x$ (c) $\ddot{x} = -9x$
2. A mass of 10 kg moves with simple harmonic motion in a straight line. When it is 2 m from the centre of oscillation, the velocity and acceleration of the body are 12 m s^{-1} and 162 m s^{-2} respectively. Calculate the:
   (a) period.
   (b) amplitude.
   (c) greatest force exerted on the particle during its motion.
3. 3. A particle moves in a straight line with simple harmonic motion of period 5 seconds and has a maximum speed of 4 m s^{-1}. Find the:
   (i) amplitude.
   (ii) speed when it is $\frac{6}{\pi}$ m from the centre.
4. A particle is performing simple harmonic motion with centre O, amplitude 6 m and period 2π seconds. Points B and C lie between O and A with OB = 1 m, OC = 3 m and OA = 6 m. Find the least time when travelling from:
   (a) A to B (b) A to C
5. A particle describing simple harmonic motion in a straight line directed to a fixed point O has a velocity of 25 m s^{-1} and an acceleration of 75 m s^{-2} when it is 3 m from O. Determine the:
   (i) period and amplitude of oscillation.
   (ii) time taken by the particle to reach O.
   (iii) velocity of the particle as it passes through O.
6. (a) A particle performing simple harmonic motion satisfies the equation $\ddot{x} + \omega^2 x = 0$, where ω is a positive constant and x the distance from the centre of motion. Show that $x = r\cos(\omega t + \Phi)$ where Φ is a constant and r is the amplitude.
   (b) A particle moving with simple harmonic motion of amplitude a metres travels from a point P, x_1 metres from the centre of motion directly to a point Q, x_2 metres

from the centre of motion in t seconds. Show that the period T of the motion satisfies the equation;

$$a^2\cos\frac{2\pi t}{T} = x_1x_2 + \sqrt{\left(a^2-x_1^2\right)\left(a^2-x_2^2\right)}.$$

7. A particle is moving with simple harmonic about a fixed point O on a line. The particle has velocity v_1 when its displacement from O is x and its velocity is v_2 when its displacement is y. Show that the period of oscillation satisfies the equation $T = 2\pi\sqrt{\left(\frac{x^2-y^2}{v_2^2-v_1^2}\right)}$.
8. A particle oscillates vertically in simple harmonic of amplitude 3 cm and frequency 5 Hz. Find the acceleration of the particle at:
   (a) the centre of motion.
   (b) maximum displacement.
   (c) a position halfway between the centre and maximum displacement.
9. A particle is moving with simple harmonic motion along the x-axis with centre of oscillation at O, at x = 4 m the speed is 6 m s^{-1} and at x = 3 m the speed is 8 m s^{-1}. Find the:
   (a) amplitude.
   (b) maximum velocity.
10. A particle moves with simple harmonic motion about a mean position O. It is initially projected from O with speed 6 m s^{-1} and just reaches a point A at 2 m from O. Find the:
    (a) distance from O after 3 s.
    (b) speed of the particle after 9 s.
    (c) time taken for the particle to be at a distance of 1 m from O for the third time.
11. A particle P executes simple harmonic motion with amplitude 3 m and period 2 s. If P is initially moving at maximum speed, determine the:
    (i) distance moved by P until its speed is half the maximum value.
    (ii) time taken by P to travel the distance in (i) above.

Exercise: 17B

1. A particle is performing simple harmonic motion with period 12 s. Its speed when 8 m from the centre of oscillation is $\frac{3}{5}$ of its maximum speed. If the particle is initially moving with maximum speed, find the:
   (i) amplitude of motion.
   (ii) time which elapses before the speed of the particle reduces to half its maximum value.
2. A particle moves with simple harmonic motion between two points A and B, 4 m apart. The greatest speed of the particle is 2π m s^{-1}. Find the:
   (i) least time taken by the particle to travel from A to a point 1 m from B.
   (ii) acceleration of the particle when at 1•848 m from B.
3. A particle executing simple harmonic motion starts from rest at point O and passes through points A and B in that order, given that OA = 0•1 m, OB = 0•2 m, and the particle passes through A and B with velocities 6 m s^{-1} and 8 m s^{-1} respectively. Calculate the:
   (a) amplitude and angular velocity of motion.
   (b) time the particle takes to move from A directly to B.
4. A body of mass 100 g moves horizontally with simple harmonic motion about a mean position O. When the body is 0•5 m from O the horizontal force on the body is 5 N. Find the periodic time of the motion.
5. A particle moves with simple harmonic motion about a mean position O. The particle has zero velocity at a point which is 50 cm from O and a speed of 3 m s^{-1} at O. Find the:
   (a) maximum speed of the particle.
   (b) amplitude of the motion.
   (c) periodic time of the motion.
6. A particle moves with simple harmonic motion about a mean position O. When the particle is 60 cm from O its speed is 1•6 m s^{-1} and when it is 80 cm from O its speed 1•2 m s^{-1}. Find the amplitude and period of the motion.
7. A particle is projected from a point A at time t = 0 and performs simple harmonic motion with A as the centre of oscillation. The amplitude of the motion is 50 cm and periodic time is 3 s. Find the:
   (a) speed of projection.
   (b) speed of the particle when t = 1 s.
   (c) speed of the particle when t = 2 s.
   (d) distance of the particle from A when t = 2 s.

8. A particle performs simple harmonic motion of period 3 s and amplitude 6 cm about centre O. Find the time it takes the particle to travel from O to a point P a distance of 3 cm from O.
9. The points A, O, B, C lie in that order on a straight line with AO = OC = 4 cm and OB = 2 cm. A particle performs simple harmonic motion of period 6 s and amplitude 4 cm between A and C. Find the time taken for the particle to travel from A to B.
10. The points A, O, B, C lie in that order on a straight line with AO = OC = 6 cm and OB = 5 cm. A particle performs simple harmonic motion of period 3 s and amplitude 6 cm between A and C. Find the time taken for the particle to travel from A to B.
11. A particle executing simple harmonic motion about a point O has speeds of $3\sqrt{3}$ m s^{-1} and 3 m s^{-1} when at distances of 1 m and 0•268 m respectively from the end point. Find the amplitude of the motion.
12. A particle moving with simple harmonic motion passes through three points A, B and C in that order with velocities 0 m s^{-1}, 2 m s^{-1} and -1 m s^{-1} respectively. Find the period and amplitude of the motion if AB = 2 metres and AC = 8 metres.

Exercise: 17C

1. A light elastic string of natural length l has one end fixed to a point O. The other end is attached to a particle of mass m. When the particle hangs in equilibrium the length of the string is $\frac{7}{4}$l. The particle is displaced from equilibrium so that it moves vertically when the string is taut. Show that the ensuing motion is simple harmonic with period $\pi\sqrt{\frac{3l}{g}}$. At time t = 0 the particle is released from rest at a point A at a distance $\frac{3}{2}$l vertically below O. Find the:
   (i) depth below O of the lowest point L of the motion.
   (ii) time taken to move from A to L.
   (iii) depth below O of the particle at time $t = \frac{1}{3}\pi\sqrt{\frac{3l}{g}}$.
2. A light elastic string of natural length l has one end fixed and to the free end is attached a particle of mass m and the system hangs vertically. If the mass is given a downward vertical displacement, show that the resulting motion is simple harmonic. If this particle is now replaced by another particle of mass βm and it is found that the period of oscillation is doubled. Find the value of β.
3. A light elastic string PQ of modulus of elasticity λ and natural length a has one end P fixed. A mass m hanging in equilibrium at the other end Q stretches the string through a distance l. Show that when the mass is pulled down a further small distance and released from rest, the resulting motion is simple harmonic of period $T = 2\pi\sqrt{\frac{l}{g}}$. The mass m is detached and replaced by a mass m' which also similarly moves with simple harmonic motion of period T'. Show that $\frac{m}{m'} = \left(\frac{T}{T'}\right)^2$ and find the period in terms of T and T' if both masses hang are together at the end of the string.
4. When a particle of mass m kg is supported by an elastic string of natural length l m the extension in the string is e_0. The particle is displaced through a distance $\frac{3}{4}e_0$ from equilibrium and allowed to make vertical oscillations.
   (a) Show that the motion is simple harmonic with period $T = 2\pi\sqrt{\frac{e_0}{g}}$.
   (b) Prove that the time the particle takes to move from its lowest point to a point $\frac{1}{2}e_0$ above the equilibrium position is $\left[\pi - \cos^{-1}\left(\frac{2}{3}\right)\right]\sqrt{\frac{e_0}{g}}$.
   (c) Find an expression for the time the particle takes to move between points at which the speeds are half of the particle's maximum speed.
5. A mass of 5 kg is suspended from a string causing an extension of 25 cm. If the mass in pulled down a further distance of 25 cm and then released, it performs simple harmonic motion. Find the:

(a) frequency of the motion.

(b) velocity when the mass is 1·25 cm above the lowest point.

6. A particle of mass 1 kg is attached to one end of a light elastic string AB of modulus 25 N and natural length 0·5 m. The other end B of the string is fixed and A is held at a distance of 0·75 m vertically below B and then released. Calculate the initial acceleration of A.

7. A light elastic string is of natural length 60 cm and modulus 3mg N. The string hangs vertically with its top end fixed and a body of mass m kg fastened to the other end. Find the extension in the string when the body hangs in equilibrium. If the body is then pulled vertically downwards a distance of 10 cm and released, show that the ensuing motion will be simple harmonic and find the time period of the motion and the maximum speed of the body.

8. A particle of mass m is attached to one end of a light elastic string of natural length l and modulus 2mg N. The other end of the string is fixed at a point A. The particle rests on a support B vertically below A, with AB = $\frac{5}{4}$l. Find the tension in the string and the reaction exerted on the particle by support B. The support B is suddenly removed. Show that the particle will execute simple harmonic motion and find the:

(i) depth below A of the centre of oscillation.

(ii) period of the motion.

Exercise: 17D

1. A particle P of mass m lies on a smooth horizontal table and is attached to two fixed points A, B on the table by two light strings of natural lengths 2l, 3l and moduli 2mg, mg respectively. If AB = 7l show that when P is in equilibrium, AP = $\frac{5}{2}$l. The particle P is held at rest at a point C in the line AB, when AC = 3l and C lies between A and B, and is then released. Show that the ensuing motion of P is simple harmonic of period $\pi\sqrt{\frac{3l}{g}}$. Find the maximum speed of the particle in the ensuing motion.

2. (a) A mass oscillates with simple harmonic motion of period one second. The amplitude of oscillation is 5 cm. Given that the particle begins from the centre of the motion, state the relationship between the displacement x of the mass and time t. Hence find the first times when the mass is 3 cm from its end position.

(b) A particle of mass m is attached by means of light strings AP and PB of the same natural length a m and moduli of elasticity mg N and 2mg N respectively to the points A and B on a smooth horizontal table. The particle is released from the midpoint of AB, where AB = 3a m. Show that the motion of the particle is simple harmonic with period $T = \left(\frac{4\pi^2 a}{3g}\right)^{\frac{1}{2}}$.

3. A particle P of mass m lies on a smooth horizontal table and is attached to two light elastic strings fixed to the table at points A and B. The natural lengths of the strings are AP = 4l, PB = 5l and their moduli of elasticity are mg and $\frac{5}{2}$mg respectively. Given AB = 12l, Show that when P is in equilibrium, AP = 6l, P is now held at C in the line AB with AC = 5l and then released. Show that the resulting motion is simple harmonic with period $4\pi\sqrt{\frac{l}{3g}}$. Find the maximum speed of the particle.

4. A light elastic string of natural length 2·4 m and modulus of elasticity 15 N is stretched between two points A and B, 3 m apart on a smooth horizontal surface. A particle of mass 4 kg is attached at the midpoint of the string and pulled 10 cm towards B and then released.

(a) Show that the resulting motion is simple harmonic.

(b) Find the period.

(c) Find the maximum speed.

5. (a) A particle of mass 2 kg is attached to one end of an elastic string of natural length 1 m and modulus 8 N lying on a smooth horizontal plane. The other end of the string is fixed to a pointA on the plane and when the string is just taut the particle is at a point B. The particle is pulled away from A until it reaches a point C, where AC = 1·5 m

and then released.

(i) Find the velocity of the particle when it passes through B.

(ii) What is its velocity when it reaches A.

(b) A particle is performing simple harmonic motion in a straight line. Its speed when at 3 m from the centre of motion is 12 m s^{-1} and its acceleration is 27 m s^{-2}. Find the:

(i) periodic time and amplitude of motion.

(ii) least time taken to reach the centre of motion.

(iii) velocity of the particle as it passes through the centre of motion.

6. A particle of mass 1·5 kg lies on a smooth horizontal table and is attached to two light elastic strings fixed at points P and Q, 12 m apart. The strings are of natural length 4 m and 5 m and their moduli are λ and 2·5λ respectively.

(a) Show that the particle stays in equilibrium at a point R midway between P and Q.

(b) If the particle is held at some point S in the line PQ with PS = 4·8 m and then released, show that the particle performs simple harmonic motion and find the:

(i) period of oscillation.

(ii) velocity when the particle is 5·5 m from P.

7. A particle of mass 2 kg is attached to one end of an elastic spring of natural length 1 m whose other end A is fixed to a point on a smooth horizontal plane. The particle is pulled across the plane from B to C, where AC = 1·5 m and is then released. If the modulus of elasticity of the spring is 10 N, show that:

(a) from C to B the particle performs simple harmonic motion with B as the centre.

(b) the time taken to travel from B to C is $\frac{\pi\sqrt{5}}{10}$ seconds.

(c) the speed at B is $\frac{\sqrt{5}}{2}$ m s^{-1}.

8. A light spring is of natural length 1 m and modulus 2 N. One end of the spring is attached to a fixed point A on a smooth horizontal surface and to the other end is attached a body of mass 0·5 kg. The body is held at rest on the surface at a distance of 1·25 m from A. Show that on release the body will move with simple harmonic motion and find the amplitude and period of the motion.

9. Two points A and B are 1 m apart on a smooth horizontal table. A light spring of natural length 75 cm and modulus 54 N has one end fastened to the table at A and the other end to a body of mass 8 kg which is held at rest at B. Show that when the body is released it moves with simple harmonic motion and find the maximum speed of the body during the motion.

10. A body of mass 2 kg is fixed to the midpoint of a light elastic string of natural length 1 m and modulus 18 N. The ends of the string are attached to two points A and B, 2 m apart on a smooth horizontal surface. The body is pulled a distance r towards A (r < 0·5 m) and released. Show that the subsequent motion is simple harmonic and find the period of the motion. If the maximum speed of the body is 1.5 ms^{-1}, find the value of r.

11. A light elastic string of natural length 1·5 m and modulus 12 N is stretched between two points A and B, 2 m apart on a smooth horizontal surface. A body of mass 2 kg is attached to the midpoint of the string, pulled 20 cm towards A and released. Show that the subsequent motion is simple harmonic and find the speed of the body when it is 88 cm from A.

12. Two fixed points A and B on a smooth horizontal table are at a distance 10a apart. A particle P of mass m lies between A and B. It is attached to A by means of a light elastic string of modulus λ and natural length 2a and to B by means of another light elastic string of modulus 2λ and natural length 5a. Let AP = x, if the particle is given a slight displacement towards B:

(i) Show that $\ddot{x} + \left(\frac{9\lambda}{10am}\right)x = \frac{3\lambda}{m}$.

(ii) Determine x when the particle is in equilibrium.

(iii) If the particle is released from rest when string AP is just taut, show that the particle moves with simple harmonic motion of amplitude $r = \frac{4}{3}a$ and period

$$T = \frac{2\pi}{3}\sqrt{\frac{10am}{\lambda}}.$$

Answers to Exercises

Exercise: 17A

1. (a) 2π s (b) π s (c) $\frac{2\pi}{3}$ s 2. (a) $\frac{2\pi}{9}$ s (b) 2•404 m (c) 1947 N

3. (i) $\frac{10}{\pi}$ m (ii) 3•2 m s^{-1}

4. (a) 0•45π s (b) $\frac{\pi}{3}$ s **5.** (i) $\frac{2\pi}{5}$ s; $\sqrt{34}$ m (ii) 0•108 s (iii) $5\sqrt{34}$ m s^{-1}

6. (a) (b) 7. 8. (a) 0 m s^{-2} (b) 29•61 m s^{-2} (c) 14•804 m s^{-2}

9. (a) 5 m (b) 10 m s^{-1}

10. (a) 0•824 m (b) 1.7528 m s^{-1} (c) $\frac{7\pi}{18}$ s

11. (i) $\frac{3\sqrt{3}}{2}$ m (ii) $\frac{1}{3}$ s

Exercise: 17B

1. (i) 10 m (ii) 2 s **2.**(i) $\frac{2}{3}$ s **(ii)** 1.5ms^{-2}

3. (a) 0•5 m; 20 rad s^{-1} (b) 0•0142 s 4. $\frac{\pi}{5}$ s 5. (a) 3 m s^{-1} **(b)** 0•5 m (c) $\frac{\pi}{3}$ s **6.** 1 m; π s **7.** (a) $\frac{\pi}{3}$ m s^{-1} (b) $\frac{\pi}{6}$ m s^{-1} **(c)** $\frac{\pi}{6}$ m s^{-1} **(d)** $\frac{\sqrt{3}}{4}$ m **8.** 0.25 s **9.** 2 s **10.** 1•22 s **11.** 2•0 m **12.** 11•24 s ; 4•2 m

Exercise: 17C

1. (i) 2l (ii) $\frac{\pi}{2}\sqrt{\frac{3l}{g}}$ (iii) $\frac{15}{8}l$ 2. β = 4 3. $T'' = \sqrt{T^2+(T')^2}$

4. (a) (b) (c) $\frac{2\pi}{3}\sqrt{\frac{e_0}{g}}$ 5. (a) 0•9965 Hz (b) 0•4887 m s^{-1}

6. 2•7 m s^{-2}(upwards) 7. 20 cm ; $\frac{2\pi}{7}$ s ; 0•7 m s^{-1}

8. $\frac{1}{2}$mg ; $\frac{1}{2}$mg (i) $\frac{3}{2}l$ (ii) $\pi\sqrt{\frac{2l}{g}}$

Exercise: 17D

1. $\sqrt{\frac{gl}{3}}$ 2. (a) x = 0•05sin 2πt ; 0•0655 s ; 0•5655 s (b)

3. $\frac{1}{2}\sqrt{3gl}$ 4. (a) (b) $\frac{4\pi}{5}$ s (c) $\frac{1}{4}$ m s^{-1}

5. (a) (i) 1 m s^{-1} (ii) 1 m s^{-1} (b) (i) $\frac{2\pi}{3}$ s ; 5 m (ii) 0•2145 s (iii) 15 m s^{-1}

6. (a) (b)(i) $2\pi\sqrt{\frac{2}{\lambda}}$ s (ii) $\frac{1}{10}\sqrt{\frac{119\lambda}{2}}$ m s^{-1}

7. (a) (b) (c) 8. 25 cm ; π s 9. 0•75 m s^{-1}

10. $\frac{\pi}{3}$ s ; 25 cm 11. 0.64 m s^{-1} 12. (i) (ii) $\frac{10}{3}a$ (iii)

18. RESULTANT AND RELATIVE MOTION

18.1 Resultant Velocity

The resultant of given velocities when they are given in vectorial notation is simply the vector sum of the velocities.

If $v_1 = ai + bj$ and $v_2 = ci + dj$ then their resultant **v** is their vector sum:

$$\mathbf{v} = v_1 + v_2$$

$$\mathbf{v} = (ai + bj) + (ci + dj) = (a + c)i + (b + d)j$$

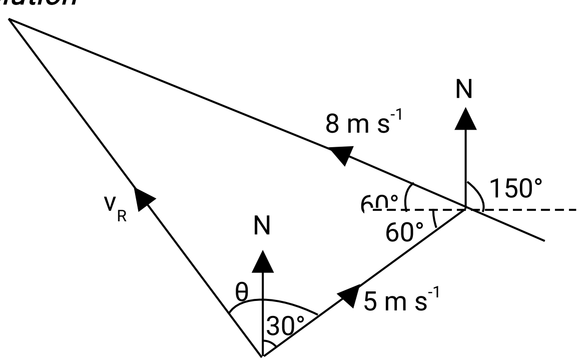
However if velocities are given in terms of magnitude and their directions by angles then:

- We can express the velocities in vector form thereafter we find the magnitude and direction of the resultant.
- If they are two velocities, we can use the parallelogram of vectors.

Example 1

A boat travelling at 5 m s^{-1} in the direction 030° in still water is blown by wind moving at 8 m s^{-1} from the bearing of 150°. Calculate the true speed and course the boat will be steered.

Solution



Let v_R be the true or resultant speed of the boat

$$v_R^2 = 8^2 + 5^2 - 2\times8\times5\times\cos 120$$

$$v_R = \sqrt{129} = 11\cdot3578 \text{ m s}^{-1}$$

$$\frac{v_R}{\sin 120} = \frac{8}{\sin\theta} \Rightarrow \frac{\sqrt{129}}{\sin 120} = \frac{8}{\sin\theta}$$

$$\theta = \sin^{-1}\left(\frac{8\sin 120}{\sqrt{129}}\right)$$

$$\theta = 37\cdot6°$$

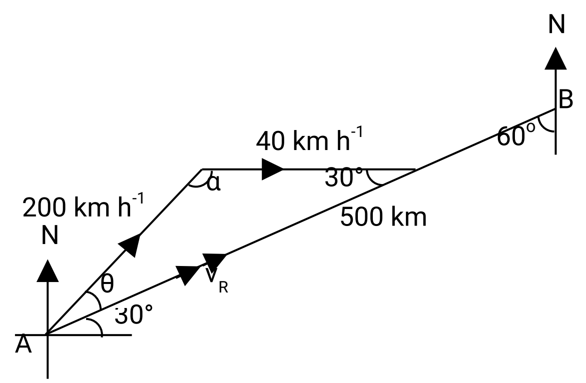
Course set = 360−(37·6−30) = 352·4°

Hence the course set is on a bearing of 352·4°.

Example 2

Two airfields A and B are 500 km apart with B on a bearing of 060° from A. An aircraft which can travel at 200 km h^{-1} in still air is to be flown from A to B. If there is a wind of 40 km h^{-1} blowing from the west, find the course that the pilot must set in order to reach B and also find to the nearest minute, the time taken.

Solution



$$\frac{200}{\sin 30} = \frac{40}{\sin\theta} \Rightarrow \theta = 5\cdot7°$$

Course set is on bearing of 60−5·7 = 054·3°

$$\alpha = 180-(30 + 5\cdot7) = 144\cdot3°$$

$$\frac{v_R}{\sin 144\cdot3} = \frac{200}{\sin 30}$$

$$v_R = 233\cdot4165 \text{ km h}^{-1}$$

$$t = \frac{500}{233\cdot4165} = 2\cdot142 \text{ hours}$$

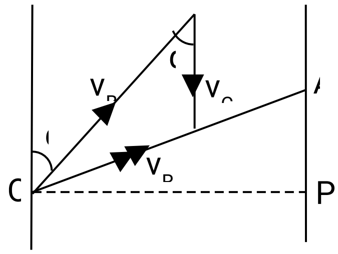
$$t = 2 \text{ hours } 8 \text{ minutes}$$

18.2 Crossing a river by boat

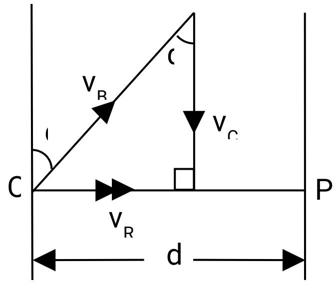
We consider crossing a river flowing between two parallel banks. Taking v_C as velocity of current (velocity with which the river flows), v_B as velocity of boat in still water and v_R as the resultant velocity(actual velocity with which the boat moves).

There are mainly four possible cases:

1. Crossing to a point upstream from starting point.



2. Crossing to a point exactly opposing the starting point.



The course set:

$$\cos\alpha = \frac{v_C}{v_B} \Rightarrow \alpha = \cos^{-1}\left(\frac{v_C}{v_B}\right)$$

Resultant velocity:

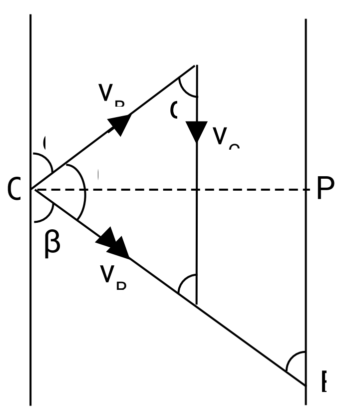
$$v_R^2 = v_B^2 - v_C^2 \Rightarrow v_R = \sqrt{v_B^2 - v_C^2}$$

Time taken for the crossing:

$$t = \frac{d}{v_R} \Rightarrow t = \frac{d}{\sqrt{v_B^2 - v_C^2}}$$

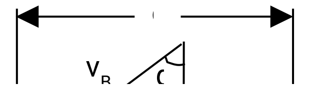
Note: This is only possible if $v_B > v_C$.

3. Crossing to a point downstream from starting point.



The solution to the problem can be obtained using the sine rule and the cosine rule.

4. Crossing the river in the shortest time.



- The course set is at an angle α to the bank.
- The resultant velocity is along the line from starting point to the target point.

$$t = \frac{OP}{v_B \sin\alpha} = \frac{d}{v_B \sin\alpha}$$

$$t_{min} = \frac{d}{(v_B \sin\alpha)_{max}}$$

But $(\sin\alpha)_{max} = 1$, when $\alpha = 90°$

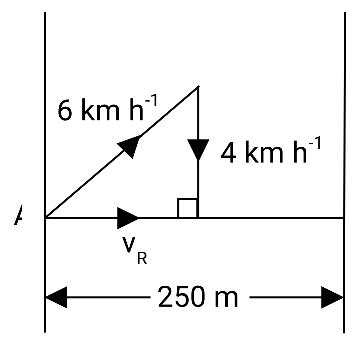
Hence $t_{min} = \dfrac{d}{v_B}$

Hence to cross the river in shortest possible time, the course is set directly across the river or normal to the bank.

Example 3

A man can row a boat in still water at 6 km h^{-1}. He wishes to cross a river to a point directly opposite his starting point. The river flows at 4 km h^{-1} and has a width of 250 m. Find the time the man would take to cross the river.

Solution:



$$v_R = \sqrt{6^2 - 4^2} = \sqrt{20} = 2\sqrt{5} \text{ km h}^{-1}$$

$$t = \frac{0{\cdot}25}{2\sqrt{5}}$$

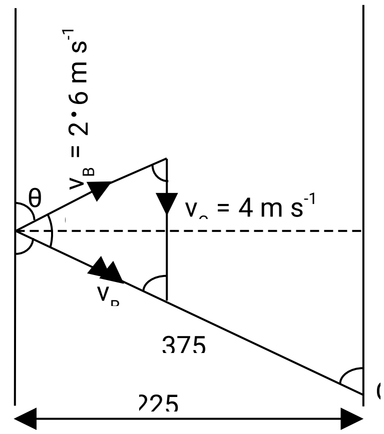
$= 0{\cdot}0559$ hours

$= 201{\cdot}25$ seconds.

Example 4

A river flows at a constant speed of 4 m s^{-1} between straight parallel banks which are 225 m apart. A boat which has a maximum speed of $2{\cdot}6\ \text{m s}^{-1}$ in still water leaves a point P on one bank and sails in a straight line to the opposite bank. Find the time the boat can take to reach a point Q on the opposite bank where PQ = 375 m and Q is downstream from P. Find also the least time the boat can take to cross the river. Find the time taken to sail from P and Q by the slowest boat capable of sailing directly from P to Q.

Solution



$$\sin\beta = \frac{225}{375} \Rightarrow \beta = 36{\cdot}9°$$

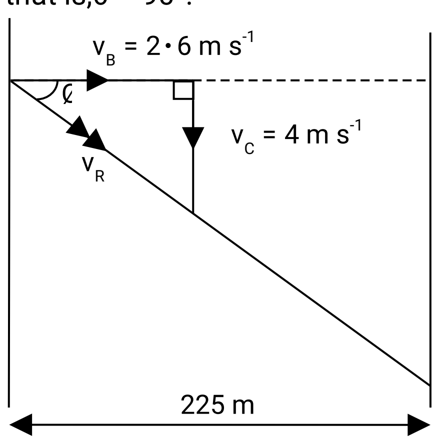
$$\frac{2{\cdot}6}{\sin\beta} = \frac{4}{\sin\alpha} \Rightarrow \frac{2.6}{\sin 36{\cdot}9} = \frac{4}{\sin\alpha} \Rightarrow \alpha = 67{\cdot}4°$$

$$\frac{v_R}{\sin\theta} = \frac{2{\cdot}6}{\sin 36{\cdot}9} \Rightarrow \frac{v_R}{\sin 75{\cdot}7} = \frac{2{\cdot}6}{\sin 36{\cdot}9}$$

$$\Rightarrow v_R = 4{\cdot}196\ \text{m s}^{-1}$$

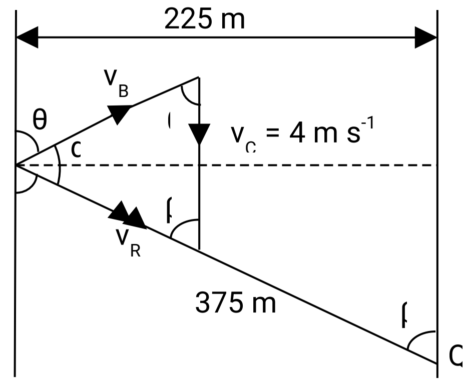
$$t = \frac{375}{4{\cdot}196} = 89{\cdot}4\ \text{s}$$

To cross in least time, the course is set normal to the bank, that is,θ = 90°.



$$t_{min} = \frac{225}{v_B} = \frac{225}{2{\cdot}6} = 86{\cdot}5\ \text{s}$$

Note: To find the time taken by the slowest boat, β remains the same since it is calculated from the triangle of distances but θ and α depend on the new speed of the boat.



$$\sin\beta = \frac{225}{375} \Rightarrow \sin\beta = 0{\cdot}6$$

$$\frac{v_B}{\sin\beta} = \frac{4}{\sin\alpha} \Rightarrow \frac{v_B}{0{\cdot}6} = \frac{4}{\sin\alpha}$$

$$v_B = \frac{2{\cdot}4}{\sin\alpha}$$

$$v_{B_{min}} = \frac{2{\cdot}4}{(\sin\alpha)_{max}}$$

$$(\sin\alpha)_{max} = 1, \text{ when } \alpha = 90°$$

$$v_{B_{min}} = \frac{2{\cdot}4}{1} = 2{\cdot}4\ \text{m s}^{-1}$$

$$v_R = \sqrt{4^2 - 2{\cdot}4^2}$$

$$v_R = 3{\cdot}2\ \text{m s}^{-1}$$

$$t = \frac{375}{3{\cdot}2} = 117{\cdot}2\ \text{s}$$

18.3 Relative and true velocity

18.3.1 Relative Velocity

For two bodies A and B moving with velocities v_A and v_B respectively, the velocity of A relative to B is the velocity of A as measured by an observer on B and is denoted by ${}_Av_B$ and given by ${}_Av_B = v_A - v_B$.

18.3.2 True Velocity

If v_B the true velocity of B is known and ${}_Av_B$ is also known, the true velocity of A, v_A can be calculated from: ${}_Av_B = v_A - v_B \Rightarrow v_A = {}_Av_B + v_B$.

Example 5

To an observer on a liner moving with velocity $(18i-17j)$ km h^{-1}, a yatch appears to have a velocity $(-8i+29j)$ km h^{-1}. To someone on the yatch the wind appears to have a velocity $(-5i-5j)$ km h^{-1}. Find the true velocity of the wind.

Solution

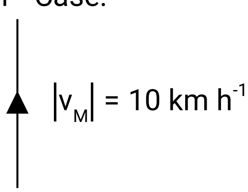
$$v_L = (18i-17j) \text{ km h}^{-1}$$
$${}_Tv_L = (-8i+29j) \text{ km h}^{-1}$$
$${}_Yv_L = v_Y - v_L$$
$$v_Y = {}_Yv_L + v_L$$
$$= (-8i+29j) + (18i-17j)$$
$$= (10i+12j) \text{ km h}^{-1}$$
$${}_Wv_Y = (-5i-5j) \text{ km h}^{-1}$$
$${}_Wv_Y = v_W - v_Y$$
$$v_W = {}_Wv_Y + v_Y$$
$$= (-5i-5j) + (10i+12j)$$
$$= (5i+7j) \text{ km h}^{-1}$$

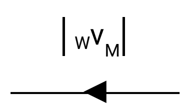
Example 6

When a man cycles due north at 10 km h^{-1} the wind appears to come from the east. When he cycles in a direction N60°W at 8 km h^{-1} it appears to come from the south. Find the velocity of the wind.

Solution:

1st Case:





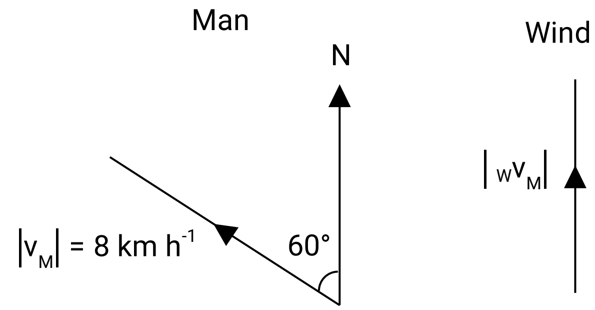
Man Wind

$v_M = \begin{pmatrix}0\\10\end{pmatrix}$ km h^{-1} Let $v_W = \begin{pmatrix}a\\b\end{pmatrix}$ km h^{-1}

$${}_Wv_M = v_W - v_M = \begin{pmatrix}a\\b\end{pmatrix} - \begin{pmatrix}0\\10\end{pmatrix} = \begin{pmatrix}a\\b-10\end{pmatrix}$$

Since ${}_Wv_M$ is due west; $b-10 = 0 \Rightarrow b = 10$

2nd Case: Note that the velocity of the man changes but the velocity of the wind remains the same.

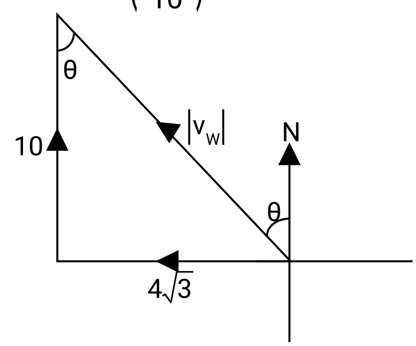


$$v_M = \begin{pmatrix}-8\sin 60\\8\cos 60\end{pmatrix} = \begin{pmatrix}-4\sqrt{3}\\4\end{pmatrix}$$

$${}_Wv_M = v_W - v_M = \begin{pmatrix}a\\10\end{pmatrix} - \begin{pmatrix}-4\sqrt{3}\\4\end{pmatrix} = \begin{pmatrix}a+4\sqrt{3}\\6\end{pmatrix}$$

Since ${}_Wv_M$ is due north $a + 4\sqrt{3} = 0 \Rightarrow a = -4\sqrt{3}$

$$v_W = \begin{pmatrix}-4\sqrt{3}\\10\end{pmatrix}$$



$$\tan\theta = \frac{4\sqrt{3}}{10} \Rightarrow \theta = 34{\cdot}7°$$

$$|v_W| = \sqrt{(4\sqrt{3})^2+10^2} = 12.166 \text{ km h}^{-1}$$

Velocity of wind 12.166 km h^{-1} from S34·7°E

Example 7

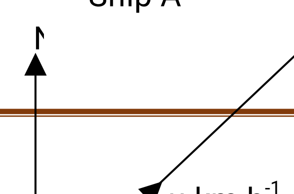
(a) Ship A is sailing with speed u km h^{-1} in a direction N30°E. A second ship B is sailing with speed v km h^{-1} in a direction Nθ°E. The velocity of ship A relative to B is due North East. Show that $u = v(\sqrt{3}+1)(\cos\theta - \sin\theta)$.

(b) Ship A changes its course to N60°E, while it continues with the same speed. Ship B continues with the same velocity. The velocity of ship A relative to B is now due East, find tan θ. (Leave your answer in surd form).

<u>Solution:</u>

(a)

Ship A

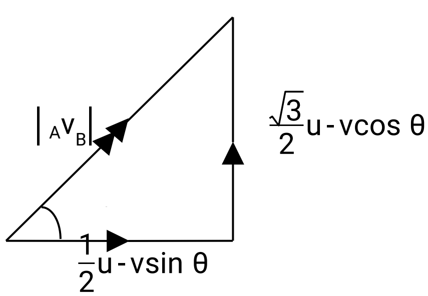


$$v_A = \begin{pmatrix} u\sin 30 \\ u\cos 30 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}u \\ \frac{\sqrt{3}}{2}u \end{pmatrix}$$

$$v_B = \begin{pmatrix} v\sin\theta \\ v\cos\theta \end{pmatrix}$$

$${}_Av_B = v_A - v_B = \begin{pmatrix} \frac{1}{2}u \\ \frac{\sqrt{3}}{2}u \end{pmatrix} - \begin{pmatrix} v\sin\theta \\ v\cos\theta \end{pmatrix} = \begin{pmatrix} \frac{1}{2}u - v\sin\theta \\ \frac{\sqrt{3}}{2}u - v\cos\theta \end{pmatrix}$$

Since ${}_Av_B$ is due north east:



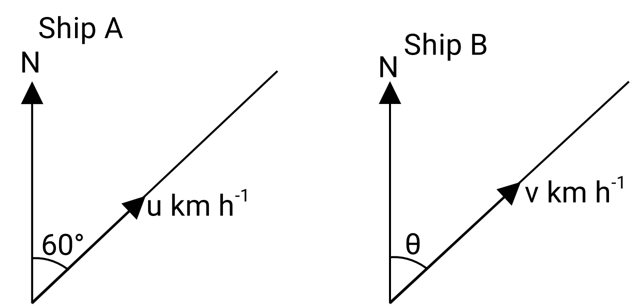
$$\tan 45 = \frac{\left(\frac{\sqrt{3}}{2}u - v\cos\theta\right)}{\left(\frac{1}{2}u - v\sin\theta\right)}$$

$$\Rightarrow \frac{1}{2}u - v\sin\theta = \frac{\sqrt{3}}{2}u - v\cos\theta$$

$$2v(\cos\theta - \sin\theta) = u\left(\sqrt{3}-1\right)$$

$$u = v(\sqrt{3}+1)(\cos\theta - \sin\theta)$$

(b)



$$v_A = \begin{pmatrix} u\sin 60 \\ u\cos 60 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2}u \\ \frac{1}{2}u \end{pmatrix}$$

$$v_B = \begin{pmatrix} v\sin\theta \\ v\cos\theta \end{pmatrix}$$

$${}_Av_B = v_A - v_B = \begin{pmatrix} \frac{\sqrt{3}}{2}u \\ \frac{1}{2}u \end{pmatrix} - \begin{pmatrix} v\sin\theta \\ v\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{3}}{2}u - v\sin\theta \\ \frac{1}{2}u - v\cos\theta \end{pmatrix}$$

Since ${}_Av_B$ is due east:

$$\frac{1}{2}u - v\cos\theta = 0 \Rightarrow u = 2v\cos\theta$$

But $u = v(\sqrt{3}+1)(\cos\theta - \sin\theta)$

Hence $2v\cos\theta = v(\sqrt{3}+1)(\cos\theta - \sin\theta)$

$$\frac{2}{\left(\sqrt{3}+1\right)} = 1 - \tan\theta \Rightarrow \tan\theta = 1 - \frac{2\left(\sqrt{3}-1\right)}{2}$$

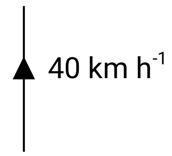
$$\tan\theta = \left(2-\sqrt{3}\right)$$

Example 8

To the driver of a car travelling due north at 40 km h^{-1} the wind appears to be blowing at 50 km h^{-1} from the direction N60°E. The wind velocity remains constant but the speed of the car is increasing. Find its speed when the wind appears to be blowing from the direction N30°E.

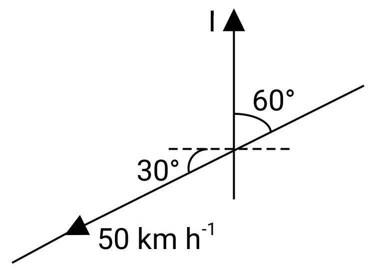
Solution

Car



$$v_C = \begin{pmatrix} 0 \\ 40 \end{pmatrix} \text{ km h}^{-1}$$

$$_Wv_C = 50 \text{ km h}^{-1} \text{ from N60°E}$$

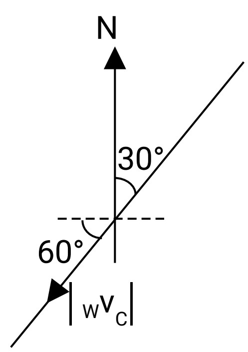


$$_Wv_C = \begin{pmatrix} -50\cos 30 \\ -50\sin 30 \end{pmatrix} = \begin{pmatrix} -25\sqrt{3} \\ -25 \end{pmatrix}$$

$$v_W = {_Wv_C} + v_C = \begin{pmatrix} -25\sqrt{3} \\ -25 \end{pmatrix} + \begin{pmatrix} 0 \\ 40 \end{pmatrix}$$

$$= \begin{pmatrix} -25\sqrt{3} \\ 15 \end{pmatrix}$$

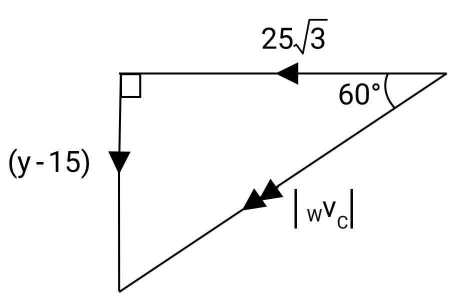
When wind appears to blow from N30°E:



$$\cdot {_Wv_C} = v_W - v_C$$

Let $v_C = \begin{pmatrix} 0 \\ y \end{pmatrix}$

$$_Wv_C = \begin{pmatrix} -25\sqrt{3} \\ 15 \end{pmatrix} - \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} -25\sqrt{3} \\ 15-y \end{pmatrix}$$



Note: For any real numbers a, b and c:

- If $a - b < 0 \Rightarrow |a-b| = b - a$
- If $a - b > 0 \Rightarrow |a-b| = a - b$
- If $c < 0 \Rightarrow |c| = -c$

$$\tan 60 = \frac{y-15}{25\sqrt{3}} \Rightarrow y = 90$$

$$v_C = \begin{pmatrix} 0 \\ 90 \end{pmatrix} \Rightarrow |v_C| = \sqrt{0^2+90^2} = 90 \text{ km h}^{-1}$$

Example 9

A man cycling at a speed v observes that when he is moving due north, the velocity of the wind appears to be $\frac{1}{2}u_1(\mathbf{i} - \sqrt{3}\mathbf{j})$. On changing his velocity to $\frac{1}{2}v(-\sqrt{3}\mathbf{i} + \mathbf{j})$, the wind velocity appears to be u_2 due East. Prove that the true velocity of the wind is $\frac{\sqrt{3}}{6}v(\mathbf{i} + \sqrt{3}\mathbf{j})$.

Solution

When $v_M = \begin{pmatrix} 0 \\ v \end{pmatrix} = v\mathbf{j}$; $_Wv_M = \frac{1}{2}u_1(\mathbf{i} - \sqrt{3}\mathbf{j})$

$$_Wv_M = v_W - v_M \Rightarrow v_W = v\mathbf{j} + \frac{1}{2}u_1\left(\mathbf{i} - \sqrt{3}\mathbf{j}\right) = \frac{1}{2}u_1\mathbf{i} + \left(v - \frac{\sqrt{3}}{2}u_1\right)\mathbf{j} \ldots\ldots\ldots(i)$$

When $v_M = \frac{1}{2}v\left(-\sqrt{3}\mathbf{i}+\mathbf{j}\right)$; $_Wv_M = u_2\mathbf{i}$

$$\text{From } {_Wv_M} = v_W - v_M \Rightarrow v_W = u_2\mathbf{i} + \frac{1}{2}v\left(-\sqrt{3}\mathbf{i}+\mathbf{j}\right)$$

$$v_W = \left(u_2 - \frac{\sqrt{3}}{2}v\right)\mathbf{i} + \frac{1}{2}v\mathbf{j} \ldots\ldots\ldots\ldots\ldots(ii)$$

From(i) and (ii)

$$\frac{1}{2}u_1\mathbf{i} + \left(v - \frac{\sqrt{3}}{2}u_1\right)\mathbf{j} = \left(u_2 - \frac{\sqrt{3}}{2}v\right)\mathbf{i} + \frac{1}{2}v\mathbf{j}$$

$$\mathbf{i} : \frac{1}{2}u_1 = u_2 - \frac{\sqrt{3}}{2}v \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (iii)$$

$$\mathbf{j} : v - \frac{\sqrt{3}}{2}u_1 = \frac{1}{2}v \Rightarrow u_1 = \frac{\sqrt{3}}{3}v$$

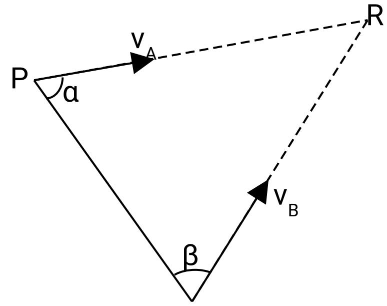
From (iii): $\frac{\sqrt{3}}{6}v = u_2 - \frac{\sqrt{3}}{2}v \Rightarrow u_2 = \frac{2\sqrt{3}}{3}v$

From (i): $v_W = \left(\frac{1}{2} \times \frac{\sqrt{3}}{3}v\right)\mathbf{i} + \left(v - \frac{\sqrt{3}}{3}v \times \frac{\sqrt{3}}{2}\right)\mathbf{j}$

$$v_W = \frac{\sqrt{3}}{6}v\left(\mathbf{i} + \sqrt{3}\mathbf{j}\right)$$

18.4 Interception and Collision

Consider two bodies A and B moving with uniform velocities v_A and v_B initially at points P and Q respectively.



For collision triangle PQR is closed.

That is; $\frac{|\vec{QR}|}{\sin\alpha} = \frac{|\vec{PR}|}{\sin\beta}$ and $\frac{v_A}{\sin\beta} = \frac{v_B}{\sin\alpha}$

Alternatively, if B is assumed to be at rest, then A moves with velocity ${}_Av_B$ given by

${}_Av_B = v_A - v_B$.

For collision ${}_Av_B$ is along the line $\vec{PQ}$. Hence if velocities and locations are given in vector form,

for a collision ${}_Av_B$ is parallel to $\vec{PQ}$ hence

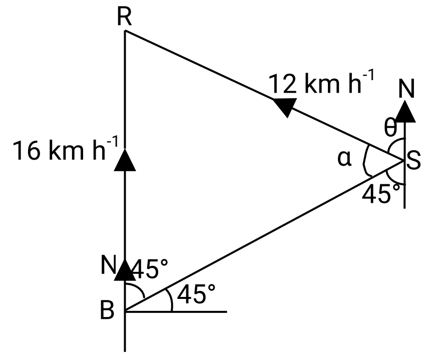
${}_Av_B = h\vec{PQ}$, where h is a constant.

The time taken for the collision is $t = \frac{|\vec{PQ}|}{|{}_Av_B|}$.

Example 10

A battle ship is steaming northwards at 16 km h^{-1} and is southwest of a submarine. Find two possible courses which the submarine could take in order to intercept the battleship if its maximum speed is 12 km h^{-1}.

Solution



For a collision:

$$\frac{12}{\sin 45} = \frac{16}{\sin\alpha} \Rightarrow \alpha = 70{\cdot}5°, 109{\cdot}5°$$

When $\alpha = 70{\cdot}5°$

$\theta = 180-(70{\cdot}5 + 45) = 64{\cdot}5°$

When $\alpha = 109{\cdot}5°$

$\theta = 180-(109{\cdot}5 + 45) = 25{\cdot}5°$

Possible courses: N64•5°W and N25•5°W

Example 11

Particles P and Q are moving with constant velocities $3\mathbf{i}-2\mathbf{j}$ and $-\mathbf{i} + 4\mathbf{j}$ respectively. Initially P has position vector $\mathbf{i} + 5\mathbf{j}$ and Q has position vector $3\mathbf{i} + 2\mathbf{j}$. Find the velocity of Q relative to P and the displacement of Q relative to P at time t. Show that the particles collide and state the value of t when the collision occurs.

Solution

$r_P(0) = \mathbf{i} + 5\mathbf{j}$ $\quad$ $v_P = 3\mathbf{i}-2\mathbf{j}$

$r_Q(0) = 3\mathbf{i} + 2\mathbf{j}$ $\quad$ $v_Q = -\mathbf{i} + 4\mathbf{j}$

$r_P(t) = r_P(0) + v_Pt = (\mathbf{i} + 5\mathbf{j}) + (3\mathbf{i}-2\mathbf{j})t$

$= (1 + 3t)\mathbf{i} + (5-2t)\mathbf{j}$

$r_Q(t) = r_Q(0) + v_Qt$

$= (3\mathbf{i} + 2\mathbf{j}) + (-\mathbf{i} + 4\mathbf{j})t = (3-t)\mathbf{i} + (2 + 4t)\mathbf{j}$

${}_Qr_{P(t)} = r_Q(t) - r_P(t) = [(3-t)\mathbf{i}+(2+4t)\mathbf{j}] - [(1 + 3t)\mathbf{i} + (5-2t)\mathbf{j}]$

${}_Qr_P(t) = (2-4t)\mathbf{i} + (6t-3)\mathbf{j}$

${}_Qv_P = (-\mathbf{i} + 4\mathbf{j})-(3\mathbf{i}-2\mathbf{j}) = (-4\mathbf{i} + 6\mathbf{j})$

When P and Q collide; ${}_Qr_P(t) = \mathbf{0}$

$(2-4t)\mathbf{i} + (6t-3)\mathbf{j} = 0\mathbf{i} + 0\mathbf{j}$

i: $2-4t = 0 \Rightarrow t = \frac{1}{2}$

j: $6t-3 = 0 \Rightarrow t = \frac{1}{2}$

Hence a collision occurs when $t = \frac{1}{2}$

Alternatively, for a collision between P and Q:

${}_Qv_P = h\,{}_Qr_P(0)$, where h is a constant.

${}_Qv_P = (-4\mathbf{i}+6\mathbf{j}) = 2(-2\mathbf{i}+3\mathbf{j})$ and

$$_Qr_P(0) = -2i + 3j$$

Hence $_Qv_P = 2\ _Qr_P(0)$ implying that P and Q collide.

Time for collision, $t = \dfrac{|_Qr_P(0)|}{|_Qv_P|} \Rightarrow t = \dfrac{\sqrt{(-2)^2+3^2}}{\sqrt{(-4)^2+6^2}}$

$$\Rightarrow t = \frac{1}{2}$$

Example 12

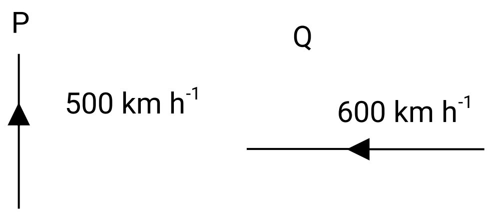
Two aircrafts P and Q are flying at the same height. P is flying north at 500 km h^{-1} while Q is flying due west at 600 km h^{-1}. When the aircrafts are 100 km apart, the pilots realise that they are about to collide. The pilot of P then changes course to 345° and maintains the speed of 500 km h^{-1}. The pilot of Q maintains his course but increases the speed. Determine the:

(a) distance each aircraft would have travelled if the pilots had not realised that they were about to collide.

(b) new speed beyond which the aircraft Q must fly to avoid collision.

Solution

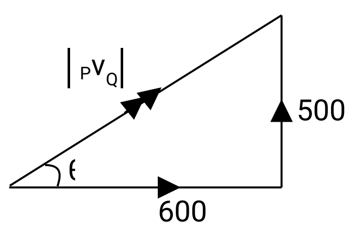
(a)



$$v_P = \begin{pmatrix}0\\500\end{pmatrix} \text{ km h}^{-1}$$

$$v_Q = \begin{pmatrix}-600\\0\end{pmatrix} \text{ km h}^{-1}$$

$$_Pv_Q = \begin{pmatrix}0\\500\end{pmatrix} - \begin{pmatrix}-600\\0\end{pmatrix} = \begin{pmatrix}600\\500\end{pmatrix}$$



$$|_Pv_Q| = \sqrt{600^2+500^2} = 781\cdot025 \text{ km h}^{-1}$$

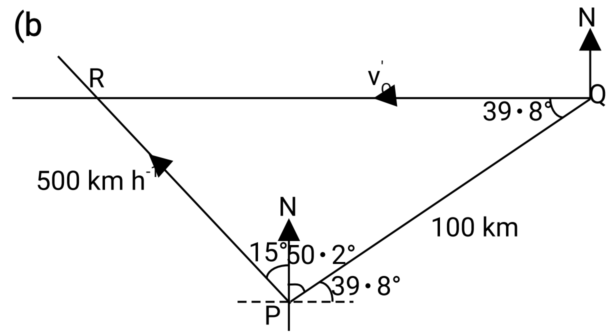
$$\tan\theta = \frac{500}{600} \Rightarrow \theta = 39\cdot8°$$

$$t = \frac{100}{781\cdot025} = 0\cdot128 \text{ hours}$$

$$s_P = 500\times0\cdot128 = 64 \text{ km}$$

$$s_Q = 600\times0\cdot128 = 76\cdot8 \text{ km}$$

(b



For collision, triangle PQR should be closed.

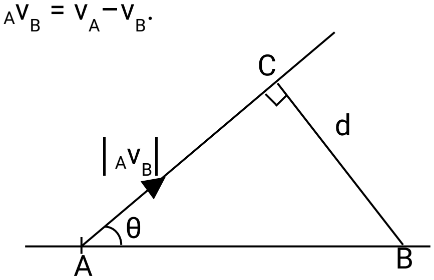
$$\frac{v_q'}{\sin 65\cdot2} = \frac{500}{\sin 39\cdot8}$$

$$v_Q' = 709\cdot08 \text{ km h}^{-1}$$

Hence to avoid collision $v_Q' > 709\cdot08$ km h^{-1}

18.5 Closest approach

Consider two bodies A and B moving with velocities v_A and v_B respectively. If B is assumed to be at rest then A moves with velocity $_Av_B$ where $_Av_B = v_A - v_B$.



The distance between A and B is closest when $\vec{BC}$ is perpendicular to $_Av_B$. The closest distance can be obtained from:

$$\sin\theta = \frac{d}{|\vec{AB}|}$$

$$d = |\vec{AB}|\sin\theta \text{ .... (i)}$$

The time taken for such a situation to occur is

$$t = \frac{|\vec{AC}|}{|_Av_B|} = \frac{|\vec{AB}|\cos\theta}{|_Av_B|} \text{ ....(ii)}$$

From (i): $d = |\vec{AB}|\sin\theta$

From $\vec{AB}\times {_Av_B} = |\vec{AB}||_Av_B|\sin\theta\,\hat{u}$, where $\hat{u}$ is a unit vector perpendicular to both $\vec{AB}$ and ${}_Av_B$.

$|\vec{AB}\times {}_Av_B| = |\vec{AB}||{}_Av_B|\sin\theta$

hence $\sin\theta = \dfrac{|\vec{AB}\times {}_Av_B|}{|\vec{AB}||{}_Av_B|}$.

This implies that $d = \dfrac{|\vec{AB}||\vec{AB}\times {}_Av_B|}{|\vec{AB}||{}_Av_B|}$

Hence $d = \dfrac{|\vec{AB}\times {}_Av_B|}{|{}_Av_B|}$.. (iii)

From (ii): $t = \dfrac{|\vec{AB}|\cos\theta}{|{}_Av_B|}$

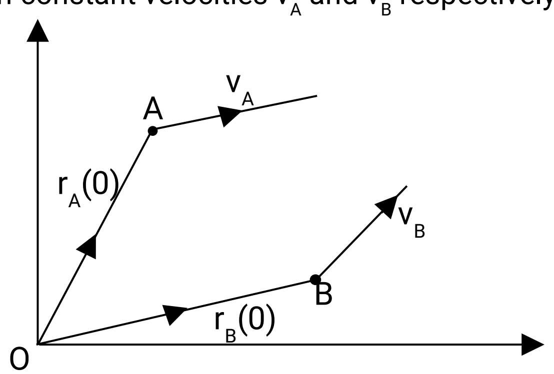
From $\vec{AB}\bullet{}_Av_B = |\vec{AB}||{}_Av_B|\cos\theta \Rightarrow \cos\theta = \dfrac{\vec{AB}\bullet{}_Av_B}{|\vec{AB}||{}_Av_B|}$

$t = \dfrac{|\vec{AB}|}{|{}_Av_B|}\times\dfrac{\vec{AB}\bullet{}_Av_B}{|\vec{AB}||{}_Av_B|}$

Hence $t = \dfrac{\vec{AB}\bullet{}_Av_B}{|{}_Av_B|^2}$ .... (iv)

Equations (i) and (ii) can be applied if the question is geometrical while (iii) and (iv) can be used when the question requires a vectorial approach.

Consider bodies A and B initially at points with position vectors $r_A(0)$ and $r_B(0)$ and are moving with constant velocities v_A and v_B respectively.



After time t:

$r_A(t) = r_A(0) + v_A\times t$

$r_B(t) = r_B(0) + v_B\times t$

${}_Ar_B(t) = r_A(t) - r_B(t)$

At closest distance:

$$\frac{d}{dt}|{}_Ar_B(t)|^2 = 0 \text{.............................. (v)}$$

Equation (v) can be used to obtain the time taken for the bodies to get closest and hence the closest distance $|{}_Ar_B(t)|_t$ can also be found.

Also since when A and B are closest, the relative displacement and relative velocity are perpendicular. At closest distance:

${}_Av_B\bullet{}_Ar_B(t) = 0$.....................................(vi)

From (vi) the time taken for A and B to be closest can be found and hence closest distances between A and B calculated.

Compleleting of squares technique can also be used to find the closest distance between two moving bodies and the time taken to attain it. If the displacement between two moving bodies A and B is given by ${}_Ar_B(t)$ then the distance d between the bodies is given by $d = |{}_Ar_B(t)|$. Since this distance is a function of time, it can be expressed in the form

$d^2 = h\left[(t-t_0)^2+d_0\right]$

$$\Rightarrow d = \sqrt{h\left[(t-t_0)^2+d_0\right]} \text{...............(vii)}$$

where h, t_0 and d_0 are constants.

Since $(t - t_0)^2 \geq 0 \Rightarrow d_{min} = \sqrt{hd_0}$ when $(t - t_0)^2 = 0 \Rightarrow t - t_0 = 0 \Rightarrow t = t_0$.

Hence using equation (vii) the closest distance between the bodies in the subsequent motion is $d_{min} = \sqrt{hd_0}$ and the time taken for this situation to occur is $t = t_0$.

Example 13

Particles A and B are moving with constant velocities $5\mathbf{i} + 2\mathbf{j}$ and $2\mathbf{i} + \mathbf{j}$ respectively. Initially the position vectors of A and B are $3\mathbf{i} + 4\mathbf{j}$ and $5\mathbf{i} + 8\mathbf{j}$ respectively. Write down expressions for r_A and r_B, the position vectors of the particles at time t. Find the vector $\vec{AB}$ at time t. Hence find the minimum distance between the particles.

$r_A(0) = (3i+4j) \qquad v_A = 5i + 2j$

$r_B(0) = (5i+8j) \qquad v_B = 2i + j$

Solution

$r_A = r_A(0) + v_A \times t$

$r_A = (3i + 4j) + (5i + 2j)t$

$= (3 + 5t)i + (4 + 2t)j$

$r_B = r_B(0) + v_B \times t$

$r_B = (5i + 8j) + (2i + j)t$

$= (5 + 2t)i + (8 + t)j$

$\vec{AB} = r_B - r_A = [(5 + 2t)i + (8 + t)j] - [(3 + 5t)i + (4 + 2t)j]$

$$\vec{AB} = (2-3t)i + (4-t)j$$

At closest distance; $\frac{d}{dt}\left|\vec{AB}\right|^2 = 0$

$\left|\vec{AB}\right|^2 = (2\text{-}3t)^2 + (4\text{-}t)^2$

$\frac{d}{dt}\left|\vec{AB}\right|^2 = 2(2-3t)\times\text{-}3 + 2(4-t)\times\text{-}1 = 20(t\text{-}1)$

At closest distance, $\frac{d}{dt}\left|\vec{AB}\right|^2 = 0$

$20t - 20 = 0 \Rightarrow t = 1$

When t = 1s

$$\left|\vec{AB}\right| = \sqrt{(2\text{-}3)^2 + (4\text{-}1)^2} = \sqrt{10}$$

Alternatively, using completing of squares technique:

From

${}_Br_A(t) = \vec{AB} = (2-3t)i + (4-t)j \Rightarrow d^2 = \left|{}_Br_A(t)\right|^2 = (2-3t)^2 + (4-t)^2$

$d^2 = 4 - 12t + 9t^2 + 16 - 8t + t^2 \Rightarrow d^2 = 10t^2 - 20t + 20$

On completing squares:

$d^2 = 10[t^2\text{-}2t+2] \Rightarrow d^2 = 10[(t\text{-}1)^2\text{-}1^2+2]$

$d = \sqrt{10[(t\text{-}1)^2+1]}$

$d_{min} = \sqrt{10\times1} \Rightarrow d_{min} = \sqrt{10}$ when $t - 1 = 0 \Rightarrow t = 1$

Example 14

Two particles A and B move with constant velocities i m s^{-1} and $(5i-3k)$ m s^{-1} respectively. At time $t = 0$, the position vectors of A and B are $(ai+bj+ck)$ m and $(i+4j-7k)$ m respectively. Show that if the particles are closest together at $t > 0$ then $4a-3c > 25$. Given the particles are closest together when $t = 2$ and at this instant the position vector of A is $(32i+7j+15k)$ m, find the values of a, b and c.

Solution

$v_A = \mathbf{i}$ m s^{-1}, $v_B = (5i-3k)$ m s^{-1}

At $t = 0$

$r_A(0) = (ai+bj+ck)$ m, $r_B(0) = (i+4j\text{-}7k)$ m

After time t:

$r_A(t) = (ai + bj + ck) + \mathbf{i}\times t = (a + t)\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

$r_B(t) = (i + 4j - 7k) + (5i - 3k)t$

$= (1 + 5t)\mathbf{i} + 4\mathbf{j} - (7 + 3t)\mathbf{k}$

${}_Br_A(t) = (4t + 1 - a)\mathbf{i} + (4 - b)\mathbf{j} - (3t + 7 + c)\mathbf{k}$

${}_Bv_A = (5i - 3k) - \mathbf{i} = 4\mathbf{i} - 3\mathbf{k}$

At closest distance ${}_Br_A(t) \bullet {}_Bv_A = 0$

$[(4t + 1 - a)\mathbf{i} + (4 - b)\mathbf{j} - (3t + 7 + c)\mathbf{k}] \bullet (4\mathbf{i} - 3\mathbf{k}) = 0$

$16t + 4 - 4a + 9t + 21 + 3c = 0$

$$t = \frac{4a\text{-}3c\text{-}25}{25}$$

Since $t > 0 \Rightarrow \frac{4a\text{-}3c\text{-}25}{25} > 0$

$4a - 3c - 25 > 0 \Rightarrow 4a - 3c > 25$

If particles are closest at t = 2 when

$r_A = (32i+7j+15k)$ m;

But when $t = 2$, $r_A = (a + 2)\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

Hence $(a + 2)\mathbf{i} + b\mathbf{j} + c\mathbf{k} = 32\mathbf{i} + 7\mathbf{j} + 15\mathbf{k}$

$\mathbf{i} : a + 2 = 32 \Rightarrow a = 30$

$\mathbf{j} : b = 7$

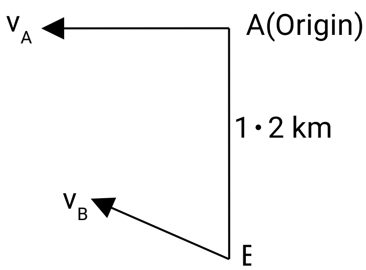
$\mathbf{k} : c = 15$

Example 15

A car A is travelling with constant velocity of 20 km h^{-1} due west and car B has a constant velocity of 16 km h^{-1} in the direction of the vector $(\text{-}4\mathbf{i} + 3\mathbf{j})$. At noon, A is 1·2 km due north of B. Take the position of A at noon as the origin and obtain expressions for the position vectors of A and B at time t hours after noon, and hence show that the position vector of A relative to B is **r** km, where $5\mathbf{r} = 6\{\text{-}6t\mathbf{i} + (1-8t)\mathbf{j}\}$. Deduce that the distance between A and B is d km, where $25d^2 = 36(100t^2 - 16t + 1)$. Hence show that the minimum separation between A and B is 720 m and find the time at which this occurs.

Solution

Car A:	Car B:
20 kmh^{-1} $v_A = -20i$ km h^{-1}	$\lvert v_B\rvert = 16$ km h^{-1} $v_B = h(-4i + 3j)$ $v_B\rvert = h\sqrt{(-4)^2+3^2}$ $16 = 5h \Rightarrow h = \frac{16}{5}$ $v_B = \frac{16}{5}(-4i+3j)$ $v_B = \frac{-64}{5}i + \frac{48}{5}j$



$$r_A(t) = (-20i)t = -20ti$$

$$r_B(t) = \frac{-6}{5}j + \left(\frac{-64}{5}i+\frac{48}{5}j\right)t = \frac{-64}{5}ti + \left(\frac{48t-6}{5}\right)j$$

$$_Ar_B(t) = -20ti - \left[\frac{-64}{5}ti+\left(\frac{48t-6}{5}\right)j\right]$$

$$\mathbf{r} = \left(\frac{64}{5}-20\right)ti + \left(\frac{6-48t}{5}\right)j$$

$$5\mathbf{r} = 6\{-6ti + (1-8t)j\}$$

Distance between A and B, d = |**r**|

$$5|\mathbf{r}| = 6\sqrt{36t^2+(1-8t)^2}$$

$$25d^2 = 36(100t^2-16t + 1)$$

At minimum separation $\frac{d}{dt}|r|^2 = 0$

$$|r|^2 = \frac{36}{25}\left(100t^2-16t+1\right)$$

$$\frac{d}{dt}|r|^2 = \frac{36}{25}(200t - 16)$$

At minimum separation

$$\frac{36}{25}(200t-16) = 0 \Rightarrow t = \frac{2}{25} \text{ hours}$$

When $t = \frac{2}{25}$ hours

$$25d^2 = 36\left[100\times\left(\frac{2}{25}\right)^2-16\times\frac{2}{25}+1\right]$$

$$d = 0•72 \text{ km} \Rightarrow d = 720 \text{ m}$$

From $t = \frac{2}{25}$ hours $= \left(\frac{2}{25}\times60\right)$ minutes = 4 minutes and 48 seconds

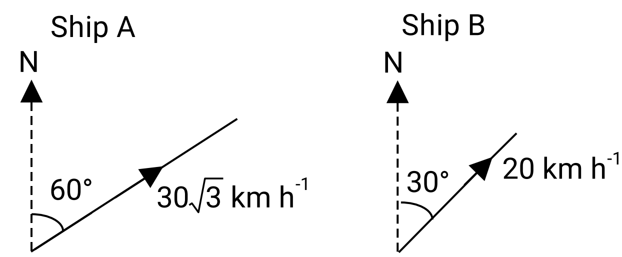
At 12.04 and 48 seconds

Example 16

A ship A is travelling on a course of 060° at a speed of $30\sqrt{3}$ km h^{-1} and ship B is travelling on a course of 030° at 20 km h^{-1}. At noon B is 260 km due east of A. Calculate the least distance between A and B, to the nearest kilometre and the time taken for this to happen.

Solution

Vector diagrams

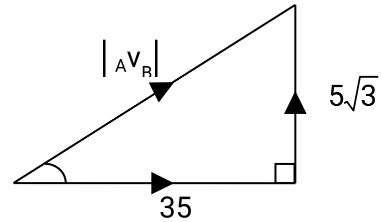


$$v_A = \begin{pmatrix}30\sqrt{3}\sin 60\\30\sqrt{3}\cos 60\end{pmatrix} \qquad v_B = \begin{pmatrix}20\sin 30\\20\cos 30\end{pmatrix}$$

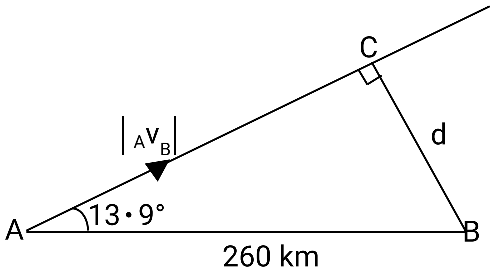
$$v_A = \begin{pmatrix}45\\15\sqrt{3}\end{pmatrix} \text{ km h}^{-1} \qquad v_B = \begin{pmatrix}10\\10\sqrt{3}\end{pmatrix} \text{ km h}^{-1}$$

$$_Av_B = \begin{pmatrix}45\\15\sqrt{3}\end{pmatrix}-\begin{pmatrix}10\\10\sqrt{3}\end{pmatrix} = \begin{pmatrix}35\\5\sqrt{3}\end{pmatrix} \text{ km h}^{-1}$$

$$|_Av_B| = \sqrt{35^2+(5\sqrt{3})^2} = 10\sqrt{13} \text{ km h}^{-1}$$



$$\tan\theta = \left(\frac{5\sqrt{3}}{35}\right) \Rightarrow \theta = 13•9°$$



$$\sin 13{\cdot}9 = \frac{d}{260}$$

$$d = 62{\cdot}45 \text{ km}$$

$$d = 62 \text{ km (to nearest km)}$$

$$t = \frac{|\vec{AC}|}{|{}_A v_B|} = \frac{\sqrt{260^2 - 62^2}}{10\sqrt{13}}$$

$$t = 7{\cdot}0 \text{ hours}$$

Example 17

Two particles P and Q initially at position vectors $(3i+2j)$ m and $(13i+2j)$ m respectively begin moving. Particles P and Q move with constant velocities $(2i+6j)$ m s^{-1} and 5j m s^{-1} respectively.

(a) Find the:
   (i) time taken for P and Q to be closest.
   (ii) bearing of P from Q when they are closest together.

(b) Given that after half the time the two particles are moving closest to each other, particle P reduces its speed to half its original speed in the direction to approach particle *Q* and the velocity of Q remains unchanged, find the direction of particle P.

Solution

(a) $r_P(0) = (3i+2j)$ m , $v_P = (2i+6j)$ m s^{-1}

$r_Q(0) = (13i+2j)$ m, $v_Q = 5j$ m s^{-1}

(i)

$r_P(t) = (3i + 2j) + (2i + 6j)t = (3 + 2t)i + (2 + 6t)j$

$r_Q(t) = (13i + 2j) + (5j)t = 13i + (2 + 5t)j$

${}_P r_Q(t) = [(3+2t)i+(2+6t)j]-[13i+(2+5t)j]$

${}_P r_Q(t) = (2t-10)i + tj$

${}_P v_Q = (2i + 6j) - 5j = (2i + j)$

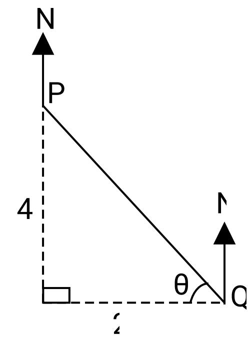
At closest distance: ${}_P r_Q \bullet {}_P v_Q = 0$

$[(2t-10)i + tj]\bullet(2i + j) = 0$

$4t - 20 + t = 0 \Rightarrow t = 4$ s

(ii) When t = 4 s

${}_P r_Q = (2\times4-10)i + 4j = (-2i + 4j)$



$$\tan\theta = \frac{4}{2} \Rightarrow \theta = 63{\cdot}4°$$

Bearing of P from Q = 270 + 63•4 = 333•4°

(b) When $t = \frac{1}{2}\times4 = 2$ s; from

${}_P r_Q(t) = (2t-10)i + tj$

${}_P r_Q(2) = (2\times2-10)i + 2j = -6i + 2j$

Let $v'_P = ai + bj$

${}_P v'_Q = ai + (b - 5)j$

For P to get as close as possible to Q, the direction of P should be perpendicular to the relative path, that is v'_p is perpendicular to ${}_P v'_Q$.

$(ai + bj)\bullet[ai + (b-5)j] = 0$

$a^2 + b^2 - 5b = 0$ ……………… (i)

$|v'_P| = \frac{1}{2}\sqrt{2^2+6^2} = \sqrt{10}$

Implying that : $a^2 + b^2 = 10$ …………… (ii)

From (i) and (ii)

$10 - 5b = 0 \Rightarrow b = 2$

$a^2 + 4 = 10 \Rightarrow a = \pm\sqrt{6}$

Hence either $v'_P = (-\sqrt{6}i+2j)$ or

$v'_P = (\sqrt{6}i+2j)$

Note that when t = 2, ${}_Q r_P = (6i - 2j)$

When $v'_P = (-\sqrt{6}i+2j)$, ${}_P v'_Q = -\sqrt{6}i - 3j$

From ${}_Q r_P(2) \bullet {}_P v'_Q = |{}_P r_Q(2)||{}_P v'_Q| \cos\alpha$

$(6i-2j)\bullet(-\sqrt{6}i-3j) = \sqrt{(36+4)}\times\sqrt{(6+9)}\cos\alpha$

$$\cos\alpha = \frac{6(1-\sqrt{6})}{10\sqrt{6}} \Rightarrow \alpha = 110{\cdot}8°$$

When $v'_P = (\sqrt{6}i+2j)$; ${}_P v'_Q = \sqrt{6}i - 3j$

From ${}_Q r_P(2) \bullet {}_P v'_Q = |{}_P r_Q(2)||{}_P v'_Q| \cos\beta$

$(6i-2j)\bullet(\sqrt{6}i-3j) = \sqrt{(36+4)}\times\sqrt{(6+9)}\cos\beta$

$$\cos\beta = \frac{6\left(1+\sqrt{6}\right)}{10\sqrt{6}} \Rightarrow \beta = 32\cdot3°$$

When $v_P' = \left(-\sqrt{6}i+2j\right)$ m s^{-1} P moves further from Q hence $v_P' = \left(\sqrt{6}i+2j\right)$ m s^{-1}

From: $\left(\sqrt{6}i+2j\right)\bullet\mathbf{i} = \sqrt{\left(\left(\sqrt{6}\right)^2+2^2\right)}\times1\times\cos Ø$

$$\cos Ø = \frac{\sqrt{6}}{\sqrt{10}} \Rightarrow Ø = 39\cdot2°$$

Hence P makes an angle of 39•2° with the horizontal.

Example 18

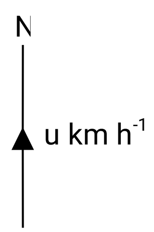
A car A is moving due north at u km h^{-1} while car B is moving at v km h^{-1} N60°W. Initially B is d km due East of A. If B passes to the south of A, show that the:

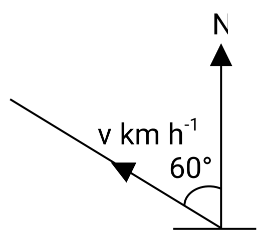
(i) shortest distance between the cars in the subsequent motion is $\frac{d(2u-v)}{2\sqrt{(u^2+v^2-uv)}}$.

(ii) cars are closest to each other after $\frac{dv\sqrt{3}}{2(u^2+v^2-uv)}$.

Solution

(i) Car A Car B



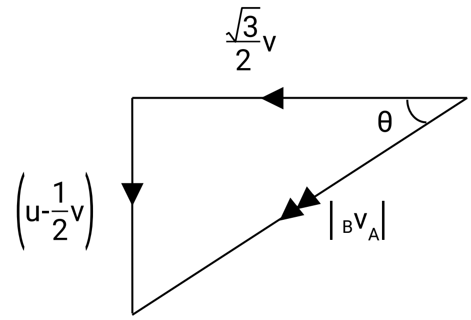


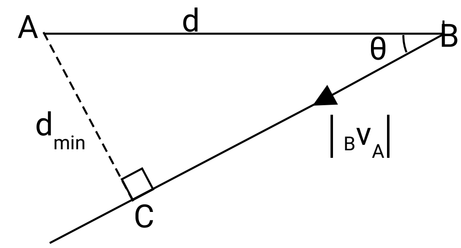
$$v_A = \begin{pmatrix}0\\u\end{pmatrix} \text{ km h}^{-1};\quad v_B = \begin{pmatrix}-v\sin 60\\v\cos 60\end{pmatrix} = \begin{pmatrix}\frac{-\sqrt{3}}{2}v\\ \frac{1}{2}v\end{pmatrix}$$

$$_Bv_A = \begin{pmatrix}\frac{-\sqrt{3}}{2}v\\ \frac{1}{2}v\end{pmatrix} - \begin{pmatrix}0\\u\end{pmatrix} = \begin{pmatrix}\frac{-\sqrt{3}}{2}v\\ \frac{1}{2}v-u\end{pmatrix}$$

$$\left|_Bv_A\right| = \sqrt{\left(\frac{-\sqrt{3}}{2}v\right)^2+\left(\frac{1}{2}v-u\right)^2}$$

$$= \sqrt{u^2+v^2-uv}$$





$$\sin\theta = \frac{d_{min}}{d} \Rightarrow d_{min} = d\times\frac{\left(u-\frac{1}{2}v\right)}{\sqrt{u^2+v^2-uv}}$$

$$d_{min} = \frac{d(2u-v)}{2\sqrt{(u^2+v^2-uv)}}$$

(ii)

$$t = \frac{\left|\overrightarrow{BC}\right|}{\left|_Bv_A\right|}$$

$$\left|\overrightarrow{BC}\right|^2 = d^2-d_{min}^2 \Rightarrow \left|\overrightarrow{BC}\right| = \frac{dv\sqrt{3}}{2\sqrt{(u^2+v^2-uv)}}$$

$$t = \frac{dv\sqrt{3}}{2\sqrt{(u^2+v^2-uv)}} \div \sqrt{u^2+v^2-uv}$$

$$t = \frac{dv\sqrt{3}}{2(u^2+v^2-uv)}$$

Example 19

An aircraft carrier A is at a point O at 0700 hours and steaming south at 30 km h^{-1}. An enemy battleship B at 8 km to the east of A appears to an observer on the carrier to be moving northwest at $30\sqrt{2}$ km h^{-1}.

(a) Determine the true velocity of B and bearing of B when they are closest together.

(b) If A is liable to be hit by a shot fired from B, when the two ships are not more than $5\sqrt{2}$ km apart, prove that *A* is within range from 0702 hours until 0714 hours.

Solution

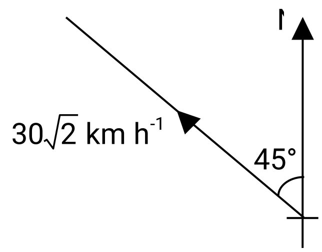
(a)

Aircraft carrier A:

$$v_A = \begin{pmatrix} 0 \\ -30 \end{pmatrix} \text{ km h}^{-1}$$

Enemy battle ship B:

$${}_Bv_A = 30\sqrt{2} \text{ km h}^{-1} \text{ N4}$$

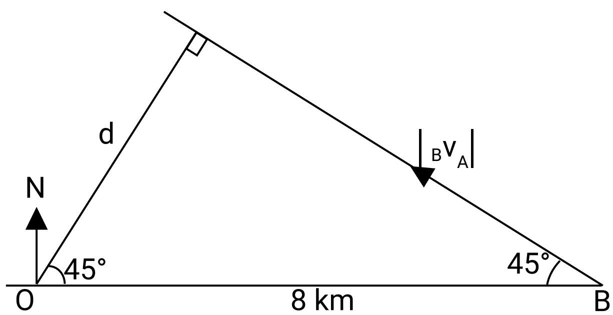


$${}_Bv_A = \begin{pmatrix} -30\sqrt{2}\sin 45 \\ 30\sqrt{2}\cos 45 \end{pmatrix} = \begin{pmatrix} -30 \\ 30 \end{pmatrix} \text{ km h}^{-1}$$

From ${}_Bv_A = v_B - v_A \Rightarrow v_B = {}_Bv_A + v_A$

$$v_B = \begin{pmatrix} -30 \\ 30 \end{pmatrix} + \begin{pmatrix} 0 \\ -30 \end{pmatrix} = \begin{pmatrix} -30 \\ 0 \end{pmatrix} \text{ km h}^{-1}$$

Velocity of battleship is 30 km h^{-1} due west

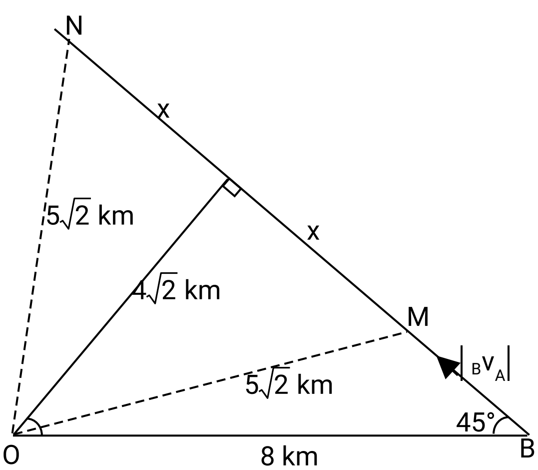


$$\sin 45 = \frac{d}{8}$$

$$d = 4\sqrt{2} \text{ km}$$

B is on a bearing of N45°E from A at closest distance

(b)



$$x^2 = (5\sqrt{2})^2 - (4\sqrt{2})^2$$

$$x = 3\sqrt{2} \text{ km}$$

$$|\overrightarrow{BM}| = 4\sqrt{2} - 3\sqrt{2}$$
$$= \sqrt{2} \text{ km}$$

Time to move from B to M

$$t = \frac{\sqrt{2}}{30\sqrt{2}} = \frac{1}{30} \text{ hours}$$

$$t = \frac{1}{30}\times 60 = 2 \text{ minutes}$$

Time taken from M to N

$$t = \frac{2\times 3\sqrt{2}}{30\sqrt{2}} = \frac{1}{5} \text{ hours}$$

$$t = \frac{1}{5}\times 60 = 12 \text{ minutes}$$

B is at M at 0702 hours and at N at 0714 hours

Hence A is within range of not more than $5\sqrt{2}$ km from 0702 hours to 0714 hours

Alternatively:

(b) ${}_Br_A = {}_Br_A(0) + {}_Bv_A \times t$

$$= \begin{pmatrix} 8 \\ 0 \end{pmatrix} + \begin{pmatrix} -30 \\ 30 \end{pmatrix} t = \begin{pmatrix} 8-30t \\ 30t \end{pmatrix}$$

When $|{}_Br_A| \le 5\sqrt{2}$

$$\sqrt{(8-30t)^2 + (30t)^2} \le 5\sqrt{2}$$

$$(8-30t)^2 + (30t)^2 \le 50$$

$$t^2 - \frac{4}{15}t + \frac{7}{900} \le 0$$

$$\left(t - \frac{2}{15}\right)^2 \le \frac{1}{100}$$

$$\frac{-1}{10} \le t - \frac{2}{15} \le \frac{1}{10}$$

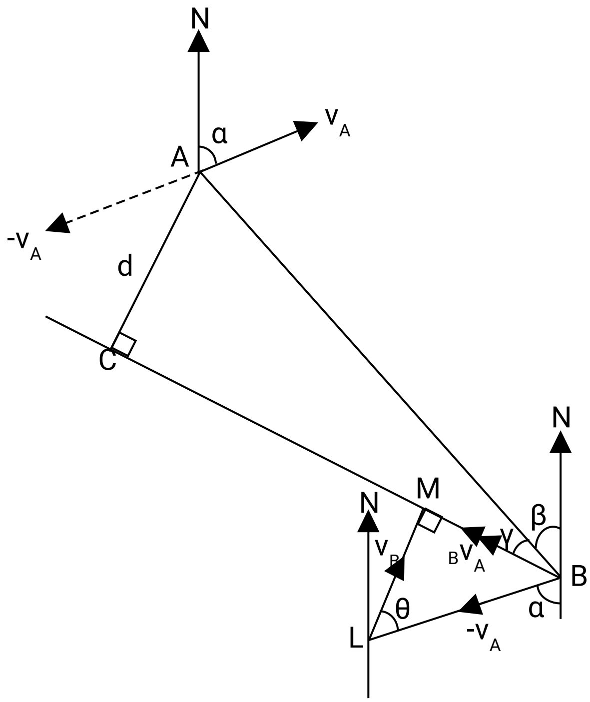
$$\frac{1}{30} \le t \le \frac{7}{30} \text{ hours}$$

$$2 \le t \le 14 \text{ minutes}$$

Hence A is within range from 0702 hours to 0714 hours

18.6 Course for closest approach

Consider a body A moving with constant speed $|v_A|$ in the direction Nα°E. If the bearing of A from B is Nβ°W and B moves with constant speed $|v_B|$, we can find the course B must set in order to get as close as possible to A, the closest distance of approach and the time taken for such a situation to occur.



Draw a diagram showing the initial positions of A and B and consider the motion of B relative to A. To pass as close as possible to A, B must travel relative to A along line BC which makes as small an angle as possible with BA. In the vector triangle BLM, LM must be perpendicular to BC.

The closest distance d is obtained from triangle ABC:

$\sin\gamma = \frac{d}{|\vec{AB}|} \Rightarrow d = |\vec{AB}|\sin\gamma$

The time for closest approach t is obtained from:

$t = \frac{|\vec{BC}|}{|{}_B v_A|}$.

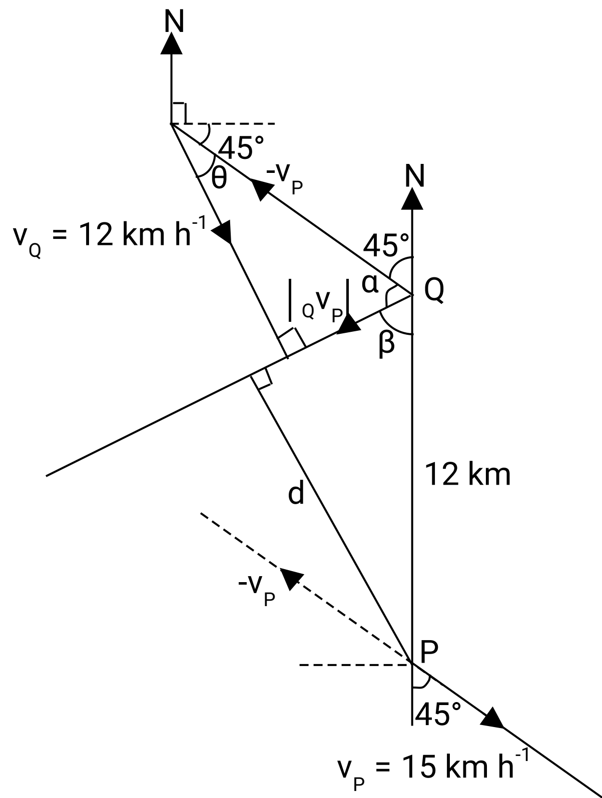
The course set by B to get as close as possible to A is obtained using triangle BLM. Note from this triangle that; $\cos\theta = \frac{|v_B|}{|v_A|}$.

Hence θ and the other known angles on the sketch can be used in finding the direction of v_B, that is, the course set by B.

Example 20

Ship P is steaming on a straight course south-east at a uniform speed of 15 km h^{-1}. Another ship Q is at a distance of 12 km north of P and steams at a speed of 12 km h^{-1}. Find the course Q must steer in order to get as close to P as possible and their minimum distance apart.

Solution



$\sin\alpha = \frac{12}{15} \Rightarrow \alpha = 53\cdot1°$

$\theta = 90 - 53\cdot1 = 36\cdot9°$

Course set by Q= 90 + 45 + 36·9 = 171·9°

Course set by Q is on a bearing of 171·9°

$\beta = 180 - (45 + 53\cdot1) = 81\cdot9°$

$\sin 81\cdot9 = \frac{d}{12}$

d = 11·88 km

Exercises

Exercise: 18A

1. The position vectors of particles P and Q are $r_P = 9t^2 i + (t^3 - 3)j$ metres and $r_Q = 2t^3 i + (4 - t^2)j$ metres at time t seconds. Calculate the relative speed of the particles at t = 2 seconds.
2. (a) To a motorist travelling due north at

40 km h^{-1} the wind appears to come from the direction N60°E at 50 km h^{-1}.
(i) Find the true velocity of the wind.
(ii) If the wind velocity remains constant but the speed of the motorist is increasing, find his speed when the wind appears to be blowing from the direction N45°E.

(b) A cyclist observes that when his velocity is u**i**, the wind appears to come from the direction **i** + 2**j** but when his velocity is u**j**, the wind appears to come from the direction -**i** + 2**j**.
(i) Prove that the true velocity of the wind is $\left(\frac{3}{4}i-\frac{1}{2}j\right)u$
(ii) Find the speed and direction of the cyclist when the wind velocity appears to be u**i**.

3. (a) An aircraft is flying with a velocity of 320 km h^{-1} in a direction S20°W. It is being blown by wind which has a velocity of 35 km h^{-1} from the south. Find the velocity of the aircraft over the ground.

(b) Jinja is south-east of Kampala. To a passenger in a Kampala-Jinja bound bus, traveling at 120 km h^{-1} a steady wind appears to be blowing from the East. On reducing its speed to 90 km h^{-1} without changing direction, the wind appears to blow from N30°E. Find the true velocity of the wind.

4. An aircraft is flying with a velocity of 100 km h^{-1} due North. It is being blown by wind which has a velocity of 40 km h^{-1} from the North west. Find the velocity of the aircraft over the ground.

5. A ship Y appears to an observer in ship X at 10 O'clock to be travelling at a speed of 20 km h^{-1} due north. After 30 minutes ship X which is travelling at a speed of 60 km h^{-1} N60°E collides with ship Y. Find the:
(i) Actual velocity of *Y*.
(ii) Distance and bearing of ship *Y* at 10 O'clock.

6. A man is walking due south at 5 km h^{-1}. The wind is blowing from the west at 8 km h^{-1}. Find the magnitude and direction of the velocity of the wind relative to the man.

7. A ship is sailing south-east at 20 km h^{-1} and a second ship is sailing due west at 25 km h^{-1}. Find the magnitude and direction of the velocity of the first ship relative to the second.

8. If particles A and B have velocity vectors 7**i**−5**j** and -2**i** + 7**j** respectively, find the magnitude of the velocity of B relative to A and the angle its direction makes with the direction of **i**.

9. Particles A and B have velocity vectors 3**i**−11**j** and 5**i** + **j** respectively. The velocity of particle C relative to A is -2**i** + 7**j**. Find in vector form the velocity of C and the velocity of B relative to C.

10. A boat A travels due west at a speed of 30 km h^{-1}. The velocity of a boat B relative to A is 14 km h^{-1} due south. Find the speed of boat B and the direction in which it is moving.

11. To an observer on a ship P steaming at 20 km h^{-1} due west, a ship Q appears to be steaming at 20 km h^{-1} N30°W. Find the true velocity of Q.

12. To a cyclist riding due east at 15 km h^{-1} the wind appears to be blowing from the north-east at 12 km h^{-1}. Find the magnitude and direction of the true velocity of wind.

Exercise: 18B

1. A boat has to sail from A to B where B is on a bearing of 110° from A. There is a current from a bearing of 335° with a speed of 15 km h^{-1}. In still water the boat would travel at 30 km h^{-1}. Determine the direction the boat will have to steer (course set) in order to sail from A to B.

2. A river 100 m wide flows at 4 km h^{-1}. A boat is to be rowed at 10 km h^{-1} relative to the stream from a point on one bank to a landing site on the opposite bank 50 m further downstream. Find the direction in which the boat must be steered and time taken to cross.

3. A river flows at 5 m s^{-1} from west to east between parallel banks which are at a distance of 300 m apart. A man rows a boat at a speed of 3 m s^{-1} in still water.
(i) State the direction in which the boat must be steered in order to cross

the river from the southern bank to the northern bank in the shortest possible time. Find the time taken and the actual distance covered by the boat forthis crossing.

(ii) Find the direction in which the boat must be steered to cross the river fromthe southern bank to the northern bank by the shortest possible route. Find the time taken and the actual distance covered by the boat for this crossing.

4. A river flows at a constant speed of $5\ m\ s^{-1}$ between straight parallel banks which are 300 m apart. A boat which has a maximum speed of $3{\cdot}25\ m\ s^{-1}$ in still water, leaves a point A on one bank and sails in a straight line to the opposite bank. Find the least time the boat can take to reach a point B on the opposite bank where AB = 500 m and B is downstream from A. Find also the least time the boat can take to cross the river. Find the time taken to sail from A to B by the slowest boat capable of sailing directly from A to B.

5. A boy can swim in still water at $v\ m\ s^{-1}$. He swims across a river flowing at $1{\cdot}2\ m\ s^{-1}$ which is 368 m wide. Find the time he takes if he travels the shortest possible distance if:
   (i) $v = 1$
   (ii) $v = 2$

6. A man wishes to row across a river to reach a point on the far bank, exactly opposite his starting point. In still water he can row at $5\ m\ s^{-1}$. The river is 100 m wide and flows at $3\ m\ s^{-1}$. Find at what angle to the bank the man must steer the boat in order to complete the crossing and the time it takes him.

7. A man wishes to row across a river to reach a point on the far bank exactly opposite his starting point. The river is 125 m wide and flows at $1\ m\ s^{-1}$. If the man can row at $3\ m\ s^{-1}$ in still water, find the direction the man must steer to complete the crossing and the time it takes him.

8. A boy wishes to swim across a river 100 m wide as quickly as possible. The river flows at $3\ km\ h^{-1}$ and the boy can swim $4\ km\ h^{-1}$ in still water. Find the time it takes the boy to cross the river and how far downstream he travels.

9. A man wishes to row a boat across a river to reach a point on the far bank that is 35 m downstream from his starting point. The man can row that boat at $2{\cdot}5\ m\ s^{-1}$ in still water. If the river is 50 m wide and flows at $3\ m\ s^{-1}$, find two possible courses the man could set and find the respective crossing times.

Exercise: 18C

1. Two planes A and B are both flying at the same altitude. Plane A is flying on a course of 010° at a speed of $300\ km\ h^{-1}$. Plane B is flying on a course of 340° at $200\ km\ h^{-1}$. At a certain time, plane B is 40 km from plane A. Plane B is then on a bearing of 060° from A. After what time will they come closest together and what will be their least distance apart.

2. At noon, a boat A is 30 km from boat B and its direction from B is 286°. Boat A is moving in the North East direction at $16\ km\ h^{-1}$ and boat B is moving in northern direction at $10\ km\ h^{-1}$. Find the closest distance between the boats and the time at which they are closest.

3. Two helicopters P and Q are flying at the same altitude with velocities $200\ ms^{-1}$ $N30°E$ and $300\ m\ s^{-1}$ N50°W respectively. Initially P and Q are 2 km apart with Q on a bearing of S70°E from P. Given that P and Q do not alter their velocities, find the minimum distance of separation between the two helicopters in the subsequent motion and the time taken for such a situation to occur.

4. A road running north-south crosses a road running east-west at a junction O. Initially Isaac is on the east-west road $1 \cdot 7$ km west of O and is running towards O at $15\ km\ h^{-1}$. At the same instant, Serena is at O running due north at $8\ km\ h^{-1}$. If Isaac and Serena do not alter their velocities, find their least distance apart in the subsequent motion and the time taken for that situation to occur.

5. Relative to an observer on ship A travelling from north, a second ship appears to travel at a speed of $5\ km\ h^{-1}$, show that if B travels due west at $3\ km\ h^{-1}$, the speed of A is $4\ km\ h^{-1}$.

6. A commander in a space fighter is tipped that there is an helicopter carrying rebels at the same altitude 10 km away from him due north and is cruising in a straight course

southeast at a uniform speed of 25 km h^{-1}. The guns of the space fighter can fire up to within a range of 2•5 km. The maximum speed of the fighter is 12 km h^{-1}. Show that whatever course the space fighter sets, it cannot get the helicopter within the range of its guns.

7. Two straight roads intersect at O one running north-south and the other west-east. A van, A travelling at a constant speed of 32 km h^{-1} due south on one road, passes through O at the instant when a motor cyclist B moving towards O at a constant speed of 40 km h^{-1} due east along the second road is 5 km from O.
   (a) Find the distances AO and BO when A and B are closest together.
   (b) If A changes course, find the time from the instant of closest approach taken by A to intercept B.

8. A motorist A and a cyclist B are travelling along straight roads which cross at right angles at a point O. A is travelling at a constant speed of 48 km h^{-1} towards the east, and B at 14 km h^{-1} towards the north. At the moment that B is passing through O, A is 400 m from O and has not yet passed O. Calculate the:
   (i) velocity of B relative to A in magnitude and direction.
   (ii) least distance between the car and cycle.
   (iii) distances of the car and cycle from O when they are nearest to each other.

9. Two cars A and B are proceeding one on each road towards the point of intersection of the two roads which meet at an angle of 60°. If the velocities of A and B are 20 km h^{-1} due east and 32 km h^{-1} S30°E and are 70 m and 40 m respectively from the cross-roads, find the relative speed of A to B and distance of the cars from the crossroads when they are nearest together.

10. A ship is travelling due east with speed 36 km h^{-1} and at noon, a patrol boat at 250 km south east of the ship, is moving with velocity of 45 km h^{-1} Nθ°W relative to the ship, where $\theta = \tan^{-1}\left(\frac{4}{3}\right)$.
   (a) Find the shortest distance between the vessels and the time at which it occurs.
   (b) If at noon, the boat sets off with the same speed to intercept the ship, find the course it should take.

11. To a ship P which starts to move at 40 km h^{-1} due N60°E, a ship Q which starts to move at 50 km h^{-1} appears to travel southwards. Given that the ships collide after 45 minutes while maintaining the given velocities, find the initial position of Q.

12. Initially two ships A and B are 65 km apart with B due east of A. A is moving due east at 10 km h^{-1} and B due south at 24 km h^{-1}. The two ships continue moving with these velocities. Find the least distance between the ships in the subsequent motion and the time taken to the nearest minute for such a situation to occur.

Exercise: 18D

1. A particle A moving with constant velocity $2\mathbf{i} + 3\mathbf{j} + 8\mathbf{k}$, passes through a point with position vector $6\mathbf{i} - 11\mathbf{j} + 4\mathbf{k}$. At the same instant, particle B passes through a point with position vector $\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$, moving with constant velocity $3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$, find the:
   (a) position and velocity of B relative to A at that instant.
   (b) closest distance between A and B in the subsequent motion.
   (c) time that elapses before the particles are closest together.

2. At noon ships A and B have position vectors $10\mathbf{i} + 3\mathbf{j}$ and $12\mathbf{i} - 15\mathbf{j}$ respectively. Both ships move with constant velocity, the velocity of A being $3\mathbf{i} + 8\mathbf{j}$ and the velocity of B relative to A being $-5\mathbf{i} + 4\mathbf{j}$. The units of distance being kilometres and those of time being hours, find an expression for the vector $\overrightarrow{AB}$, t hours after noon. Hence the shortest distance between the ships. Find also the position vectors of the ships A and B when they are closest together.

3. At noon two ships A and B have the following position and velocity vectors:

Position vector **Velocity vector**

Ship A	$10i + 5j$	$-2i + 4j$
Ship B	$2i - j$	$2i + 7j$

Where the speeds are measured in kilometres per hour and the distances in kilometres.

(i) Find the position vectors of A and B after time t hours.

(ii) Show that the two ships will collide, and give the time of the collision.

(iii) Determine how far ship A will have travelled between noon and the time of collision.

4. A particle P starts from a point with position vector $2\mathbf{j} + 2\mathbf{k}$ with velocity $\mathbf{j} + \mathbf{k}$. A second particle Q starts at the same time from a point whose position vector is $-11\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}$ with velocity $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. Find;
   (a) the time when the particles are closest together.
   (b) the shortest distance between the particles.
   (c) how far each particle has travelled by this time.

5. An object P passes through a point with position vector $3\mathbf{i} - 2\mathbf{j}$ with constant velocity $\mathbf{i} + \mathbf{j}$. At the same instant an object Q moving with constant velocity $4\mathbf{i} - 2\mathbf{j}$ passes through a point with position vector $\mathbf{i} + 4\mathbf{j}$. Find the:
   (a) displacement of P relative to Q after t seconds.
   (b) time when P and Q are closest together and their closest distance.

6. A point R(0,1,-2) is at a shore of the Indian ocean. At 1.00 p.m two ships P and Q are sighted at points (5,-3,4) and (7,5,-2) moving with constant velocities $(2i+5j+3k)$ km h^{-1} and $(-3i-15j+18k)$ km h^{-1} respectively.
   (a) Find the relative velocities of ship Q to P.
   (b) If the velocities of the ships remain constant, show that the ships will collide.
   (c) Find the distance from point R to the point where the collision occurs.

7. Two particles A and B are initially at (1,1,2) and (4,5,1) respectively. They start moving simultaneously with constant velocities $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $-4\mathbf{j} + 3\mathbf{k}$ respectively. Find the:
   (i) position vector of B relative to A at any time.
   (ii) minimum distance between the particles and the position vector of A relative to B at this instant.

8. A destroyer sights a ship travelling with constant velocity 5j, whose position vector at the time of sighting is $2000(3\mathbf{i} + \mathbf{j})$ relative to the destroyer, distances being in m and speeds in m s^{-1}. The destroyer immediately begins to move with velocity $k(4\mathbf{i} + 3\mathbf{j})$, where k is a constant, in order to intercept the ship. Find k and the time to interception. Find also the distance between the vessels when half the time of interception has elapsed.

9. At 11.30 a.m, aircraft A has a position vector $(-100i+220j)$ km and a velocity of $(300i+400j)$km h^{-1}. At 11.45 a.m, aircraft B has a position vector $(-60i+355j)$ km and a velocity $(400i+300j)$ km h^{-1}. Show that if these velocities are maintained, the aircrafts crash into each other. Find the position vector of the crash.

10. Two ships A and B have the following position vectors **r** and velocity vectors **v** at times stated.

$r_A = (-2i+3j)$ km

$v_A = (12i-4j)$ km h^{-1} at 11.45 a.m

$r_B = (8i+7j)$ km

$v_B = (2i-14j)$ km h^{-1} at 12 noon.

Assuming the ships do not alter their velocities, find their least distance of separation. If ship B has guns with a range of up to 2 km, find for what length of time A is within range.

11. At time t = 0, the position vectors and velocity vectors of two particles A and B are as follows: $r_A = \begin{pmatrix}1\\2\\3\end{pmatrix}$ m, $v_A = \begin{pmatrix}-6\\0\\1\end{pmatrix}$ m s^{-1},

$r_B = \begin{pmatrix}4\\-14\\1\end{pmatrix}$ m, $v_B = \begin{pmatrix}-5\\1\\7\end{pmatrix}$ m s^{-1}.

Find the:
   (a) position of B relative to A at time t.
   (b) value of t when A and B are closest

together.

(c) least distance between A and B.

Exercise: 18E

1. A ship A whose full speed is 40 km h^{-1} is 20 km due west of ship B which is travelling uniformly with speed 30 km h^{-1} due north. The ship A travels at full speed on a course chosen so as to intercept B as soon as possible. Find the direction of this course and calculate to the nearest minute the time A would take to reach B. When half of this time has elapsed the ship A has engine failure and thereafter proceeds at half speed. Find the course which A should set in order to approach as close as possible to B and calculate the distance of closest approach.

2. A, B and C move in a plane with constant velocities and at time $t = 0$, the position vectors of A, B and C relative to origin O are $\mathbf{i} + 3\mathbf{j}$, $9\mathbf{i} + 9\mathbf{j}$ and $6\mathbf{i} + 13\mathbf{j}$ respectively. The velocity of C relative to A is $7\mathbf{i}-10\mathbf{j}$ and that of C relative to B is $9\mathbf{i}-12\mathbf{j}$.
   (a) Find the velocity of B relative to A.
   (b) Show that A and B do not collide and find their shortest distance apart and time when A and B are this distance apart.

3. Motor boat B is travelling at a constant speed of 10 m s^{-1} due east and motor boat A is travelling at a constant speed of 8 m s^{-1}. Initially A and B are 600 m apart with A due south of B. Find the course that A should set in order to get as close as possible to B. Find this closest distance and time taken for the situation to occur.

4. At 8.00 a.m two boats A and B are 5•2 km apart with A due west of B, and B travelling due north at a steady speed of 13 km h^{-1}. If A travels with constant speed of 12 km h^{-1}, show that for A to get as close as possible to B, A should set a course Nθ°E where $\sin\theta = \frac{5}{13}$. Find this closest distance and time at which it occurs.

5. A battleship commander is informed that there is a lone cruiser positioned 40 km away from him on a bearing N70°W. The guns on a battleship have a range of up to 8 km and the top speed of the battleship is 30 km h^{-1}. The cruiser maintains a constant velocity of 50 km h^{-1} N60°E. Show that whatever course the battleship sets, it cannot get the cruiser within range of the guns.

Exercise: 18F

1. An aircraft is flying at 400 km h^{-1} in still air from town A 750 km due east of town B. If the wind is blowing from north-west at 50 km h^{-1}, determine the course and time taken for the flight.

2. To a cyclist moving in the direction S50°W at 30 km h^{-1}, wind appears to come from the direction S60°E at 60 km h^{-1}. Find the true speed of the wind.

3. Given that ${}_A v_B = \begin{pmatrix}4\\10\end{pmatrix}\text{ m s}^{-1}$ and ${}_B v_C = \begin{pmatrix}5\\2\end{pmatrix}\text{ m s}^{-1}$, find:
   (i) ${}_A v_C$
   (ii) $|{}_A v_C|$
   (iii) θ if the direction of ${}_A v_C$ is Nθ°E.

4. If the resultant of $(a\mathbf{i}+b\mathbf{j})\text{ m s}^{-1}$ and $(b\mathbf{i}-a\mathbf{j})\text{ m s}^{-1}$ is $(10\mathbf{i}-4\mathbf{j})\text{ m s}^{-1}$, find the values of a and b.

5. The resultant of two velocities is a velocity of 10 km h^{-1} N30°W. If one of the velocities is 10 km h^{-1} due west, find the magnitude and direction of the other.

6. A pilot has to fly his aircraft from airport A to airport B, 100 km due east of A. In still air the aircraft flies at 125 km h^{-1}. If there is wind of 35 km h^{-1} blowing from the north, find the course that the pilot must set in order to reach B and the time the journey takes.

7. When swimming in a river a man finds that he has a maximum speed v when swimming downstream and u when swimming upstream.
   (a) Find an expression for his maximum speed when swimming in still water.
   (b) If the river is of width s, show that the shortest time in which the man can swim across is $\frac{2s}{u+v}$ and that such a

crossing would take him a distance of $\frac{s(v-u)}{u+v}$ downstream from his starting point.

(c) If he wishes to swim as quickly as possible from a point on one bank to a point exactly opposite on the other bank, show that he must swim in a direction that makes an angle $\cos^{-1}\left(\frac{v-u}{u+v}\right)$ with the bank and that the crossing will take a time $\frac{s}{\sqrt{uv}}$.

8. (a) The velocities of two ships P and Q are $(i+6j)$ km h^{-1} and $(-i+3j)$ km h^{-1} respectively. At a certain instant the displacement of P relative to Q is $(7i+4j)$ km. Find the:
   (i) relative velocity of ship P to Q.
   (ii) magnitude of displacement between ships P and Q after 2 hours.

   (b) The position vectors of two particles are: $r_1 = (4t + 3t^2)\mathbf{i} + (t^2-2t)\mathbf{j}$ and $r_2 = (10 + 5t)\mathbf{i} + (4-2t)\mathbf{j}$ respectively. Show that the two particles will collide and find their velocities at the time of collision.

9. (a) Kampala is south of Gulu town. To a passenger in a Gulu-Kampala bound bus travelling at 110 km h^{-1}, the wind appears to be blowing in the direction 240°. When the bus reduces speed to 90 km h^{-1} without changing direction, the wind appears to be blowing in the direction 210°. Find the true speed and direction in which the wind is blowing.

   (b) Two particles A and B start at the same time from the points with position vectors $\lambda\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ and $7\mathbf{i}-2\mathbf{j} + 4\mathbf{k}$ with constant velocities $\mathbf{i}-\mathbf{j} + \mathbf{k}$ and $-2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$. If the distance between A and B is least when t = 2, find the value of λ.

10. A motor boat set out at 11.00 a.m from a position $-6\mathbf{i}-2\mathbf{j}$ relative to a marker buoy and travels at a steady speed of $\sqrt{53}$ on a direct course to intercept a ship. The ship maintains a steady velocity of $3\mathbf{i} + 4\mathbf{j}$ and at 12 noon it is at a position $3\mathbf{i}-\mathbf{j}$ from the buoy. Find the:
    (i) velocity of the motor boat.
    (ii) time of interception and position vector of point of interception from the marker buoy.

11. (a) The position vector of one particle relative to another after t seconds is given by $[(2t-5)i+(10-t)j]$ m. Determine the respective relative velocity, hence or otherwise calculate the shortest distance between the particles.

    (b) A ship travelling at 40 km h^{-1} S80°E is initially at a point 25 km northwest of a patrol vessel. The patrol is capable of reaching a maximum speed of 30 km h^{-1}. Show that the patrol vessel can take two courses to intercept the ship and determine the difference in times of interception.

12. A battleship and a cruiser are initially 16 km apart with the battleship on a bearing N35°E from the cruiser. The battleship travels at 14 km h^{-1} on a bearing S29°E and the cruiser at 17 km h^{-1} on a bearing of N50°E. The guns on the battleship have a range of up to 6 km. Find the;
    (a) least distance between the cruiser and battleship in the subsequent motion.
    (b) length of time for which the battleship has a cruiser within range of its guns.

13. Two particles P and Q move with constant velocities of $(4i+j-2k)$ m s^{-1} and $(6i+3k)$ m s^{-1} respectively. Initially P is at a point with a position vector given by $(-i+20j+21k)$ m and Q is at a point with position vector $(i+3k)$ m. Find the:
    (a) time for which the distance between P and Q is least.
    (b) distance of P from the origin at the time when the distance between P and Q is least.
    (c) least distance between P and Q.

Answers to exercises

Exercise: 18A

1. 20 ms^{-1} **2.(a)** **(i)** 45.82 kmhr^{-1} N70.9°W **(ii)** 58•3013 km h^{-1} (b) (i) (ii) $\frac{u\sqrt{5}}{4}$ from $\mathbf{i} + 2\mathbf{j}$

3.(a) 287.36 km^{-1} S22.4°W **(b)** 99.2026 km h^{-1} S31.2°E **4.** 77•0918 km h^{-1} N21 • 5°E **5.** (i) 72.111 km^{-1} in direction N46 • 1°E (ii) 10 km ; Y is initially due south of X 6. 9•434 km h^{-1} N58 • 0°E

7. 41•62 km h^{-1} ; S70•1°E **8.** 15; 126•9° **9.** **i** - 4**j** ; 4**i** + 5**j**

10. 33•106 km h^{-1}; S65.0°W **11.** $20\sqrt{3}$ km h^{-1} N60°W **12.** 10•698 km h^{-1} ; S37•5°E

Exercise: 18B

1. 089•3° 2. 95•6° to the bank ; 36 s 3. (i) due north; 100 s; 583 m **(ii)** N36•9°W; 125 s; 500 m **4.** 95•2 s ; 92•3 s ; 125 s

5. (i) 665•7 s (ii) 230 s **6.** 53.1°; 25 s 7. 70.5° to the bank; 44•2 s

8. 90 s ; 75 m 9. Upstream at 45•6° to the bank or upstream at 24•4° to the bank; 28 s ; 48•4 s

Exercise: 18C

1. 0•2425 hours; 8.14 km 2. 11•54 km; 2.26 p.m 3. 571 m ; 5•8 seconds

4. 800 m ;5 min. 18 sec 5. 6. 2.8km; >2.5km 7. (a) 2•439 km ; 1•952 km (b) 0•0975 hours

8. (i) 50 km h^{-1} towards N73•7°W (ii) 112 m (iii) 31•36 m ; 107•52 m

9. 28 km h^{-1}; 40•41 m ; 7•35 m 10. (a) 35•355 km at 5.30 p.m (b) N25•5°E

11. 42•042 km north of P 12. 60 km ; 58 minutes

Exercise: 18D

1. (a) -5**i** + 9**j** + **k** ; **i** + **j** - 15**k** (b) 10•318 (c) $\frac{11}{227}$ **2.** (2-5t)**i** + (4t-18)**j** ; $2\sqrt{41}$ km ; (16i+19j) km ; (8i+9j) km

3. (i) [(10-2t)i+(5+4t)j]; [(2+2t)i+(7t-1)j] (ii) $r_A = r_B$ at t = 2 hours (iii) $4\sqrt{5}$ km 4. (a) 6•2 (b) 5•0794 (c) $|s_P| = 8•768$; $|s_Q| = 18•6$ 5. (a) (2-3t)**i** + (3t-6)**j** (b) $\frac{4}{3}$ s ; $2\sqrt{2}$ m

6. (a) (-5i-20j+15k) km h^{-1} (b) $r_P = r_Q$ at $t = \frac{2}{5}$ hours (c) 9•46 km

7. (i) (3-2t)**i** + (4-5t)**j** + (t-1)**k** (ii) 1•3038 ; -1•2**i** + 0•5**j** + 0•1**k**

8. 3 ; 500 s ; $1000\sqrt{10}$ m 9. $r_A = r_B$ at $t = \frac{7}{20}$ hours ; (80i+460j) km

10. $\sqrt{2}$ km ; 12 minutes 11. (a) $\begin{pmatrix} 3+t \\ -16+t \\ -2+6t \end{pmatrix}$ (b) $t = \frac{25}{38}$ s (c) 15•892 m

Exercise: 18E

1. N41•4°E ; 45 minutes ; N48•2°E ; 7•45 km 2. (a) -2**i** + 2**j** (b) $7\sqrt{2}$; $t = \frac{1}{2}$

3. N53•1°E ; 360 m ; 80 s 4. 2 km at 8.57 a.m and 36 seconds

5. $d_{min} = 9•066$ km > 8 km

Exercise: 18F

1. N84•9°W ; 2 hours 4 minutes 2. 75•7 km h^{-1} 3. (i) $\begin{pmatrix} 9 \\ 12 \end{pmatrix}$ m s^{-1} (ii) 15 m s^{-1} (iii) 36•9° 4. 7 ; 3 5. 10 km h^{-1} N30°E 6. N73•7°E ; 50 minutes

7. (a) $\frac{u+v}{2}$ (b) (c) 8. (a) (i) (2i+3j) km h^{-1} (ii) 14•866 km
(b) $r_1 = r_2$ at t = 2 ; v_1 = 16**i** + 2**j** ; v_2 = 5**i** - 2**j**

9. (a) 121•244 km h^{-1} in the direction 188•2° (b) $-\frac{19}{3}$ 10. (i) (7i+2j) m s^{-1}

(ii) 12.30 p.m ; $\left(\frac{9}{2}i+j\right)$ m

11. (a) $(2i-j)$ m s^{-1} ; $3\sqrt{5}$ m (b) N4•9°E ; N85•1°E ; 1•38 hours

12. (a) 5.46km (b) 12 minutes and 27 seconds

13. (a) 2.2 s (b) 24•141 m (c) 28•7965 m

19 MISCELLANEOUS EXERCISES

Exercise 1

1. Given that **a** = $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$, **b** = $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and **c** = $\begin{pmatrix} 22 \\ -11 \end{pmatrix}$, find:
   (i) a unit vector perpendicular to **a** and **b**.
   (ii) the value of constants m and n for which m**a** + n**b** = **c**.
   (iii) the scalar product of **a** and **b**, hence find the angle between the direction of **a** and the direction of **b**.

2. Two points A and B lie on a horizontal surface. A particle is projected vertically upwards from A with an initial speed of 20 ms^{-1}. One second later another particle is projected vertically upwards from B with an initial speed of 17.5ms^{-1} Calculate at the instant at which the two particles are at the same vertical height above the surface the:
   (a) time which has elapsed since the first particle was projected from A.
   (b) speeds of the two particles.

3. A train travels between two stations 3•9 km apart in 6 minutes, starting and finishing at rest. During the first $\frac{3}{4}$ minutes the acceleration is uniform, for the next $3\frac{3}{4}$ minutes the speed is constant and for the remainder of the journey the train is retarded uniformly. Sketch a speed-time graph of the journey and hence or otherwise calculate the:
   (i) maximum speed in km h^{-1} attained by the train.
   (ii) acceleration of the train during the first $\frac{3}{4}$ minute stating the units.

4. A motorist starting from rest accelerates uniformly to a speed of v m s^{-1} in 9 s. He maintains this speed for another 50 s and then applies the brakes and decelerates uniformly to rest. His uniform deceleration is numerically equal to three times his previous acceleration.
   (i) Sketch the velocity-time graph.
   (ii) Calculate the time during which deceleration takes place.
   (iii) Given that the total distance is 840 m, calculate the value of v.
   (iv) Calculate the initial acceleration.

5. To a cyclist riding due north at 3 m s^{-1} the wind appears to be blowing from the east. If the cyclist doubles his speed but does not change his direction, the wind appears to be blowing from N60°E. Find the true wind speed and direction. The cyclist now turns around and cycles due south at 3 m s^{-1}. Calculate the apparent wind direction.

6. A constant force of 35 N, always acting in the same horizontal direction causes a particle of mass 2 kg to move along a rough horizontal plane. The particle passes two points A and B, 4 m apart, with speeds of 6 m s^{-1} and 10 m s^{-1} respectively. The frictional resistance to motion is constant. Calculate the:
   (i) acceleration of the particle.
   (ii) magnitude of the frictional resistance.
   (iii) distance of the particle from A, 4 s after it has passed A.

7. A breakdown truck of mass 2000 kg is towing a car of mass 1000 kg by means of a rope up an incline of 1 in 20. The resistances due to friction on each vehicle are proportional to the masses of the vehicles. The engine of the truck exerts a tractive force of 3600 N when moving up the hill at a steady speed of 18 km h^{-1}. Show that the tension in the rope is 1200 N. The rope breaks and the two vehicles continue to move up the hill. Calculate:

(i) how much time elapses before the car comes momentarily to rest.
(ii) how far the car travels in this time.

8. Masses of 1 kg and 2 kg are attached to the ends of a long light string which passes over a light pulley supported by a frictionless horizontal axis. If the tension in the string is T, write down the equation of motion of each mass, and hence find the:

   (i) tension in the string.

   (ii) time taken for the heavier mass to fall from rest a distance of 1•5 m.

9. An elastic string of natural length l and modulus of elasticity λ, is stretched to length l + x. As a result, the tension in the string is mg and the energy stored in it is E. Find x and λ in terms of E,g,l and m.

10. The engine of a car is working at a constant rate of 6 kW in driving the car along a straight horizontal road at a constant speed of 54 km h^{-1}. Find the resistance to motion of the car.

Exercise 2

1. A car of mass 800 kg is towing a trailer of mass 300 kg along a straight horizontal road, resistances which are constant are 600 N for the car and 240 N for the trailer. Calculate the tractive force exerted by the car and the tension in the coupling between the car and the trailer in each of the following cases:

   (a) when both are travelling at constant velocity.

   (b) when both are accelerating at 2•5 m s^{-2}, calculate the power developed by the car when travelling at a constant velocity of 15 m s^{-1}.

2. A car of mass 1000 kg moves with its engine shut off down a slope of inclination θ, where $\sin\theta = \frac{1}{20}$ at a steady speed of 15 m s^{-1}. Find the resistance to the motion of the car. Calculate the power delivered by the engine when the car ascends the same inclination at the same steady speed, assuming that resistance to motion is unchanged. [Take g = 10 m s^{-2}]

3. The frictional resistance to motion of a car of mass 1000 kg is kv newtons, where v m s^{-1} is its speed and k is a constant. The car ascends a hill of inclination arc $\sin\left(\frac{1}{10}\right)$ at a steady speed of 8 m s^{-1}, the power exerted by the engine being 9•76 kW. Prove that the numerical value of k is 30. Find the steady speed at which the car ascends the hill if the power developed by the engine is 12•8 kW. When the car is travelling at this speed, the power exerted by the engine is increased by 2 kW. Find the immediate acceleration of the car.

4. A body is projected upwards from a point on a horizontal plane with a speed of 40 m s^{-1} at an angle of 60° to the horizontal. The point of projection is at a horizontal distance of 40 m from the foot of a vertical wall which is 10 m high and motion takes place in a plane perpendicular to the wall. Calculate the:
   (a) vertical height by which the body clears the wall.
   (b) greatest height above the horizontal plane reached by the body.
   (c) time of flight of the body.
   (d) horizontal distance beyond the wall at which the body strikes the plane.

5. Show that the acceleration of an object moving along a straight line may be written as $v\frac{dv}{ds}$. A vehicle of mass 2500 kg moving on a straight course is subjected to a single resisting force in the line of motion of magnitude kv newtons, where v m s^{-1} is the velocity and k is a constant. At 100 km h^{-1} this force is 2000 N, the vehicle is slowed down from 100 km h^{-1} to 50 km h^{-1}. Find the:
   (i) value of k.
   (ii) distance travelled.
   (iii) time taken.

6. (a) A particle of mass m, moves in a horizontal straight line under the action of a resisting force of magnitude mkv^2, where v is the velocity and k is a positive constant. When t = 0, v = u and x = a, where x is the displacement from the origin at time t, find the expressions for:
   (i) v in terms of x
   (ii) v in terms of t

(iii) x in terms of t.

(b) A body of mass 12 kg is propelled along a horizontal surface by a constant horizontal force of 480 N against a variable resistance of $30v^2$ N, where v is the speed of the body. The body starts from rest at an origin O. Find an expression for v in terms of s, the displacement of the body from O and sketch the velocity-time graph.

7. A particle P of mass 8 kg describes simple harmonic motion with O as the centre and has a speed of 6 m s^{-1} at a distance of 1 m from O and a speed of 2 m s^{-1} at a distance of 3 m from O.
   (a) Find the:
      (i) amplitude of the motion.
      (ii) period of the motion.
      (iii) maximum speed of P.
      (iv) time taken to travel directly from O to one extreme point B of the motion.
   (b) Determine the magnitude of:
      (i) acceleration of P when at a distance of 2 m from O.
      (ii) the force acting on P when at a distance of 2 m from O.
   (c) Write down an expression for the displacement of P from O at any time t, given that P is at O at t = 0. Hence or otherwise, find the time taken to travel directly from O to a point C between O and B and at a distance of 1 m from O. Find also the time taken to go directly from C to the point D between O and B and at a distance 2 m from O. [Answers may be left in a form involving inverse trigonometric functions].

8. A particle is initially at rest at the point $(2,2)$ has acceleration $\begin{pmatrix} t \\ 3 \end{pmatrix}$ m s^{-2} after *t* seconds. Find vector expressions for its velocity and position vector after t seconds. After how many seconds is it moving in a direction inclined at 45° to the *x*-axis?

9. (a) An elastic string of natural length l_0 and modulus 16g N has one end fastened to a fixed point O. The other end of the string is attached to a particle of mass 4kg. If the particle is released from rest at O, find the distance it falls before coming instantaneously to rest at point A. Find its speed when it is remaining with distance $\frac{1}{4}l_0$ to reach A.
   (b) A particle of mass 2 kg is released from rest at a point A on the outer surface of a smooth fixed sphere of centre O and radius 0.6m. Given that OA makes an angle θ with the upward vertical, find an expression for the speed at which the particle is travelling when it leaves the surface of the sphere.

10. A force $F_1 = (4i+2j)$ N, state the magnitude of F_1. A second force F_2 has magnitude $8\sqrt{5}$ N and acts in a direction given by **i** + 2**j**, state the force vector F_2. Hence calculate the resultant force F_R of these two forces. Find the acceleration that this resultant would produce on a mass of 3 kg. The mass was initially at rest at the point with position vector 2**i** - 4**j**. If the two forces continue to act on the mass, show that the point with position vector 18**i** + 20**j** lies on the path traced out by the mass.

Exercise 3

1. (a) At time t, the position vector **r** of the point P with respect to the origin *O* is given by $\mathbf{r} = a(\sin pt\,i + j)$, where a and p are constants. Show that the vector $\ddot{\mathbf{r}} + p^2\mathbf{r}$ is constant during the motion.
   (b) A particle moves along a straight line with its acceleration at any instant given by $a = \alpha + \beta v^2$, where v is the velocity of the particle at that instant and α and β are constants. Initially the particle is at rest at a point O on the line. If the displacement of the particle from O during the motion is x, prove that $v^2 = \frac{\alpha}{\beta}(e^{2\beta x}-1)$.

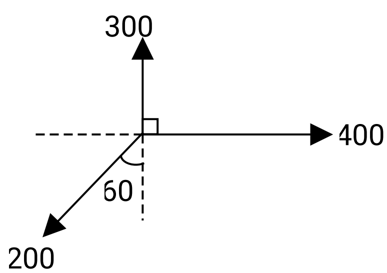
2. (a) The following horizontal forces pass through point O; 5 N in the direction 000°, 1 N in a direction 090°, 4 N in a direction 225° and 6 N in a direction 315°. Find the magnitude and direction of their resultant. Two further horizontal forces are introduced to act at O; P N in a direction 135° and Q N in a direction 225°. If

the complete set of forces is now in equilibrium, calculate the values of P and Q.

(b) A set of horizontal forces of magnitude 20 N, 12 N and 30 N act on a particle in the directions due south, due east and N40°E respectively. Find the magnitude and direction of a fourth force which holds the particle in equilibrium.

3. The resultant of a force 2P N in a direction 060° and a force of 10 N in a direction 180° is a force of $P\sqrt{3}$ N. Calculate the value of P and the direction of the resultant. A third force of 25 N concurrent with the other two and in the same plane is added so that the resultant of the system is in the direction 180°. Find the direction in which the third force is applied and find the magnitude of the resultant.

4. (a) Find the resultant of the system of coplanar forces shown below and the angle it makes with the 400 N force.



(b) Two forces *P* and *Q,* inclined at an angle of 120° have a resultant of magnitude $P\sqrt{7}$. Calculate the magnitude of Q in terms of P.

5. Three forces act through the point with position vector 3**i** + 2**j**. F_1 has magnitude 15 N and acts in the direction 3**i** + 4**j**. F_2 has magnitude $3\sqrt{2}$ N and acts in the direction **i**−**j**. F_3 has magnitude $4\sqrt{5}$ N and acts in the direction 2**i** + **j**. Express the three forces in terms of the unit vectors **i** and **j**. Hence find the resultant F_R of the three forces in terms of the unit vectors. If **i** and **j** are unit vectors in the directions of the x and y-axes, the unit of the length being the metre, illustrate the resultant F_R graphically. Using the graph, calculate the magnitude of the moment of F_R about the origin.

6. Two forces F_1 and F_2 of magnitude $3\sqrt{5}$ N and $\sqrt{5}$ N act through the point with position vector $2\mathbf{i}+\mathbf{j}$ in directions $\mathbf{i}+2\mathbf{j}$ and $\mathbf{i}-2\mathbf{j}$ respectively. Calculate F_1 and F_2 and hence find F_R the resultant of these forces. Draw a diagram to illustrate F_R in component form hence calculate the magnitude of the moment of F_R about a point with position vector **i**. What do you deduce from this result?

7. A particle P of mass m is released from rest at a point A on the surface of a fixed smooth sphere of centre O. The radius OA is inclined at an angle of 30° to the vertical. Show that while the particle remains in contact with the surface, the reaction on the particle is $mg\left(3\cos\theta-\sqrt{3}\right)$, where θ is the angle between OP and the upward vertical. Show that to the nearest degree, the value of θ when the particle leaves the surface of the sphere is 55°.

8. A uniform rod AB of mass 10 kg rests in a vertical plane with end A in contact with a smooth vertical wall. The end B is below A. The rod is inclined at 60° to the vertical and is held in equilibrium by a light string attached to B and to a point C on the wall vertically above A, show in a diagram the forces acting on the rod and hence that the distance CA is half the length of the rod. Find the:
   (i) inclination of the string to the vertical.
   (ii) tension in the string.

9. A particle P of mass 7m is placed on a rough horizontal table, the coefficient of friction between P and the table is μ. A force of magnitude 2mg acting upwards at an acute angle θ to the horizontal is applied to P and equilibrium is on the point of being broken by the particle sliding on the table. Given that $\tan\theta = \frac{5}{12}$ find the value of μ.

10. A uniform rigid rod AB of length 2a and weight W is smoothly jointed at A to a uniform rod AC of length $2a\sqrt{3}$ and weight W'. The rods rest in a vertical plane with B and C on a smooth horizontal plane, equilibrium being maintained by a light inextensible string of length 4a joining B and C. Prove that the tension in the string is $\frac{\sqrt{3}}{8}(W+W')$. Prove further that the reaction at A on the rod AC makes an angle of

$\tan^{-1}\left(\frac{3W'-W}{(W+W')\sqrt{3}}\right)$ with the horizontal.

Exercise 4

1. (a) Forces of 5 N and 3 N act along the sides $\vec{AB}$, $\vec{AC}$ respectively of an equilateral triangle ABC of side 12 m. Find the magnitude and direction of their resultant. The line of action of the resultant intersects BC at D. Find the distance BD.

   (b) Two forces have magnitudes 5 N and P N. If the resultant force has a magnitude 6 N and acts at an angle of 40° to the 5N force, find value of P.

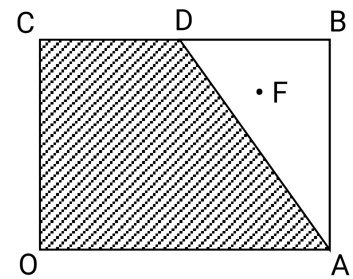
2. A rectangle is defined by four points A(0, 0), B(5, 0), C(5, 3) and D(0, 3), distances measured in metres. Forces of magnitude 6 N, 8 N, 4 N and 2 N act along AB, BC, CD and DA respectively in the directions indicated by the order of the letters. Calculate the:
   (a) magnitude of the resultant of the system of forces.
   (b) angle the line of action of the resultant makes with the x-axis.
   (c) line of action of the resultant cuts the x-axis at (a, 0) find the value of a.

3. The square ABCD has each side of length 6 m. Forces of magnitude 1, 2, 8, 5, $5\sqrt{2}$ and $2\sqrt{2}$ N acting along AB, BC, CD, DA, AC and DB respectively, in the directions indicated by the order of the letters. Prove that these forces are equivalent to a couple. Calculate the magnitude and sense of the couple.

4. A rectangle ABCD has AB = 3 cm and BC = 4 cm. Forces, all measured in newtons and of magnitudes 2, 4, 6, 8 and k act along AB, BC, CD, DA and AC respectively, their directions being indicated by the order of letters. The resultant of the forces is parallel to BD. Find k and show that the resultant has magnitude $\frac{5}{6}$ N. Find the distance from A to the line of action of the resultant.

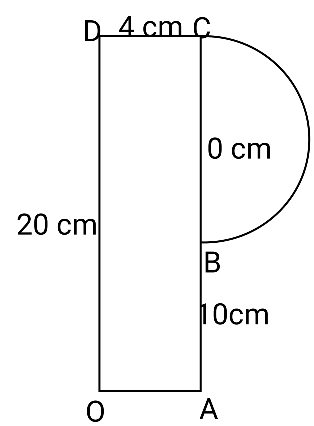
5. A uniform rectangular lamina ABCD is of mass 3M, AB = DC = 4 cm and BC = AD = 6 cm. Particles each of mass M, are attached to the lamina at B, C and D. Calculate the distance of the centre of mass of the loaded lamina from:
   (a) AB (b) AD

6. The diagram shows a square OABC of side a. The midpoint of BC is D. Show that with respect to OA and OC as axes, the coordinates of the centroid F of the triangular region ABD are $\left(\frac{5}{6}a, \frac{2}{3}a\right)$. Find the coordinates of the centre of mass of a uniform lamina in the form of the figure OADC.



If the figure OADC is suspended from C, show that the angle CD makes with the horizontal is $90° - \tan^{-1}\left(\frac{10}{7}\right)$.

7. The figure below shows a lamina formed by welding together a rectangular metal sheet of length 20 cm and width 4 cm and a semi-circular metal sheet of the same material which has a diameter of 10 cm.



   (a) Find the distance of the centre of gravity of the lamina from OD and OA.

   (b) The lamina is freely suspended from B, find the angle BC makes with the vertical in equilibrium.

8. A uniform lamina ABCD has the shape of a trapezium with DC parallel to AB and

$B\hat{A}D = 90°$, $AB = 3a$, $CD = 2a$ and $AD = 2b$. Find the distance of its centre of gravity from AB and AD. The lamina is freely suspended from B and hangs in equilibrium. If $a = 7$ cm, $b = 13$ cm, find the inclination of AB to the vertical.

9. A particle moves along a straight line ABC. It starts from rest at A and moves with constant acceleration of $2\ m\ s^{-2}$ until it reaches B. It then moves from B to C with constant acceleration of $1\ m\ s^{-2}$. The time taken to travel from A to B and from B to C are each equal to T seconds. Show that $5AB = 2BC$. Given that the distance AC is 56 m, calculate T.

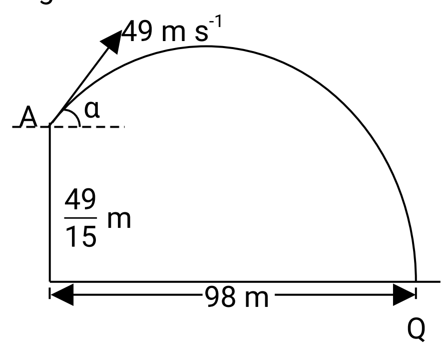
10. A car starts from rest at time, $t = 0$ seconds and moves with uniform acceleration of magnitude $2.3\ m\ s^{-2}$ along a straight horizontal road. After T seconds when its speed is $v\ m\ s^{-1}$, it immediately stops accelerating and maintains this steady speed until it hits a brick wall to come instantly to rest. The car has travelled a distance of 776.25 m in 30 s.
    (a) Sketch a speed-time graph for the motion.
    (b) Write an expression for v in terms of T.
    (c) Show that $T^2 - 60T + 675 = 0$.
    (d) Hence find the value of T and the value of v.

Exercise 5

1. (a) An aircraft flies in a straight line from A to B, where AB = 500 km and B is due north of A. The speed of the aircraft in still air is $250\ km\ h^{-1}$ and the wind is blowing in the direction 290° at $30\ km\ h^{-1}$. Find the:
    (i) course set by the pilot of the aircraft.
    (ii) time taken to the nearest minute for the journey from A to B.
   (b) A wind is blowing at $24\ km\ h^{-1}$ from the direction 270°. Find the magnitude and direction of the wind's velocity relative to a cyclist travelling at $16\ km\ h^{-1}$ in the direction 030°.

2. A car of mass 900 kg travelling on a straight level stretch of a motorway accelerates uniformly from $36\ km\ h^{-1}$ to $108\ km\ h^{-1}$ in 10 seconds. The engine provides a constant driving force F newtons and the total resistance to motion of the car is constant and equal to R newtons. Given that when the car is travelling at $50\ km\ h^{-1}$ the power being used to provide the driving force is 40 kW, find the value of F. Hence calculate the value of R.

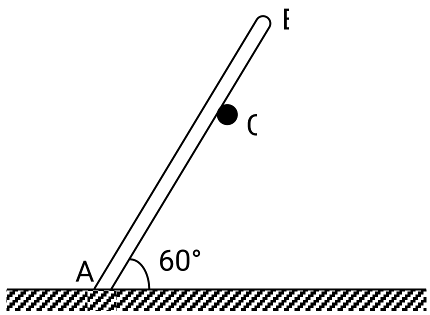
3. A golf ball is projected with a speed of $49\ m\ s^{-1}$ at an angle of elevation α from a point A on the first floor of a golf driving range. Point A is at a height of $\frac{49}{15}$ m above a horizontal ground. The ball first strikes the ground at a point Q which is at a horizontal distance of 98 m from point A as shown in the diagram below.



   (a) Show that $6\tan^2\alpha - 30\tan\alpha + 5 = 0$.
   (b) Hence find to the nearest degree the two possible angles of projection.
   (c) Find to the nearest second the least time of direct flight from A to Q.

4. A particle of mass 3 kg moves on a smooth horizontal table under the action of a variable horizontal force whose value at time t seconds is $6\cos t\mathbf{i} - 3e^{-t}\mathbf{j}$ newtons. When $t = 0$ the particle has velocity $\mathbf{j}\ m\ s^{-1}$ and is at a point with position vector $(3\mathbf{i}-\mathbf{j})$ m. Find the:
   (a) velocity of the particle at time t.
   (b) position vector **r** of the particles at time t, show that for large values of t, $\mathbf{r} = (5-2\cos t)\mathbf{i}$.

5. The diagram shows a uniform plank AB of weight W and length 4a whose lower end A rests on a rough horizontal ground. The plank is inclined at 60° to the horizontal and rests in equilibrium supported by a smooth peg at C, where AC = 3a.



Find in terms of W:

(a) the force exerted on the plank by the peg at *C*.

(b) the vertical and horizontal components of the force exerted on the plank by the ground at A.

(c) Given that equilibrium is limiting, find the coefficient of friction between the plank and the ground.

6. (a) A particle of mass 10kg hangs from a fixed point O by a light inextensible string. It is pulled aside by a horizontal force *P* and rests in equilibrium at an angle of 60° to the vertical. Find the:

   (i) horizontal force P.

   (ii) tension in the string.

   (b) Particles of mass 4 kg, 6 kg and 8 kg are located at position vectors $(2i\text{-}j)$, $(3i+2j)$ and $(i\text{-}2j)$ respectively. Find the location of their centre of gravity.

7. A uniform beam AB has length 6 m and weight 200 N. The beam rests in a horizontal position on two supports at points C and D, whereAC = DB = 1 m. Two children, Rose and Tom, each of weight 500 N, stand on the beam with Rose standing twice as far from the end B as Tom. The beam remains horizontal and in equilibrium and the magnitude of the reaction at D is three times the magnitude of the reaction at *C*. Find how far Tom is standing from the end B.

8. *A* and B are two points on level ground 60 m apart. A particle is projected from A towards B with an initial speed of 30 m s^{-1} at 45° to the horizontal. At the same time a particle is projected from B towards A with an initial speed of 50 m s^{-1}. If they collide, find the:

   (i) angle of projection of the second particle.

   (ii) time of collision.

   (iii) height at which the collision occurs.

9. Two particles of mass 2 kg and 3 kg are connected by a light inextensible string passing over a fixed smooth pulley. Initially the system is at rest with the strings taut and vertical with both particles at a height of 2 m above the ground. When the system is released, find the time which elapses before the 3 kg mass hits the ground and the maximum height reached by the 2 kg mass above the ground.

10. A light elastic string of natural length 1 m is fixed at one end and a particle of weight 4 N is attached to the other end. When the particle hangs freely in equilibrium the length of the string is 1·4 m. The string is now held at an angle α to the vertical by a horizontal force of 3 N acting on the particle. Find the value of α and the new length of the string.

Exercise 6

1. (a) A particle is projected vertically upwards with speed u, after time T a second particle is projected vertically upwards from the same point with speed 3u. Given that the particles collide when the first particle is at its maximum height, prove th

   $T = \frac{2u}{g}(\sqrt{2}\text{-}1)$.

   (b) A particle of mass 4 kg moves under the action of a force **F**. At time t the momentum of the particle is 8cos t**i** - 12sin t**j**.

   (i) Find **F** in terms of t.

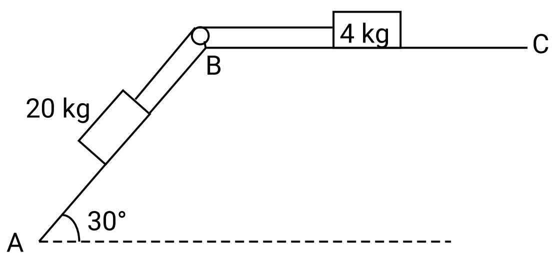
   (ii) Write down the velocity of the particle at time t and find the smallest positive value of t for which the particle is moving in the direction of the vector **j**.

   (iii) Given that when t = 0 the position vector of the particle is $(i\text{-}j)$, find its distance from the origin when $t = \frac{\pi}{2}$.

2. A car of mass 600 kg is travelling round a circular bend of radius 30 m. Given that the coefficient of friction between the car and the

road is $0 \cdot 3$, determine the maximum safe speed for the car if the road is:
(i) unbanked.
(ii) banked at 20° to the horizontal.

3.



(a) Plane ABC is made of two surfaces, a rough horizontal surface BC (coefficient of friction $0 \cdot 215$) and a smooth inclined surface AB. Boxes of masses 20 kg and 4 kg are placed on the plane as shown above. If the system is released from rest, determine the acceleration of the boxes.

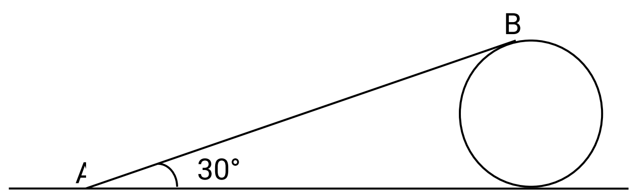
(b) A particle of mass 2 kg is projected with a speed of $10\ m\ s^{-1}$ up a rough plane inclined at 30° to the horizontal from point A, 3 m from the bottom of the plane. If the particle comes to rest at B and the angle of friction along the plane is $\tan^{-1}\left(\frac{1}{4}\right)$;
(i) Find the potential energy of the particle at B and kinetic energy at A. Hence calculate the work done against friction.
(ii) Show that the particle will not remain on the plane.

4. (a) A uniform beam AB of weight 30 N is suspended by two strings at A and B. The beam is in equilibrium at 30° to the horizontal when the strings at A and B make angles of 30° and 60° with the beam respectively and A is lower than B. Find the tensions in the strings.
(b) A rectangular uniform lamina ABCD has sides AB = 4a and AD = 3a. The corner at D is folded so that AD is alongside AB. A square of side a is removed from the corner B. Find the distance of the centre of gravity of the resulting body from B.

5. (a) A train comes to rest from a speed of $20\ m\ s^{-1}$ in a distance of 500 m. If the magnitude of retardation is directly proportional to the speed, find the magnitude of the retardation when the train's speed is $5\ m\ s^{-1}$.

(b) Vehicle A moves with constant velocity of $(400i+150j+250k)\ m\ s^{-1}$ while vehicle B moves with constant velocity of $(100i+200j+50k)\ m\ s^{-1}$, when A is at point $(-3 \cdot 3, 1, 0 \cdot 5)$ km and B at point $(2 \cdot 7, 0, 4 \cdot 5)$ km, the driver of A notices that they are going to collide and immediately reduces speed and changes direction. Find when and where the collision would occur.

6. (a) The diagram below shows a uniform rod AB of mass 8 kg freely hinged at A. It rests in equilibrium with end B on a smooth fixed cylinder.

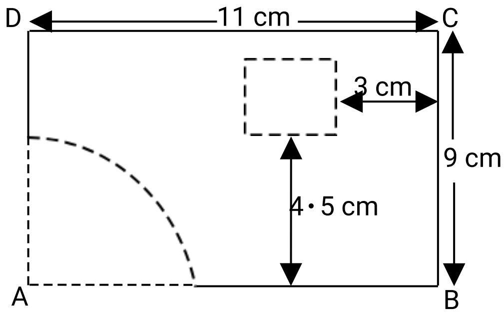


If the rod is inclined at 30° to the horizontal, calculate the magnitudes of the normal reactions at A and B.

(b) A smooth hemisphere of radius $0 \cdot 6$ m is fixed with its plane face on a rough horizontal floor. A uniform rod AB of length 1 m and weight 10 N rests in equilibrium in a vertical plane with end A on the floor and a point C on the rod in contact with the hemisphere, where AC = $0 \cdot 8$ m. Given that the rod is on the point of slipping, find the reaction at C and the coefficient of friction between the rod and the floor.

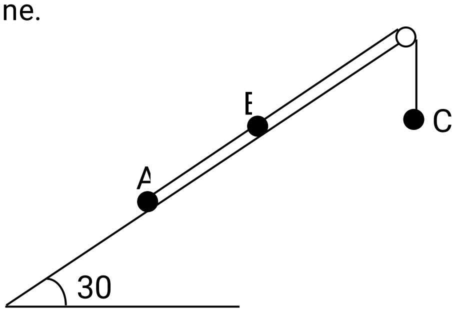
7. A particle P of mass 2 kg is attached to one end of a light elastic string of natural length 2 m and modulus of elasticity 98 N and the other end of the string is attached to a fixed point A. If P is raised to the same level as A and allowed to fall vertically under gravity, calculate the:
(i) maximum extension of the string.
(ii) velocity of P when it is 2 m below A for the first time.

8. The figure below shows a uniform lamina ABCD with a square of side 3 cm and a quadrant of radius $4 \cdot 5$ cm cut off.



Find the distance of the centre of gravity from the sides AD and AB.

9. The diagram below shows particles A, B and C of masses 10 kg, 8 kg and 2 kg respectively, connected by light inelastic strings. The string connecting B and C passes over a smooth light pulley fixed at the top of the plane.



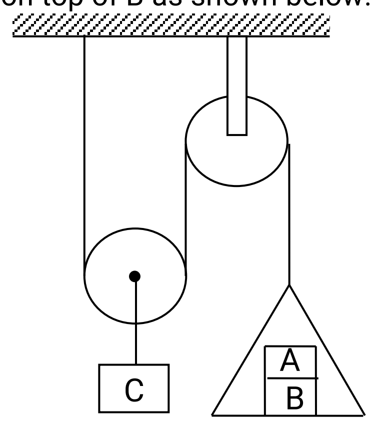
Given that the coefficients of friction between the plane and A and B are 0·22 and 0·25 respectively, calculate the:
   (i) acceleration of the system.
   (ii) tensions in the strings.

10. (a) A boy who is on a hunting mission aims at a monkey which is moving up and down a tall tree situated 100 m away. If the boy is capable of throwing a stone at $\sqrt{(200g)}$ m s^{-1}, show that the maximum height at which the monkey can be hit is 75 m from the bottom of the tree.
   (b) Two particles are projected simultaneously from two points P_1 and P_2 on a level ground and a distance of 75 m apart, the first particle is projected vertically upwards from P_1 with an initial speed of u m s^{-1} and a second particle is projected from P_2 towards P_1 with an angle of projection θ to the horizontal. If the particles collide when they are at their greatest height above the level P_1P_2, prove that $\tan\theta = \dfrac{u^2}{75g}$.

Exercise 7

1. A particle P leaves the origin and moves with a velocity of $2\mathbf{i} + 6\mathbf{j}$ while a particle Q starts from the origin with initial velocity $4\mathbf{i} - 8\mathbf{j}$ and moves with acceleration $\mathbf{i} + \mathbf{j}$.
   (i) Find the position vectors of P and Q after 2 seconds.
   (ii) Calculate the time at which the velocities of P and Q are perpendicular.

2. One end of a light inextensible string is attached to a ceiling. The string passes under a smooth light pulley carrying a weight C and then over a smooth fixed light pulley. To the free end of the string is attached a light scale pan in which two weights A and B are placed with A on top of B as shown below.



The portions of the string not in contact with the pulley are vertical. Each of the weights A and B has a mass M and weight C has a mass kM. If the system is released from rest, find the acceleration of the movable pulley and the scale pan and show that the scale pan will ascend if k > 4. When the system is moving freely, find the:
   (i) tension in the string.
   (ii) reaction between the weights A and B.

3. (a) A particle P of mass 3 kg moves such that the displacement **s**, at time t seconds is given by $\mathbf{s} = \begin{pmatrix} 3+\sin 2t \\ \cos 2t \end{pmatrix}$ m.
   (i) Calculate the value of t when P crosses the x-axis for the first time.
   (ii) Show that the speed of P is constant.

(iii) Calculate the force acting on P when $t = \frac{\pi}{2}$.

(b) A body starts from rest at origin O when t = 0 and moves along a straight line with acceleration given by $a = \frac{7}{36}t$, where t is the time in seconds. The acceleration continues until t = 6, where upon it ceases and the body is retarded to rest. During this retardation the acceleration is given by $a = -\frac{t}{4}$. Find the value of t when the body comes to rest and the displacement of the body from O at that time.

4. Two forces P and Q act on a particle. The force P has magnitude 7 N and acts due north. If the resultant of P and Q is a force of magnitude 10 N acting in a direction with a bearing of 120°, find the:
   (i) magnitude of Q.
   (ii) direction of Q.

5. A particle of mass 1·5 kg is attached to one end of a spring of natural length 1 m and modulus of elasticity 6 N. The other end O, of the spring is attached to a point on a smooth horizontal surface. If the particle is held at rest on the surface with distance between the particle and O being 125 cm and then released from rest, show that the particle performs simple harmonic motion and find the maximum acceleration.

6. A pigeon is released from a point which is at a distance of 80 m on a bearing of 300° from a shooter. The pigeon travels with a constant speed of 40 m s^{-1} on a bearing of 050°. Given that as soon as the pigeon is released, the gun is fired and the bullet has a speed of 200 m s^{-1}. Determine:
   (a) the direction in which the gun must be fired so as to hit the pigeon.
   (b) how long it takes the bullet to hit the bird.

7. A smooth bead of mass 0·2 kg is threaded on a smooth circular wire of radius r metres which is held in a vertical plane. If the bead is projected from the lowest point on the circle with speed $\sqrt{3rg}$. Find the:
   (a) speed of the bead when it has gone one-sixth of the way round the circle.
   (b) force exerted on the bead by the wire at this point.

8. A car of mass 1000 kg accelerates uniformly from 0 to 20 m s^{-1} in 10 s. Calculate the power developed when it has been accelerating for 5 s.

9. A uniform rod XY of length 2 m and weight W is hinged to a vertical post at X. It is supported in a horizontal position by a string attached at Y and to a point Z vertically above X. A weight w is hung from Y.
   (a) If the reaction at the hinge is normal to YZ, show that the length of the string YZ is $2\sqrt{\frac{2(W+w)}{W}}$.
   (b) Find the tension in the string.

10. A ship P travelling at $40\sqrt{2}$ km h^{-1} due north-east is initially 50 km, N75°W of a boat travelling northwards.
   (a) If the vessels collide, find the speed of the boat.
   (b) Calculate the shortest distance between the vessels if the boat travels north westwards at the same speed as in (a) above.

Exercise 8

1. Four forces represented by **i** - 4**j**, 3**i** + 6**j**, -9**i** + **j** and 5**i** - 3**j** act at points with position vectors 3**i** - **j**, 2**i** + 2**j**, -**i** - **j** and -3**i** + 4**j** respectively.
   (a) Show that the forces reduce to a couple and find its magnitude.
   (b) If the fourth force is removed and the first force is moved to the point **i** - 8**j**. Find the magnitude and direction of the resultant force.

2. A particle initially at a point O(0,0) moves along the x-y plane such that its displacement from O at time t is given by $\mathbf{r} = 3t^2\mathbf{i} + (4t-6)\mathbf{j}$. Find the:
   (a) distance from the point O(0,0),
   (b) speed,
   (c) magnitude of acceleration, at time t = 4.

3. Two particles A and B move in a straight line B being 18 m in front of A, B starts from rest with an acceleration of 3 m s^{-2} and A starts in pursuit with a velocity of 10 m s^{-1} and an acceleration of 2 m s^{-2}.
   (a) Prove that A will overtake and pass B

after an interval of 2 s.

(b) Show that B will in turn overtake A after a further interval of 16 s.

4. A particle is projected with a speed of 36 m s^{-1} at an angle of 40° to the horizontal from a point $0\cdot5$ m above level ground. It just clears a wall which is 70 m on the horizontal plane from the point of projection. Find the:
   (a) (i) time taken for the particle to reach the wall.
   (ii) height of the wall.
   (b) maximum height reached by the particle above the level of projection.

5. A mass of 12 kg rests on a smooth inclined plane which is 6 m long and 1 m high. The mass is connected by a light inextensible string, which passes over a smooth pulley fixed at the top of the plane, to a mass of 4 kg which is hanging freely 1 m above the ground. With the string taut, the system is released from rest. Determine the:
   (a) acceleration of the system.
   (b) tension in the string.
   (c) velocity with which the 4 kg mass hits the ground.
   (d) time the 4 kg mass takes to hit the ground.

6. A bullet is fired from a point P, 50 m above the ground with a speed of 140 m s^{-1} and it hits the ground at point Q, 200 m horizontally from the point of projection. Find the:
   (a) two possible angles of projection.
   (b) corresponding times of flight.
   (c) angle at which the bullet hits the ground in each case.

7. Ship A is travelling on a course 060° at a speed of $30\sqrt{3}\text{ km h}^{-1}$ and ship B is travelling at 20 km h^{-1}. At noon, B is 260 km due east of A. Find the:
   (a) course B must take to come as close as possible to A.
   (b) time when A and B are closest together and the shortest distance.

8. ABCD is a square of side 2 m. Forces of magnitudes 3 N, 5 N, 7 N and 2 N act along the sides DA,AB, BC and CD respectively. Calculate the:
   (a) magnitude of the resultant of the forces and the angle it makes with AD.
   (b) sum of moments of the forces about A.
   (c) equation of the line of action of the resultant.
   (d) distance from A to the point where the line of action of the resultant of the forces cuts DA produced.

9. (a) Forces of magnitude 4P, 3P, 2P and P act along AB, BC, CD and DA, where ABCD is a square of side 2a;
   (i) find the resultant force.
   (ii) determine the equation of the line of action of the resultant and the distance from B of the point at which it cuts AB produced.
   (b) It is required to replace the given system in (a) above with a single force at the centre of the square together with a couple. Determine the single force and the moment of the couple.
   (c) If instead a couple is added to the system in (a) above and the line of action of the enlarged system passes through B, find the moment and sense of the couple.

10. Forces of 2P, 4P, 3P and 6P act along the sides AB, BC, CD and DA respectively of a square of side a.
   (i) Find the magnitude and direction of the resultant force.
   (ii) Show that the equation of the line of action of the resultant in (i) above is $2x - y + 7a = 0$.
   (iii) Determine the moment of the couple that would transfer the resultant to the line $2x - y + 9a = 0$.

Exercise 9

1. A circular track of radius r is banked at an angle θ to the horizontal. Show that a vehicle will have no tendency to slip when driven around the track with speed $v = \sqrt{(rg\tan\theta)}$. If the coefficient between the tyres and the road surface is μ, prove that the maximum speed possible on the track is $v_{max} = \sqrt{\dfrac{rg(\sin\theta+\mu\cos\theta)}{\cos\theta-\mu\sin\theta}}$.

2. Two uniform rods AB and AC each of length 2 m are smoothly jointed at A and stand in a vertical plane. The ends B and C of the rods rest on a smooth horizontal surface. A string

1 m long joins the midpoints of AB and AC. If each rod weighs 30 N and the system rests in equilibrium, calculate the:

(a) reactions at B and C.

(b) tension in the string.

(c) reaction at A.

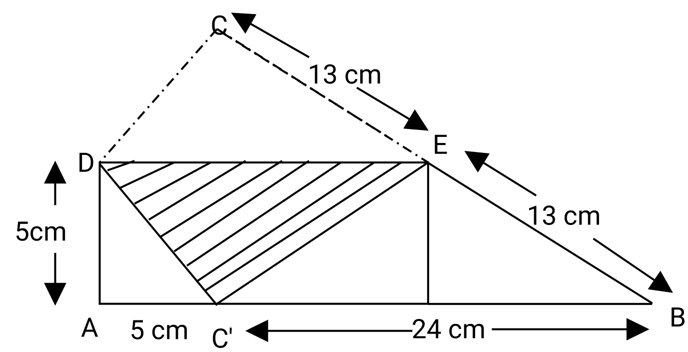
3. Two uniform rods AB and BC of equal length and weight W and 3W respectively are jointed at B. Rod AB is smoothly hinged on a horizontal ground at A while rod BC rests on the same ground, coefficient of friction $\frac{1}{5}$ between the rod and the ground. If rod BC is in limiting equilibrium, prove that it is inclined at $\tan^{-1} 2$ to the horizontal.

4. An engine of a car of mass 60 tonnes, working at 540 kW is pulling a train of mass 480 tonnes up an inclination of $\sin^{-1}\left(\frac{1}{140}\right)$. Frictional resistances amount to $\frac{1}{100}$ of the total weight of the car and the train. Find the:
   (a) acceleration of the train when its speed is 18 km h^{-1}.
   (b) maximum speed.

5. (a) A particle of mass 2 kg moves from rest at the origin under the action of two forces. One force P has magnitude 22 N and acts in the direction of the vector $2\mathbf{i}-6\mathbf{j}+9\mathbf{k}$. The other force Q has magnitude 30 N and acts in the direction of the vector $4\mathbf{j}-3\mathbf{k}$. Find the acceleration of the particle and the distance it travels in the first 3 seconds of its motion.
   (b) A particle of mass 2 kg is acted upon at time t by a force $\mathbf{F}=8\mathbf{i}-(4\cos t)\mathbf{j}+2t\mathbf{k}$. When t = 0, the velocity of the particle is $6\mathbf{i}$. Find the expression for the velocity of the particle and the power at a time t.

6. A riffle fired two shells with the same speed from a point O and they landed at a point A on the same horizontal level as O. Given that the greatest heights of the shells are in the ratio 4:1;
   (i) find the angles of projection.
   (ii) show that the ratio of OA to R_{max} is 4:5 (where R_{max} is the maximum range with the same speed of projection).

7. (a) Particles of mass 4 kg, 7 kg, 5·5 kg and 4·5 kg have position vectors $(3\mathbf{i}-\mathbf{j})$, $(2\mathbf{i}+4\mathbf{j})$, $(-\mathbf{i}-\mathbf{j})$ and $(-3\mathbf{i}+4\mathbf{j})$ respectively. Find the position vector of the centre of mass of the system.
   (b) The figure below shows a uniform lamina ABCD from which ABED is formed with C' lying along AB after folding section DEC to DEC'.



Find the centre of gravity of the lamina ABED from AD and AB.

8. A sports car starts from rest and accelerates at 3 m s^{-2} for 10 s and then travels at a constant speed. Five seconds after it starts, a racing car follows it starting from rest from the same point and accelerating at 4 m s^{-2} for 10 s and then travelling at a constant speed. Find the:
   (i) time (measured from the moment the sports car starts) which elapses before the racing car catches up with the sports car.
   (ii) distance they have both travelled at this moment.

9. (a) A particle of mass 0·2 kg and velocity $5\mathbf{i}+7\mathbf{j}$ collides with a particle of mass 0·3 kg and velocity $2\mathbf{i}-3\mathbf{j}$. If the particles couple together, find the;
   (i) common velocity.
   (ii) loss in kinetic energy.
   (b) An inelastic pile of weight W N is driven a distance of a metres into the ground by a hammer of weight wN falling h metres before hitting the pile. Show that the average resistance of the ground is $\frac{(W+w)^2a+w^2h}{(W+w)a}$.

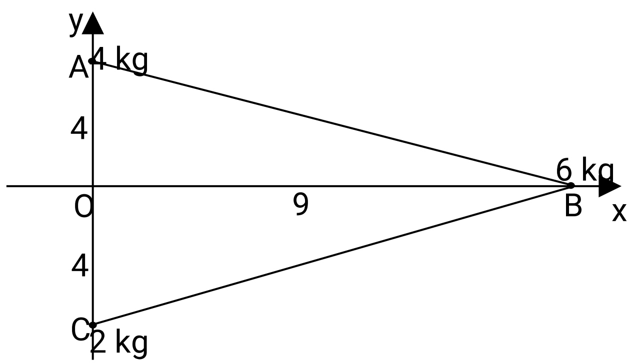
10. (a) A circular track of radius r is banked at an

angle of β to the horizontal. A motorcyclist travels around the track at a speed v without slipping. If the coefficient of friction between the tyres and the track is μ, show that the minimum value of v is given by $v_{min}^2 = \frac{rg(\sin\beta - \mu\cos\beta)}{(\cos\beta + \mu\sin\beta)}$.

(b) A particle of mass m describes complete vertical circles on the end of a light inextensible string of length r. Given that the speed of the particle at the lowest point is twice the speed at the highest point, find the:
   (i) speed of the particle at the lowest point.
   (ii) tension in the string when the particle is at its highest point.

Exercise 10

1. A uniform ladder of length 2a and mass m kg rests with one end against a smooth vertical wall and the other end on a horizontal ground. The ladder is inclined at θ° to the vertical. The bottom of the ladder is attached to a horizontal string and the other end of the string attached to the wall.
   (a) A man of mass 4 kg stands on the ladder a distance of $\frac{1}{2}a$ from the bottom of the ladder. Find the:
      (i) tension in the string.
      (ii) normal reaction on the ground and the wall.
   (b) If the maximum tension which the string can bear without breaking is 4mgtan θ, find how far up the ladder the man can climb safely.
2. An elastic string of natural length 2 m has its upper end fixed and a body of mass 1·2 kg attached to its other end Q. If the modulus of elasticity is 24 N and the system rests vertically in equilibrium.
   (i) Find the extension in the string.
   (ii) If the end Q is pulled vertically downwards to R, where QR = 0•2 m, find the initial acceleration of the body when it is released from this position.
3. A square ABCD of side 2 m is subjected to forces of magnitude 1, 2, 3, 1, $3\sqrt{2}$ and $p\sqrt{2}$ newtons acting along AB, BC, CD, AD, AC and DB respectively in the directions indicated by the order of the letters with AB horizontal. Given that the direction of the resultant is parallel to AC, find the:
   (a) value of p.
   (b) total moment about A of the forces acting on the lamina.
   (c) distance from A to a point E where the line of action of the resultant cuts AB.
4. A particle of mass 5 kg is placed on a smooth plane inclined at $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ to the horizontal.
   Find the:
   (i) magnitude of the force acting horizontally required to keep the particle in equilibrium
   (ii) normal reaction
5. A particle is executing simple harmonic motion with amplitude 2 metres and period 12 seconds.
   (a) Calculate the maximum speed of the particle.
   (b) If initially the particle was moving at a maximum speed, find the:
      (i) distance moved by the particle until its speed is half the maximum value.
      (ii) time taken by the particle to travel this distance.
6. At 9.00 a.m, a fishing boat is 10 km on a bearing of 110° from a Ferry, travelling with a speed of 8 km h^{-1} on a bearing of 060°. If the fishing boat has a top speed of 6 km h^{-1}, find the:
   (a) course set by the fishing boat if it is to get as close to the Ferry as possible.
   (b) closesest distance between the two vessels and the time at which it occurs.
7. The figure below shows a triangular lamina ABC.



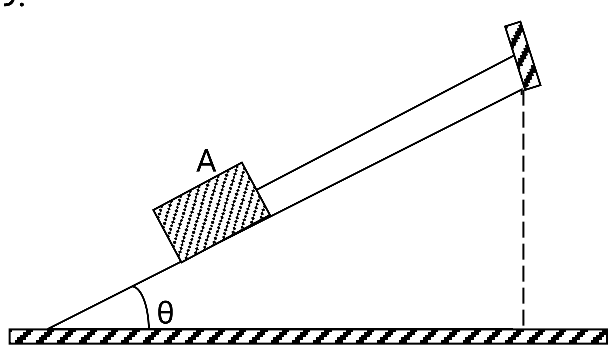
The co-ordinates of A, B and C are (0, 4), (9, 0) and (0, -4) respectively. If the particles of mass 4 kg, 6 kg and 2 kg are attached at A, B and C respectively.

(i) Calculate the co-ordinates of the centre of mass of the three particles.

(ii) The centre of mass of the combined system consisting of the three particles and the lamina has coordinates (4, λ). If lamina ABC is uniform and of mass m kg, calculate the value of m and λ.

8. The combined system is now freely suspended from O and hangs at rest, determine the angle between AC and the vertical. A car decelerated from a speed of 20 m s^{-1} to rest in 8 seconds, falling short of its parking slot by 20 m. By how much longer should the car have decelerated from the same speed so as to just reach the parking slot?

9.



A particle A of mass 5 kg is kept at rest on a rough inclined plane of angle, $\theta = \tan^{-1}\left(\frac{3}{4}\right)$, by an inelastic string parallel to the line of greatest slope of the plane. Given that the coefficient of friction between the particle and the plane is $\frac{1}{2}$, find the minimum tension in the string.

10. Initially a particle travels with a speed of 20 m s^{-1} in the direction -4**i** + 3**j** and 5 sec later its speed is 26 m s^{-1} in the direction 12**i**−5**j**. Calculate the acceleration of the particle.

Exercise 11

1. (a) A particle of mass 0·8 kg executes simple harmonic motion of amplitude 0·4 m about a central point O. Given that the particle is projected from O and the period of motion is 6 seconds, find the:

   (i) speed after 2 seconds.

   (ii) force acting at t = 1 s.

   (b) An elastic string of natural length 0·5 m and modulus of elasticity 9·8 N carries a particle of mass 0·5 kg. Calculate the:

   (i) energy stored in the string when the system is in equilibrium

   (ii) the maximum speed, when the particle is stretched by 0·2 m below its equilibrium position and then allowed to make vertical oscillations. A car of mass 2000 kg developing a constant power of 20 kW ascends a slope 1 in 49 with a maximum speed of 10 m s^{-1}. The non−gravitational resistance to the motion of the car is directly proportional to the car's speed.

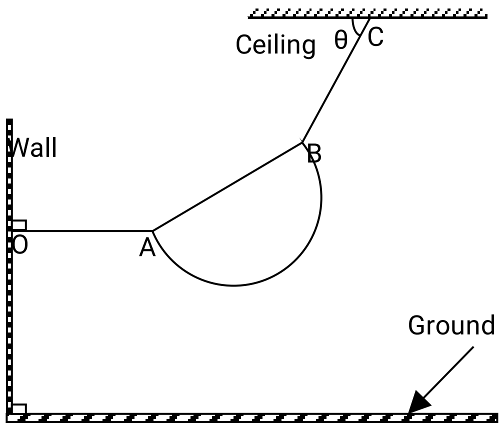
   (a) Calculate the;

   (i) non−gravitational resistance at maximum speed.

   (ii) acceleration of the car at a speed of 5 m s^{-1}

   (b) If the car descends this slope while developing the same power of 20 kW, calculate the maximum speed the car attains.

2.



The diagram shows a uniform solid hemisphere of mass 2·4 kg balanced by two inelastic strings. A horizontal string OA is attached to a vertical wall and a second string BC making an angle θ with the horizontal is attached to a ceiling as shown in the diagram. The strings and the line of action of the weight of the hemisphere lie in a vertical plane, show that:

(a) $\tan\theta = \frac{48}{55}$.

(b) the tension in string BC is 35·77 N, hence find the tension in the string OA.

3. (a) Two particles A and B are moving in the same direction on parallel horizontal tracks. At a certain point, the particle A, travelling with a speed of $7 ms^{-1}$ and accelerating uniformly at $1.5\ ms^{-2}$ overtakes B travelling at $3\ ms^{-1}$ and accelerates uniformly at $2{\cdot}5\ m\ s^{-2}$.

(i) Calculate the time which elapses before B overtakes A.

(ii) If after this time, B then ceases to accelerate and continues to move at constant speed, calculate the time taken by A to overtake B again.

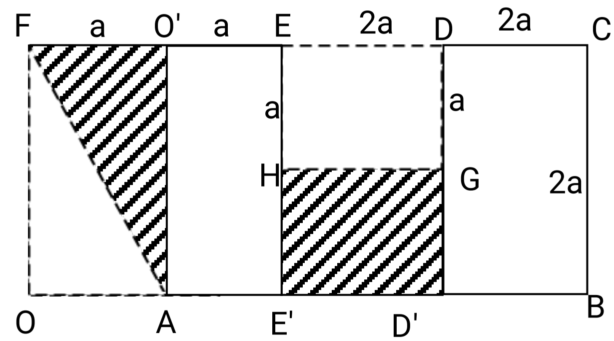
(b) A particle of mass 5 kg moves so that its position vector after t seconds is $r = [(\cos 2t)i + (3+4\sin 2t)j]$ m. Find the:

(i) speed of the particle when $t = \frac{\pi}{3}$.

(ii) acceleration and force acting on the particle when $t = \frac{\pi}{2}$.

4. (a) Masses of 8 kg, 3 kg, 7 kg, 10 kg and 12 kg are located at points with position vectors $-i + 3j, 2i + 4j, -2i - 4j, -2i - 3j$ and $5i + 3j$ respectively. Find the position vector of their centre of gravity.

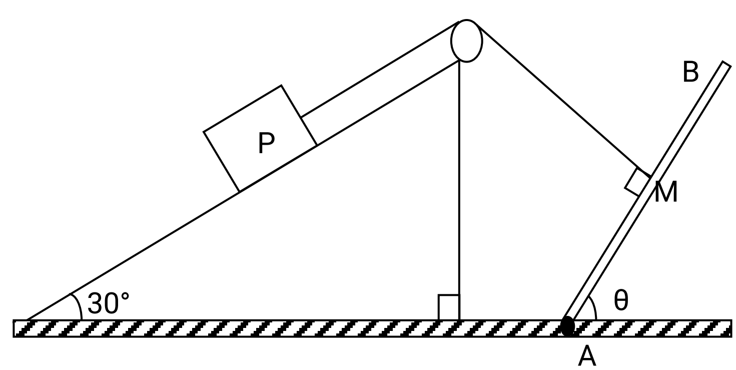
(b) A uniform rectangular lamina of length 6a and width 2a has triangle OAF cut and fixed to form a composite triangle O'AF and E is folded to E' while D is folded to D' as shown in the figure below.



Find the centre of gravity of the new lamina AE'D'BCDGHEO'F. Taking OF and OB as reference axes.

5. The points O, A, B and C lie in a straight line such that AB = 28 m and BC = 72 m. A particle moving with constant acceleration starts from rest at O and passes through A, B and C, its velocities at B and C being $9\ m\ s^{-1}$ and $15\ m\ s^{-1}$ respectively. Find the velocity of the particle at A and the time it takes to travel from A to C.

6.



An inextensible string connects particle P of mass 8 kg to a uniform rod AB of mass 5 kg.

The rod is smoothly hinged to a point A on the ground and the string is attached at right angles to the rod at its midpoint M. Given that P is in limiting equilibrium on a rough inclined plane, coefficient of friction is $\frac{\sqrt{3}}{12}$ while the rod makes an angle θ with the ground. Calculate the:

(a) minimum tension in the string,

(b) and the corresponding

(i) value of θ.

(ii) reaction at A.

7. A car of mass 300 kg moving at 144 km h^{-1} collides with a stationary trailer of mass 900 kg there by losing 15% of its momentum. If the car decelerates at 6 m s^{-2} after collision, calculate the:

(i) velocity of the trailer after collision.

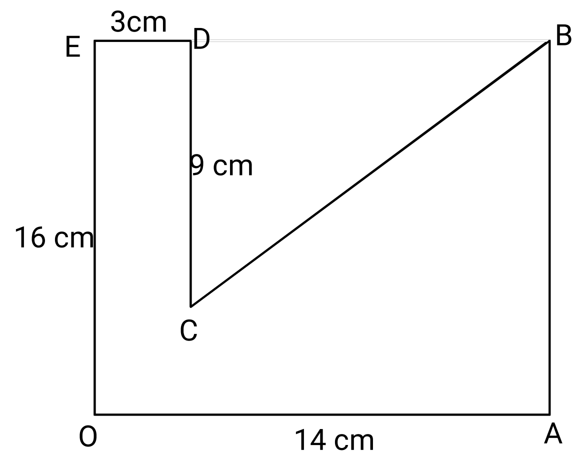
(ii) distance the car moves before coming to rest.

(iii) deceleration force.

8. (a) A particle is projected horizontally from a point 2•5 m above a horizontal surface. The particle hits the surface at a point which is horizontally 10 m from the point of projection. Find the speed of projection.

(b) A and B are two points such that B is vertically h metres above A. From A, a particle is projected vertically upwards with velocity u, at the same instant another particle is projected with velocity v vertically upwards from B. If the particles collide at a point C above B, Prove that $\overline{AC} = \frac{uh}{u-v} - \frac{gh^2}{2(u-v)^2}$.

9. (a) Find the centre of gravity of the lamina shown below.



(b) If the lamina is suspended from O, find the angle OB makes with the vertical.

Exercise 12

1. (a) A train travelling uniformly on a level track at 80 km h^{-1} begins to ascend a hill of slope 1 in 75. The tractive force exerted by the engine is constant and equal to $\frac{1}{100}$ of the weight of the train, the frictional resistance is constant and equal to $\frac{1}{150}$ of the weight of the train. How far will the train ascend the hill before coming to a halt.

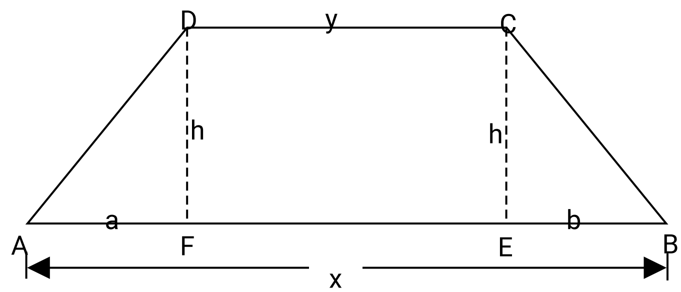
(b) To one end of a light inextensible string passing over a smooth fixed pulley is attached a particle of mass 8 kg and to the other end a light pulley. Over this pulley passes a light inextensible string to the ends of which are attached particles of masses 5 kg and 3 kg. If the system is released from rest, find the accelerations of the masses and the tensions in the strings.

2. The initial velocity of a body moving with constant acceleration is $(5i-2j)$ m s^{-1}. After 5 seconds the particle has a speed of 52 m s^{-1} in the direction 5i + 12j. Find the acceleration of the body.

3. A boy of weight 7W climbs a uniform ladder AB of weight 2W whose end A is in contact with a rough horizontal surface and B is

resting against a smooth vertical wall. When he is $\frac{1}{5}$ of the way up, the ladder is about to slide. If the coefficient of friction between the ladder and the ground is $\frac{4}{15}$ find the inclination of the ladder to the horizontal.

4. A car of mass 750 kg moves along a horizontal road against a total resistance of 240 N. If the car engine is working at a constant rate of 12 kW, find the:
   (i) maximum speed of the car.
   (ii) acceleration of the car when its speed is 30 m s^{-1}.
5. The speed of a motorcycle reduced from 90 km h^{-1} to 18 km h^{-1} in a distance of 120 m. Find its speed when it had covered a distance of 80 m.
6. Forces of 7 N and 4 N act away from a common point and make an angle θ° with each other. Given that the magnitude of their resultant is 10·75 N, find the:
   (i) value of θ.
   (ii) direction of the resultant.
7. A particle of mass m kg is released at rest from the highest point of a smooth solid spherical object of radius a metres. Find the angle to the vertical at which the particle leaves the sphere.
8. (a) Particles of masses 5 kg,2 kg,3 kg and 2 kg act at points with position vectors $3\mathbf{i}-\mathbf{j}, 2\mathbf{i}+3\mathbf{j}, -2\mathbf{i}+5\mathbf{j}$ and $-\mathbf{i}-2\mathbf{j}$ respectively. Find the position vector of their centre of gravity.
   (b) The figure ABCD below shows a metal sheet of uniform material cut in shape of a trapezium. $\overline{AB} = x$, $\overline{CD} = y$, $\overline{AF} = a$, $\overline{EB} = b$ and h is the vertical distance between AB and CD.



Show that the centre of gravity of the sheet is at a distance $\frac{h}{3}\left[\frac{3y+a+b}{x+y}\right]$ from side AB.

9. Two balls A and B of masses m_1 and m_2 respectively lie on a smooth surface in a straight line. If the balls are projected with velocities u_1 and u_2 respectively in the same direction, show that on colliding elastically, then A will move with velocity v_1 given by $v_1 = \frac{(m_1-m_2)u_1+2m_2u_2}{(m_1+m_2)}$.
10. A particle of mass m is free to rotate at the end of a light inextensible string fixed at its other end, if the length of the string is r and the particle is projected horizontally from its lowest position with speed $\sqrt{6rg}$ to traverse a vertical circle. Find the:
    (a) least tension.
    (b) maximum tension in the string.

Answers

Exercise 1

1. (i) **k** (ii) m = 5 ; n = 1 (iii) -4 ; 100·3° 2. (a) 3·0685 s (b) $|v_A| = 10 \cdot 07 \text{ m s}^{-1}$

$|v_B| = 2 \cdot 77 \text{ m s}^{-1}$ 3. (i) 48 km h^{-1} (ii) 3840 km h^{-2} 4. (i) (ii) 3 s (iii) 15 m s^{-1}

(iv) $\frac{5}{3} \text{ m s}^{-2}$ 5. 6 m s^{-1} ; N60°W;

N40·9°W 6. (i) 8 m s^{-2} (ii) 19 N (iii) 88 m 7. (i) 4·1667 s (ii) 10·4167 m 8. 1 kg mass: T - g = a ; 2 kg mass: 2g - T = 2a

(i) 13·0667 N (ii) 0·9583 s 9. $x = \frac{2E}{mg}$; $\lambda = \frac{m^2g^2l}{2E}$ 10. 400 N

Exercise 2

1. (a) 840 N; 240 N (b) 3·59 kN; 990 N; 12·6 kW 2. 500 N ; 15 kW 3. 10 m s^{-1} ; 0.2ms^{-2} 4. (a) 39.682 m (b) 61·2245 m (c) 7.0696 s (d) 101·392 m 5. (i) k = 72 (ii) 482.2531m (iii) 24.068 s **6.(a)**(i) $v = ue^{-k(x-a)}$ (ii) $v = \left(\frac{u}{1+kut}\right)$ (iii) $x = a + \frac{1}{k}\ln(1+kut)$ (b) $v = 4\left(1-e^{-5s}\right)^{\frac{1}{2}}$ 7. (a) (i) $\sqrt{10}$ m (ii) π seconds (iii) $2\sqrt{10}$ m s^{-1} (iv) $\frac{\pi}{4}$ seconds (b) (i) 8 m s^{-2} (ii) 64 N (c) $x = \sqrt{10}\sin 2t$; $\frac{1}{2}\sin^{-1}\left(\frac{\sqrt{10}}{10}\right)$ $\frac{1}{2}\left[\sin^{-1}\left(\frac{\sqrt{10}}{5}\right)-\sin^{-1}\left(\frac{\sqrt{10}}{10}\right)\right]$ 8. $\mathbf{v} = \begin{pmatrix}\frac{1}{2}t^2\\ 3t\end{pmatrix}$; $\mathbf{s} = \begin{pmatrix}\frac{1}{6}t^3+2\\ \frac{3}{2}t^2+2\end{pmatrix}$; 6 seconds

9. (a) $2l_0$; $\frac{\sqrt{5gl_0}}{2}$ (b) $\sqrt{0{\cdot}4g\cos\theta}$ 10. $2\sqrt{5}$ N ; $(8i+16j)$ N ; $(12i+18j)$ N ; $(4i+6j)$ m s^{-2}

Exercise 3

1. (a) (b) 2. (a) 8·8318 N;316·6°;P = 8·8284 N;Q = 0·2426 N (b) 31·425 N;S84·6°W 3. P = 10 N ; 090° ; 223·9° ; $5\sqrt{13}$ N 4. (a) 302·384 N ; 41·4° (b) Q = 3P 5.
$F_1 = (9i+12j)$N;$F_2 = (3i-3j)$N;$F_3 = (8i+4j)$N;$F_R = (20i+13j)$N;1 N m Clockwise
6. $F_1 = (3i+6j)$N ; $F_2 = (i-2j)$N ; $F_R = (4i+4j)$N ; 0 N m, No turning effect produced about the point 7. 8. (i) 40·9° (ii) 129·64 N 9. $\frac{8}{27}$ 10.

Exercise 4

1. (a) 7 N ; 81·8° below BC ; 4·5 m (b) 3·878

2. (a) $2\sqrt{10}$ N(b) 71·6° (c) 8·6667 m

3. Since X = 0, Y = 0 and G ≠ 0 the system is equivalent to a couple ;

48 N m Anticlockwise. 4. $k = \frac{35}{6}$; 0·432 m 5. (a) $\frac{7}{2}$cm (b) $\frac{7}{3}$cm 6. $\left(\frac{7}{18}a, \frac{4}{9}a\right)$
7. (a) 3·3572 cm ;11·6463 cm (b) 21·3° 8. $\frac{14}{15}b$; $\frac{19}{15}a$; 45° 9. 4 s
10. (a) (b) v = 2·3T (c) (d) T = 15 s ; v = 34·5 m s^{-1}

Exercise 5

1. (a) (i) 006·5° (ii) 116·0 minutes (b) 21·166 km h^{-1} in direction 130·9°

2. F = 2880 N ; R = 1080 N 3. (a) (b) 10° ; 78° (c) 2 s

4. (a) $\mathbf{v} = \left(2\sin t\mathbf{i}+e^{-t}\mathbf{j}\right)$ (b) $\left(5-2\cos t\right)\mathbf{i} - e^{-t}\mathbf{j}$ 5. (a)

$\frac{1}{3}W$ (b) $\frac{5}{6}W$; $\frac{\sqrt{3}}{6}W$ (c) $\frac{\sqrt{3}}{5}$

6. (a) (i) $98\sqrt{3}$ N (ii) 196 N (b) $\frac{1}{9}(17i-4j)$ 7. 1•2 m 8. (i) 25•1°

(ii) 0•9024 s (iii) 15•15 m 9. $\frac{10}{7}$ s ; 4•4 m

10. 36•9° ; 1•5 m

Exercise 6

1. (a) (b) (i) **F** = -8sin t**i** - 12cos t**j** (ii) 2cos t**i** - 3sin t**j** ; t = $\frac{\pi}{2}$ s (iii) |**r**| = 5 m

2. (i) 9•3915 m s^{-1} (ii) 14•8032 m s^{-1} 3. (a) 3•732 m s^{-1} (b) (i) 99•17 J ; 100 J ; 30•224 J (ii) 4. (a) $15\sqrt{3}$ N ; 15 N (b) 2•1a 5. (a) $\frac{1}{5}$ m s^{-2} (b) t = 20 s ; (4700i+4000j+5500k) m 6. (a) 49 N ; 33•948 N (b) 5 N ; $\frac{1}{2}$

7. (i) 1•7266 m (ii) 6•261 m s^{-1}

8. 6.15 cm; 4•87 cm 9. (i) 1•648 m s^{-2}

(ii) Tension in AB:13•848 N ; Tension in BC:22•896 N 10. (a) (b)

Exercise 7

1. (i) $r_P = (4i+12j)$ m ; $r_Q = (10i-14j)$ m (ii) 5 s

2. $g\left(\frac{k-4}{k+8}\right)$ downwards; $2g\left(\frac{k-4}{k+8}\right)$ upwards (i) $\left(\frac{6kMg}{k+8}\right)$ (ii) $\left(\frac{3kMg}{k+4}\right)$

3. (a) (i) $\frac{\pi}{4}$ s (ii) 2 m s^{-1} (iii) $\begin{pmatrix}0\\12\end{pmatrix}$N (b) 8 s ; 10•6667 m 4. (i) 14•7986 N

(ii) On a bearing of 144•2° 5. $\ddot{x} = -4x$; 1 m s^{-2} 6. (a) 310•8° (b) 0•38 s

7. (a) $\sqrt{2rg}$ (b) 4•9 N 8. 20 kW 9. (a) (b) $T = \sqrt{\frac{(W+w)(W+2w)}{2}}$ 10. (a) 50•718 km h^{-1} (b) 15•534 km

Exercise 8

1. (a) 26 N m (b) 5•831 at 149•0° to the direction of i 2. (a) 49•0306 (b) 24•331

(c) 6 3. (a) (b) 4. (a) (i) 2•5383 s (ii) 27•67 m (b) 27•32 m

5. (a) 1•225 m s^{-2} (b) 34•3 N (c) 1.565 m s^{-1} (d) 1•278 s

6(a) 87•2° above horizontal ; 11•2° below horizontal (b) 28•9 s ; 1•456 s (c) 87•3° below horizontal ; 16•8° below horizontal 7. (a) 352•6° (b) 5.23 p.m ; 33•32 km 8. (a) 5 N ; 36•9° (b) 18 N m in sense ABCD (c) 4x - 3y = 18 (d) 6 m 9. (a) (i) $2\sqrt{2}$P at 45° above AB (ii) y = x - 5a ; 3a (b) $2\sqrt{2}$P through centre and parallel to resultant ; 10aP in sense ABCD (c) 6aP in sense DCBA

10.(i) $\sqrt{5}$P at 63•4° below BA (ii) (iii) 2aP in sense ABCD

Exercise 9

1. 2. (a) 30 N ; 30 N (b) $10\sqrt{3}$ N (c) $10\sqrt{3}$ N horizontally **3.** **4.** (a) 0.032 ms^{-2} (b) 5•9524 m s^{-1} **5.** (a) (2i+6j) m s^{-2} ; $9\sqrt{10}$ m (b) (6+4t)**i** - (2sin t)**j** + $\frac{1}{2}t^2$**k**;

$t^3 + 32t + 4\sin 2t + 48$

6. (i) $63 \cdot 4°$; $26 \cdot 6°$ (ii) 7. (a) $\frac{1}{3}i + \frac{73}{42}j$

(b) $10 \cdot 566$ cm ; $2 \cdot 566$ cm

8. (i) 25 s (ii) 600 m 9. (a) (i) $(3 \cdot 2i+j)$
(ii) $6 \cdot 54$ J (b)

10. (a) (b) (i) $4\sqrt{\frac{rg}{3}}$ (ii) $\frac{1}{3}mg$

Exercise 10

1. (a) (i) $\frac{1}{2}(m+2)g\tan\theta$ (ii) $(m+4)g$ $\frac{1}{2}(m+2)g\tan\theta$ (b) $\frac{7}{4}$ma from the bottom

2. (i) $0 \cdot 98$ m (ii) 2 m s^{-2} 3. (a) $\frac{5}{2}$ (b) 5 N m in sense ABCD (c) $\frac{10}{7}$ m

4. (i) $\frac{49\sqrt{3}}{3}$ N (ii) $\frac{98\sqrt{3}}{3}$ N 5. (a) $\frac{\pi}{3}$ m s^{-1} (b) (i) $\sqrt{3}$ m (ii) 2 s

6. (a) On a bearing of $018 \cdot 6°$ (b) $0 \cdot 246$ km ; 10.53 a.m

7. (i) $\left(\frac{9}{2}, \frac{2}{3}\right)$ (ii) 6 kg ; $\frac{4}{9}$ (iii) $6 \cdot 3°$

8. 2 s 9. $9 \cdot 8$ N 10. $(8i-4 \cdot 4j)$ m s^{-2}

Exercise 11

1. (a) (i) $\frac{\pi}{15}$ m s^{-1} (ii) $0 \cdot 304$ N (b) (i) $0 \cdot 6125$ J (ii) $1 \cdot 252$ m s^{-1}

2. (a) (i) 1600 N (ii) $1 \cdot 4$ m s^{-2} (b) $12 \cdot 5$ m s^{-1} 3. (a) (b) $26 \cdot 95$ N

4. (a) (i) 8 s (ii) $\frac{16}{3}$ s (b) (i) $\sqrt{19}$ m s^{-1} (ii) $4i$ m s^{-2}; $20i$ N 5. (a) $\frac{3}{5}i + \frac{7}{20}j$

(b) $\frac{109}{36}a$; $\frac{8}{9}a$ 6. 5 m s^{-1} ; 10 s 7. (a) $29 \cdot 4$ N (b) (i) $53 \cdot 1°$ (ii) 39.2 N along AB 8. (i) 2 m s^{-1} (ii) $96 \cdot 333$ m (iii) 1800 N

9. (a) 14 m s^{-1} (b) 10. (a) 7.095 cm from OE ; $6 \cdot 582$ cm from OA (b) $5 \cdot 9°$

Exercise 12

1. (a) $2 \cdot 52$ km (b) 8 kg mass: $0 \cdot 316$ m s^{-2}(downwards); 3 kg mass: $2 \cdot 845$ m s^{-2} (upwards) ; 5kg mass: $2 \cdot 213$ m s^{-2}(downwards); $75 \cdot 87$ N ; $37 \cdot 935$ N

2. $(3i+10j)$ m s^{-2} 3. 45° 4. (i) 50 m s^{-1} (ii) $\frac{16}{75}$ m s^{-2} 5. 15 m s^{-1}

6. (i) $25 \cdot 5°$ (ii) $9 \cdot 2°$ to the 7 N force 7. $41 \cdot 8°$ 8. (a) $\left(\frac{11}{12}i+j\right)$ (b)

9. 10. (a) mg (b) 7mg

MATH 2

In numerical work, take g to be 9.8 ms-2

SECTION A (40 MARKS)

Answer all *questions in this section.*

1. The numbers a,b,8,5,7 have mean and varience 2. Find the value of and b if a>2 (5 marks).

2. A class performed an experiment to estimate the diameter of a circular object. A sample of five students had the following results in centimetres;

   3.12, 3.16, 2.94, 3.33 and 3.0

   (a) mean

   (b) standard deviation

3. Show that the root of the equation $2x - 3\cos(\frac{x}{2}) = 0$ lies between 1 and 2

   (5 marks)

4. The table below shows the values of a function f(x).

x	1.8	2.0	2.2	2.4
f(x)	0.532	0.484	0.436	0.384

   Use linear interpolation to find the value of

   (a) *f(2.08)*

   (b) *x* corresponding to *f(x)* = 0.5 (5 marks)

5. Agroup of 20 people played a game. The table below shows frequency distribution of their scores

Scores	1	2	4	x
Number of people	2	5	7	6

Given that the mean score is 5, find the:

a). value of x

b). varience of the distribution. (5 marks)

SECTION B (60 Marks)

6. The table below shows the marks obtained by students in a physical test.

Marks (%)	**Frequency**
25 – 29	9
30 – 34	12
35 – 39	10
40 – 44	17
45 – 49	13
50 – 54	25
55 – 59	18
60 – 64	14
65 – 69	8
70 – 74	8

(a) Draw a histogram and use it to estimate the modal mark. (04 marks)

(b) Find the:

(i) Mean mark, (05 marks)

Standard deviation. (03 marks

7. The cumulative frequency table below shows the ages in years of employees of a certain company.

Age (years)	<15	<20	<30	<40	<50	<60	<65
Cumulative frequency	0	17	39	69	87	92	98

(a) (i) Use the data in the table to draw a cumulative frequency curve

(Ogive),

(ii) Use the curve to estimate the semi- interquartile range.

(06 marks)

(b) Calculate the mean age of the employees.

(06 marks)

8. Show that the root of the equation $f\ (x) = e^x + x^3 - 4x = 0$ lies between 1 and 2. By using linear interporation find the root of the equation to two decimal places.

1. The table below shows the values of a continuous function *f* with respect to ***t.***

t	0	0.3	0.6	1.2	1.8
F(t)	2.72	3.00	3.32	4.06	4.95

Using linear interpolation find:

(a) *f (t)* when *t = 0.9,*

(b) *t* when *f (t)* = 4.48.

(12 marks)

/

SMASK

S.6 REVISION WORK TERM 1 APPLIED MATHEMATICS P425/2

1. A ship P is moving due west at $12kmh^{-1}$. The velocity of a second ship Q relative to P is $15kmh^{-1}$ in a direction 30^0 west of south.
Find the velocity of ship Q.

2. (a) Show graphically that the equation $e^{2x} + 4x - 5 = 0$ had only one real root between 0 and 1.
(b) Use the Newton – Raphson iterative method to find the root of the equation in (a) above giving your answer correct to 2 decimal places.

3. A tennis player hits a ball at a point 0, which is at a height of 2m above the ground and at a horizontal distance 4m from the net, the initial speed being in a direction of 45^0 above the horizontal. If the ball just clears the net which is 1m high,
   (a) Show that the equation of path of the ball is $16y = 16x - 5x^2$.
   (b) Calculate the;
      (i) distance from the net at which the ball strikes the ground.
      (ii) magnitude and direction of the velocity with which the ball strikes the ground.

4. A random variable X has a continuous probability density function (p.d.f) given by

$$f(x) = \begin{cases} c(x+1) & 0 < x < 2 \\ \dfrac{3cx}{2} & 2 \le x < 4 \\ 0 & \text{Elsewhere} \end{cases}$$

Where c is a constant.

a) Sketch the graph of $f(x)$. Hence find the value of c.
b) Determine the cumulative density function (c.d.f) F(x). Hence find the
   (i) Median
   (ii) $P(1.5 < X \le 3)$

5. Given that $M = 8.542, N = 4.6$ rounded off to the given number of decimal places.
a) State the maximum possible error in M and N.

   Determine the interval with in which the value of $M - \frac{M}{N}$ lies.
b) Show that the maximum relative error in the function $\sqrt{MN}$ is given by $\frac{1}{2}\left(\frac{|\Delta M|}{M} + \frac{|\Delta N|}{N}\right)$ where $|\Delta M|$ and $|\Delta N|$ are the respective errors in M and N. Hence calculate the percentage relative error in finding $\sqrt{MN}$.

6. Given that $\tan 30° = 0.577$ and $\tan^{-1}(1.321) = 52.87°$, use linear interpolation or extrapolation to find the value of;
   a) $\tan^{-1}(0.865)$
   b) $\tan 60°$

7. A and B are events such that $P(A) = \frac{8}{15}$, $P(B) = \frac{1}{3}$, and $P(A/B) = \frac{1}{5}$. Find the probability that;
   (i) neither A nor B occurs
   (ii) Event B does not happen if event A has occurred

8. ABCD is a parallelogram. Forces of magnitude 6N, FN and $13\sqrt{2}$ N act along the sides AB, AD and AC respectively in the direction indicated by the order of the letters. Given that angle ADC = 60°, find the value of F and the size of angle BAC.

9. A class contains 18 girls of which 6 are left handed, 12 boys of which 5 are left handed. A group of 5 pupils is selected at random from the class. Find the
   i) probability that at most 2 are girls
   ii) expected number of left handed pupils in the group.

10. A particle Q moves such that its displacement from the origin at any time t is given by $(r = 4t^2\mathbf{i} + 2t\mathbf{j} - 5\mathbf{k})$m. Find the
   (i) Speed of the particle at t = 3s
   (ii) acceleration of the particle at any time.

11. The price index of an article in 2000 based on 1998 was 130.The price index for the article in 2005 based on 2000 was 80. Calculate the;
a) price index of the article in 2005 based on 1998
b) price of the article in 1998 if the price of the article was 45,000 in 2005.

12. Forces of magnitude **5N** and **PN** are acting away from each other at
an angle of **60°**. Given that their resultant is **7N**, find the:
(i) value of **P**
(ii) angle **P** makes with the resultant

13. A discrete random variable X has a cumulative distribution function (c.d.f) given below

x	1	2	3
F(x)	β	4β	9β

Find the value of β and the mean of x

S.6 REVISION WORK TERM 1 APPLIED MATHEMATICS P425/2

14. A particle of weight 20N is on a rough plane. Given that a force of 10N acting away from the plane at 20° to the line of greatest slope is just sufficient to prevent it from sliding down and the frictional force experienced by the particle is 8N, find the angle of inclination of the plane and the reaction between it and the particle.

15. (a) On the same axes, draw a curve $y = xe^x$ and a line $y = x + 1$ to show that the function $xe^x - x - 1 = 0$ has a root in the interval 0 and 1.

(b) Use the Newton Raphson method to find the root of the equation correct to 3 decimal places.

16. The marks scored in an exam are normally distributed with mean 56 and standard deviation 14.2. Find the probability that a candidate picked at random scored

(i) between 62 and 72 marks.

(ii) at least 40 marks.

17. Forces of $(3i+13j)N$, $(2i-j)N$ and $(-i-4j)N$ act at points with position vectors $(i+j)M$, $(3i+2j)M$, and $(-3i+5j)M$. Find the;

(a) magnitude of the resultant force.

(b) the equation of its line of action.

18. $X \sim B(n, p)$ with mean 5 and standard deviation 2. Find the value of n and p.

19. Find the approximate value of $\int_0^1 \frac{1}{x^2+\sin x}dx$ using six-ordinates.

20. (a) Show that the iterative formula based on Newton Raphson formula for finding the root of the equation

$X = \sqrt[4]{N}$ is given by $x_{n+1} = \frac{1}{4}\left(3x_n + \frac{N}{x_n^{\,3}}\right)$ n=1, 2, 3...

(b) Construct a flow chart that

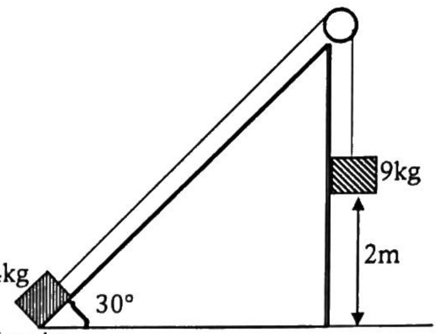
(i) Reads N and the first approximation x_0

(ii) Computes and prints the root to three decimal places or after exactly 4 iterations

Taking $N = 85, x_0 = 3.0$ perform a dry run for the flow chart. Give your root correct to **three** decimal places.

21. Particles of mass 4, 6, 3 and 5kg act at the vertices A, C, B and D respectively of a rectangle ABCD, with $AB = 6\ cm$ and $AD = 4\ cm$. Find the coordinates of the centre of mass of the particles.

22. Two masses of 4 kg and 9 kg are connected by a light inextensible string passing over a smooth pulley at the edge of the smooth inclined plane of inclination 30° as shown below. If the system is released from rest,



Calculate the

(i) acceleration of the system

(ii) tension in the string

(iii) Reaction on the pulley at the edge of the inclined plane.

(iv) Maximum displacement of the 4kg mass up the plane.

23. The table below shows the ages in years of workers on a certain firm.

Age (years)	15–	20–	30–	35–	50–	60–	65–	70–
Frequency density	0.4	0.5	1.6	0.2	0.3	0.8	1.0	0.0

a) Represent the above data on a histogram, hence estimate the modal age.

b) Calculate the

(i) Mean age

(ii) Variance

24. The brakes of a train, which is travelling at $108km^{-1}$ are applied as the train passes point A. The brakes produce a constant retardation of magnitude $3f$ ms^{-2} until the speed of the train is reduced to $36kmh^{-1}$. The train travels at this speed for a distance and is then uniformly accelerated at f ms^{-2} until it again reaches a speed of $108kmh^{-1}$ as it passes point B. The time taken by the train in travelling from A to B, a distance of 4km, is 4 minutes.

(a) Sketch the speed-time graph for this motion

(b) Calculate;

(i) the value of f

(ii) the distance travelled at $36kmh^{-1}$

END